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FUZZY LABELING TREE

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Abstract: This paper extends the fuzzy labeling concept to fuzzy tree and the resultant graph is called as fuzzy labeling tree. Different properties of fuzzy labeling trees are discussed and also bipartite fuzzy labeling graph is defined and some of its properties are discussed. It contains an algorithm for finding the fuzzy spanning subgraph F of a fuzzy labeling tree G, such that G^* is complete and also the same algorithm can be used for finding the maximum spanning, strong arcs and fuzzy bridges of any fuzzy labeling graph.

AMS Subject Classification: 03E72, 05C72, 05C78
Key Words: fuzzy bridge, fuzzy labeling, fuzzy labeling tree, fuzzy spanning subgraph

1. Introduction

Fuzzy relation on a set was first defined by Zadeh in 1965 [9]. A fuzzy set is defined mathematically by assigning to each possible individual in the universe of discourse a value, representing its grade of membership, which corresponds to the degree, to which that individual is similar or compatible with the concept represented by the fuzzy set. Based on Zadeh's fuzzy relation the first definition

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of a fuzzy graph was introduced by Kaufmann in 1973. Fuzzy graphs have many more applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Azriel Rosenfeld in 1975 [7] developed the structure of fuzzy graphs and obtained analogs of several graph theoretical concepts like bridges and trees. Further Sunitha [8] established some characterization of fuzzy trees using fuzzy bridges and fuzzy cut nodes. Nagoorgani et. al [5] introduced the concept of bipartite fuzzy graph.

This note is a further contribution on fuzzy labeling. Fuzzy labeling for fuzzy tree is called a fuzzy labeling tree. In this paper the definition of fuzzy labeling graph is elaborated as compared to [6] and the definition of fuzzy labeling tree is defined and some of its properties are discussed. And also bipartite fuzzy labeling graph is defined and some of its results were discussed. An algorithm for finding the fuzzy spanning sub graph has been constructed and it is explained with an example.

All the definition and notations were followed as in [1, 2, 3, 4].

2. Preliminaries

A *fuzzy graph* $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where for all $u, v \in V$, we have $\mu(u, v) \leq \sigma(u)\Lambda\sigma(v)$. A *pathP* in a fuzzy graph is a sequence of distinct nodes $v_1, v_2, \ldots, v_n$ such that $\mu(v_i, v_{i+1}) > 0; 1 \leq i \leq n$; here $n > 1$ is called the *length of the path P*. The consecutive pairs (v_i, v_{i+1}) are called the *edge of the path.* A path P is called *a cycle* if $v_1 = n$ and $n \geq 3$. The *strength* of a path P is defined as $\underset{i=1}{\overset{n}{\Lambda}} \mu(v_i, v_{i+1})$. Let $G : (\sigma, \mu)$ be a fuzzy graph. Two nodes that are joined by a path are said to be connected. The *strength of connectedness* between two vertices u and v is $\mu^{\infty}(u, v) = \sup\{\mu^k(u, v)/k = 1, 2, \ldots\}$ where $\mu^k(u, v) = \sup\{\mu(uu_1)\Lambda\mu(u_1u_2)\Lambda \ldots \Lambda\mu(u_k - 1v)/u_1, \ldots, u_k - 1 \in V\}$. An arc of a fuzzy graph is called *strong* if its weight is at least as great as the strength of the connectedness of its end nodes when it is deleted. A path consisting of only strong arcs is said to be a strong path in G. An edge is called a *fuzzy bridge* of G if its removal reduces the strength of connectedness between some pair of nodes in G. A node is a fuzzy cut node of $G = (\sigma, \mu)$ if removal of it reduces the strength of the connectedness between some other pair of nodes. A connected fuzzy graph $G = (\sigma, \mu)$ is a fuzzy tree if it has a fuzzy spanning subgraph $F = (\sigma, \nu)$ which is a tree, where for all arcs (u, v) not in $F, \mu(u, v) < \nu^{\infty}(u, v)$. Equivalently, there is a path in F between

