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MINIMUM COVERING HARARY ENERGY OF A GRAPH

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Abstract: In this paper we computed minimum covering Harary energies of a star graph, complete graph, crown graph, bipartite graph and cocktail party graphs. Upper and lower bounds for minimum covering Harary energies are also established.

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1. Introduction

The concept of energy of a graph was introduced by I. Gutman [11] in the year 1978. Let G be a graph with n vertices $\{v_1, v_2, ..., v_n\}$ and m edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

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of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

For details on the mathematical aspects of the theory of graph energy see the reviews [12], papers [6, 7, 13] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [15, 17], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [9, 14].

The Harary matrix of G is the square matrix of order n whose (i, j)-entry is $\frac{1}{d_{ij}}$ where d_{ij} is the distance between the vertices v_i and v_j. Let $\rho_1, \rho_2, ..., \rho_n$ be the eigenvalues of the Harary matrix of G. The Harary energy HE is defined by $HE = HE(G) := \sum_{i=1}^{n} |\rho_i|$. Detailed studies on distance energy and Harary energy can be found in [5, 10, 21, 23, 8, 22].

Recently Prof. Chandrashekar Adiga et al. [1] have defined the minimum covering energy, $E_C(G)$ of a graph G which depends on its particular minimum cover C. Motivated by this paper, we introduced the concept of minimum covering Harary energy $HE_C(G)$ of a graph G and computed minimum covering Harary energies of a star graph, complete graph, crown graph, bipartite graph and cocktail party graphs. Upper and lower bounds for $HE_C(G)$ are also established.

Further, studies on minimum covering energy, maximum degree energy, minimum dominating energy, minimum covering distance energies can be found in [1, 2, 18, 19, 20] and the references cited there in.

2. Definitions and Example

2.1. The Minimum Covering Energy of a Graph

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. A subset C of V is called a covering set of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a minimum covering set. Let C be a minimum covering set of a graph G. The minimum covering matrix of G is the $n \times n$ matrix defined by $A_C(G) := (a_{ij})$, where $a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise} \end{cases}$

The characteristic polynomial of $A_C(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A_C(G))$. The minimum covering eigenvalues of the graph G are the eigenvalues of $A_C(G)$. Since $A_C(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The minimum covering energy of G is then defined as $E_C(G) = \sum_{i=1}^{n} |\lambda_i|$.

2.2. The Minimum Covering Harary Energy of a Graph

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. Let C be a minimum covering set of a graph G. The minimum covering Harary matrix of G is the $n \times n$ matrix defined by $H_C(G) := (h_{ij})$,

where $h_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{if } i = j \text{ and } v_i \notin C \\ \frac{1}{d(v_i, v_j)} & \text{otherwise} \end{cases}$

The characteristic polynomial of $H_C(G)$ is denoted by $f_n(G, \rho)= det(\rho I - H_C(G))$. The minimum covering Harary eigenvalues of the graph G are the eigenvalues of $H_C(G)$. Since $H_C(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$. The minimum covering Harary energy of G is defined as $HE_C(G) := \sum_{i=1}^{n} |\rho_i|$

Note that the trace of $H_C(G) = |C|$.

In this paper, we are interested in studying mathematical aspects of the minimum covering Harary energy of a graph. The application of minimum covering Harary energy in other branches of science have to be investigated.

Example 1. The possible minimum covering sets for the following graph G (see Figure 1) are i) $C_1=\{v_1, v_2, v_5\}$, ii) $C_2=\{v_2, v_4, v_5\}$, iii) $C_3=\{v_1, v_3, v_5\}$.

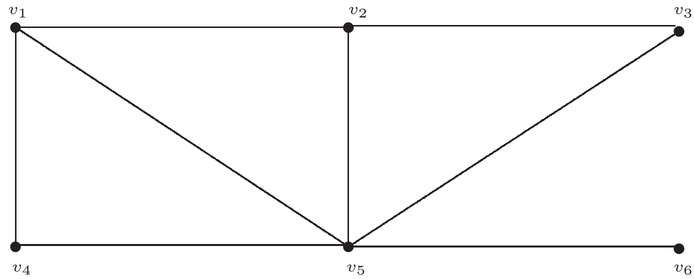

Figure 1

i) $H_{C_1}(G) = \begin{pmatrix} 1 & 1 & 1/2 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1/2 & 1 & 1/2 \\ 1/2 & 1 & 0 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1/2 & 0 & 1 & 1/2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 0 \end{pmatrix}$

Characteristic equation is $\rho^6 - 3\rho^5 - \frac{27}{4}\rho^4 - 2\rho^3 + \frac{23}{16}\rho^2 + \frac{3}{8}\rho - \frac{1}{8} = 0$.
Minimum covering Harary eigenvalues are $\rho_1 \approx .3090169944, \rho_2 \approx -.8090169944$, $\rho_3 \approx .2577609956, \rho_4 \approx 4.560355576, \rho_5 \approx -.5641882616,\ \rho_6 \approx -.7539283099$.
Minimum covering Harary energy, $HE_{C_1}(G) \approx 7.254267133$.

ii) $H_{C_2}(G) = \begin{pmatrix} 0 & 1 & 1/2 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1/2 & 1 & 1/2 \\ 1/2 & 1 & 0 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1/2 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 0 \end{pmatrix}$

Characteristic equation is $\rho^6 - 3\rho^5 - \frac{27}{4}\rho^4 - \frac{5}{4}\rho^3 + \frac{39}{16}\rho^2 + \frac{13}{16}\rho - \frac{1}{16} = 0$.

Minimum covering Harary eigenvalues are $\rho_1 \approx .6486997113, \rho_2 \approx .5939853888, \rho_3 \approx 4.524661925, \rho_4 \approx -.5693526951, \rho_5 \approx -.6596347450, \rho_6 \approx -.9545298444$.
Minimum covering Harary energy, $HE_{C_2}(G) \approx 7.950864309$.
$\therefore$ Minimum covering Harary energy depends on the covering set.

3. Minimum Covering Harary Energy of some Standard Graphs

Definition 3.1. The Cocktail party graph is denoted by $K_{n\times 2}$, is a graph having the vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and the edge set $E = \{u_iu_j, v_iv_j : i \neq j\} \bigcup \{u_iv_j, v_iu_j : 1 \leq i < j \leq n\}$.

Theorem 3.1. *The minimum covering Harary energy of Cocktail party graph $K_{n\times 2}$ is $(n-1)+\sqrt{4n^2+4n-7}$.*

Proof. Let $K_{n\times 2}$ be the Cocktail party graph with vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$. The minimum covering set is $C=\bigcup_