u and v whose strength exceeds $\mu(u,v)$. Let $h(\mu) = \vee\{\mu(x,y)/\mu(x,y) \in \mu^*\}$ is called height of G. The *degree of a vertex* u in G is defined by $d_G(u) = \sum_{u\neq v}\mu(uv) = \sum_{uv\in E}\mu(uv)$. Two nodes of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them. A subset S of V is said to be a fuzzy independent set of if any two nodes of S are fuzzy independent G. A fuzzy graph G is fuzzy bipartite if it has a spanning fuzzy subgraph $H = (\tau,\pi)$ which is bipartite where for all edges (u,v) not in H, weight of (u,v) in G is strictly less than the strength of pair (u,v) in H. A fuzzy bipartite graph G with fuzzy bipartition (V_1,V_2) is said to be a complete fuzzy bipartite if for each node of V_1, every node of V_2 is a strong neighbor. A fuzzy subgraph (τ,ν) spans the fuzzy graph (σ,μ) if $\sigma=\tau$ and $\nu(u,v) = \begin{cases}\mu(u,v), & \text{if } u,v\in\nu^*\\ 0, & \text{otherwise.}\end{cases}$

3. Properties of Fuzzy Labeling tree

Definition 3.1 ([6]). A graph $G = (\sigma,\mu)$ is said to be a fuzzy labeling graph, if $\sigma : V \to [0,1]$ and $\mu : V\times V \to [0,1]$ is bijective such that the membership value of edges and vertices are distinct and $\mu(u,v) < \sigma(u)\Lambda\sigma(v)$ for all $u,v\in V$.

Definition 3.2 ([6]). A cycle graph G^* is said to be a fuzzy labeling cycle graph if it has fuzzy labeling.

Definition 3.3. A graph $G = (\sigma,\mu)$ is said to be a fuzzy labeling tree, if it has fuzzy labeling and a fuzzy spanning subgraph $F = (\sigma,\nu)$ which is a tree, where for all arcs (u,v) not in $F, \mu(u,v) < \nu^\infty(u,v)$.

Proposition 3.4. If G is a fuzzy labeling tree, then the arcs of F are fuzzy bridges of G.

Proof. Let G be a fuzzy labeling tree and F be its spanning subgraph. Let (x,y) be an arc in F. Then $\mu'^\infty(x,y) < \mu(x,y) \le \mu^\infty(x,y)$, which implies the arc (x,y) is a fuzzy bridge of G. Conversely, if (x,y) is not a fuzzy bridge of G then $\mu'^\infty(x,y) > \mu^\infty(x,y) \ge \mu(x,y)$, which implies (x,y) is not an arc of F. □

Proposition 3.5. Every fuzzy labeling graph is a fuzzy labeling tree.

Proof. Let G be any fuzzy labeling graph. Since μ is bijective, each and every vertex of G will have at least one arc as fuzzy bridge. Therefore a spanning subgraph F will exist, such that whose arcs are fuzzy bridges. Hence by Proposition 3.4, every fuzzy labeling graph is a fuzzy labeling tree. □

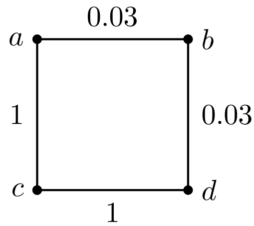

Figure 1: A connected fuzzy graph.

Remark 3.6. Proposition 3.5 is not true for general fuzzy graph. For example Fig. 1 is a fuzzy graph but not a fuzzy tree.

Proposition 3.7. If $G(\sigma, \mu)$ is a fuzzy labeling tree then its spanning subgraph $F(\sigma, \nu)$ is also a fuzzy labeling graph.

Proof. Let $G(\sigma, \mu)$ be a fuzzy labeling tree, by the definition of fuzzy labeling σ and μ are bijective in G. since F is its fuzzy spanning subgraph of $G, \mu = \nu$ if $(u, v) \in \nu^*$, which implies bijection is preserved in F. Hence F is a fuzzy labeling graph. □

Remark 3.8. All the properties of fuzzy labeling graph hold good for fuzzy labeling tree. As in the fuzzy graph, here also internal nodes of F are cut nodes since the arcs are fuzzy bridges. Here also the fuzzy spanning subgraph F is unique and which is the maximum spanning tree. And as μ is bijective one cannot conclude that the lower weighted arc will not be there in F. For example, consider Fig. 5, in which arc (x, y) is in F but (x, v) is not in F.

Proposition 3.9 ([6]). Let G be a fuzzy labeling cycle such that G^* is a cycle, then it has $(n-1)$ bridges.

Proposition 3.10. If G is a fuzzy labeling tree and F is its spanning subgraph, then $(G-F)^*$ is a tree.

Proof. Let G be a fuzzy labeling tree, such that G^* is not a tree. By the definition of fuzzy labeling tree there exists a spanning subgraph F, which is a tree. By Proposition 3.4, the arcs of F are fuzzy bridges of G. Therefore $(G-F)^*$ contains no fuzzy bridge. By Proposition 3.9, fuzzy labeling cycles have $(n-1)$ fuzzy bridges. Therefore $(G-F)^*$ contains no cycle. Hence $(G-F)^*$ is a tree. □

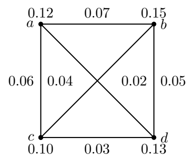

Figure 2: Fuzzy labeling tree G.

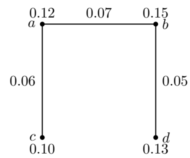

Figure 3: Spanning subgraph F.

Remark 3.11. Proposition 3.10 is not true if G^* is complete.

Proposition 3.12 ([6])**.** Every fuzzy labeling graph has at least one weakest arc.

Proposition 3.13. Let G be a fuzzy labeling tree and F be its spanning subgraph such that G^* is complete. Then $d_G(u) \neq d_F(u)$ for all $u, v \in V$.

Proof. Since G^* is complete, it will contains many cycles. By Proposition 3.12, every cycle have one arc as weakest arc, which will not be there in F, since the arcs of F are fuzzy bridges. Hence $d_G(u) \neq d_F(u)$ for all $u, v \in V$. □

Example 3.14 illustrates that, $d_G(u) \neq d_F(u)$ for all $u, v \in V$.

Example 3.14. In $G, d(a) = 0.17, d(b) = 0.14, d(c) = 0.13 d(d) = 0.10$ and in $F, d(a) = 0.13, d(b) = 0.12, d(c) = 0.06 d(d) = 0.05$.

Remark 3.15. The above proposition is not true for general fuzzy tree and other fuzzy labeling trees.

Proposition 3.16 ([6])**.** Let G be a fuzzy labeling graph such that G^* is a cycle then it has $(n-1)$ bridges.

Proposition 3.17. If G is a fuzzy labeling tree such that G^* is a cycle then its spanning subgraph F has $(n-1)$ fuzzy bridges.

Proof. It follows from Proposition 3.16. □

Remark 3.18. Proposition 3.17 is true for all fuzzy labeling trees.

4. Strong Arcs on Fuzzy Labeling tree

Proposition 4.1. Let G be a fuzzy labeling tree such that G^* is complete. Then every fuzzy bridge of G is strong and the converse is also true.

Proof. Let (u, v) be an arc of a fuzzy spanning subgraph F, which is a fuzzy bridge by Proposition 3.4 Therefore by definition $\mu(u, v) \geq \nu^{\infty}(u, v) = {\mu'}^{\infty}(u, v)$. Thus (u, v) is a strong arc of G. conversely let (u, v) be a strong arc of G, then $\mu(u, v) \geq {\mu'}^{\infty}(u, v) = \nu^{\infty}(u, v)$. Thus (u, v) is an arc of F. Hence (u, v) is a fuzzy bridge of G. □

Proposition 4.2. Let $G = (\sigma, \mu)$ be a fuzzy labeling tree and $F = (\sigma, \nu)$ be the fuzzy spanning subgraph of G, for all (x, y) not in F, then $\nu^{\infty}(x, y) \neq$ height of G.

Proof. Let us choose an arc (x, y) not in F, which implies $(x, y) \in G$ and which is not a fuzzy bridge of G, since the arcs of F are fuzzy bridges of G. By the definition of fuzzy labeling tree, if (x, y) is not in F, then $(x, y) < \nu^{\infty}(x, y)$. Since F is a tree, there exist only one path between x and y. Therefore the strength of connectedness between x and y is equal to the weight of the weakest arc. Since μ and ν are bijective, there exists only one weakest arc. This implies $\nu^{\infty}(x, y)$ is not equal to maximum of μ_i's. Hence $\nu^{\infty}(x, y) \neq$ height of G. □

Proposition 4.3 ([2])**.** Every bridge is strong, but a strong arc need not be a bridge.

Proposition 4.4. If G is a fuzzy labeling tree, such that G^* is complete with $|V| \geq 5$, then the spanning subgraph F has at least two disjoint strong paths.

Proof. Choose any two nodes of G say $v, w \in V$. Since G is a fuzzy labeling tree, G is connected also G^* is complete. Therefore there exist at least four disjoint paths between v and w. Now choose a path ρ_i if the arcs of ρ_i are not strong then $\mu(v_{i-1}, v_i) < {\mu'}^{\infty}(v_{i-1}, v_i)$. Hence all arcs have weights greater than $\mu(v_{i-1}, v_i)$. Then choose another path ρ_j and repeat the process. Evidently, the process cannot be repeated arbitrarily. Thus a strong path will exist between v and w. Since $|V| \geq 5$, it is possible to choose other nodes and the process can be repeated to find the strong path.

By Proposition 4.3, every bridge is strong; therefore the strong arc of G will exist in F. Since F is tree there exist no cycle. Hence, evidently we will get at least two disjoint strong paths in F. □

Proposition 4.5. If G is a fuzzy labeling tree then there exists a unique strong path between any two nodes of G.

Proof. If G^* is a tree, then it is done. Now choose a path (u, v) from a fuzzy labeling tree G, such that $\mu(u_i, v_i) > 0$ for all $1 \leq i \leq n$. If the arcs are strong, then $\mu(u_i, v_i) \geq \mu'^{\infty}(u_i, v_i)$. Hence each arc of $\mu(v_{i-1}, v_i)$ is greater than its strength of connectedness. Similarly choose another path between u and v, which is possible since G is connected. But as μ is bijective, getting another strong path is not possible. Hence strong path between any two nodes is unique. □

5. Bipartite Fuzzy labeling trees

Definition 5.1. A fuzzy labeling graph $G = (\sigma, \mu)$ is bipartite if the node set V can be partitioned into two nonempty sets V_1 and V_2 such that V_1 and V_2 are fuzzy independent sets.

Proposition 5.2. If G is a connected fuzzy labeling graph then there exists a strong path between any pair of nodes.

Proposition 5.3. Every fuzzy labeling tree is fuzzy bipartite graph.

Proof. Since G is a fuzzy labeling tree, it is connected. By Proposition 5.2, there exists a strong path between any two nodes of G. Therefore there exist a fuzzy independent set V_1 and V_2, such that the strong arc of the path have one node in V_1 and other in V_2. If G have a strong cycle then bipartite is not possible but strong cycle of any length will not exist in G, since μ is bijective. □

Proposition 5.4. If G is a fuzzy labeling tree such that G^* is $K^*_{1,n}$, then G is a complete bipartite graph.

Proof. It is trivial that G is a fuzzy labeling tree if G^* is a tree. Therefore $K^*_{1,n}$ is a fuzzy labeling tree which is also a complete bipartite graph because $K^*_{1,n}$ graph can be partition into two non empty independent set V_1 and V_2 such that $V_1 = \{v\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. All the arcs of G are strong arc. Therefore the node $v \in V_1$ is a strong neighbor of $v_1, v_2, \ldots, v_n\} \in V_2$. □

Corollary 5.5. *Every fuzzy labeling graph is not a complete bipartite graph.*

Also $K^*_{2,n}$, is not a complete bipartite graph.

Proposition 5.6 ([8])**.** Let $G = (\sigma, \mu)$ be a fuzzy graph such that G^* is a cycle. Then a node is a fuzzy cut node of G if and only if it is a common node of two fuzzy bridges.

Proposition 5.7. If G is a fuzzy labeling graph with $n \geq 4$, it has at least one node as cut node in each independent set.

Proof. Since G is a fuzzy labeling tree it has a fuzzy spanning subgraph F such that F^* is a tree. Therefore all the node of G will exist in F. By Proposition 5.3 the nodes of G can be partitioned into two non empty fuzzy independent sets V_1 and V_2. By Proposition 3.4, the arcs of F are fuzzy bridges which are also strong, since every fuzzy bridge is strong. Now choose any path $\{u, v, w, x, \}$, then there exist two internal nodes v and w. by Proposition 5.6, v and w are fuzzy cut nodes. Since v and w are strong neighbor, $v \in V_1$ and $w \in V_2$. $\square$

6. Algorithm for finding the spanning subgraph F of a fuzzy labeling tree G, such that G^* is complete

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Step 1. Consider a fuzzy labeling tree such that G^* is complete with $|V| = n$.

Step 2. Choose a cycle arbitrarily and remove the weakest arc (there exists only one weakest arc, since μ is bijective).

Step 3. Repeat step 2 until no cycle remains.

Step 4. The resulting graph is the spanning subgraph F of a fuzzy labeling graph G, whose arcs are fuzzy bridges.

Example 6.1. The above algorithm is illustrated with the following fuzzy labeling tree.

Example 6.2. Step 1: Fig. 4 is a fuzzy labeling tree with $|V| = 5$ and G^* is complete.
Steps 2 and 3: Choose a cycle $uvwu$ and remove the weakest arc (u, v). Similarly choose the remaining cycles and remove the weakest arc.
Step 4: The resulting graph Fig. 5 is a spanning subgraph F of G with $(n - 1)$ bridges, which is a tree.

Remark 6.3. C-Program has been written for the above algorithm. Also note that the same algorithm can be used for finding the spanning subgraph F of any fuzzy labeling tree G and note that we can use the same algorithm for finding fuzzy bridge of any fuzzy labeling graph. But it does not hold good for general fuzzy graph. This is explained with an example in Fig. 6.

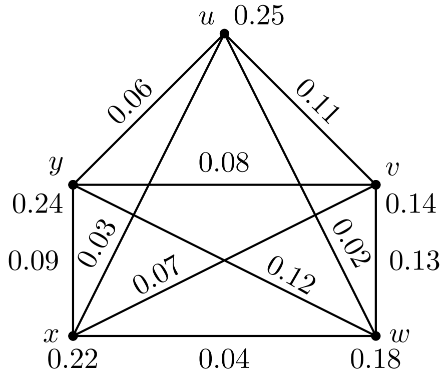

Figure 4: A fuzzy labeling tree G.

If we use the above algorithm, then the fuzzy bridges are either $(b, d), (a, b)$ or $(b, d)(a, d)$ or $(b, d), (a, c)$. But (b, d) is the only fuzzy bridge.

7. Concluding remarks

This note is an extension of fuzzy labeling to fuzzy labeling tree. Fuzzy labeling tree is a special case of fuzzy tree. Therefore the applications of fuzzy tree may also fit for fuzzy labeling tree. As it is a new concept it can be extended further. The remaining work will be discussed in the forthcoming papers.

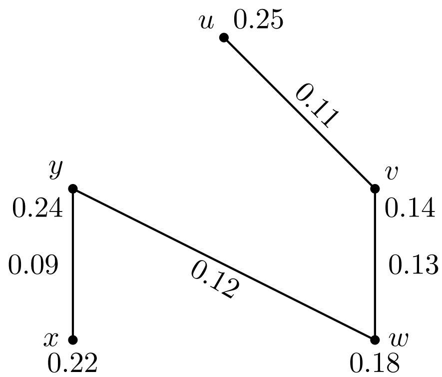

Figure 5: Fuzzy spanning subgraph G.

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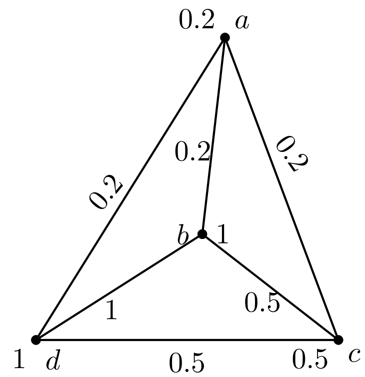

Figure 6: A fuzzy labeling graph G.

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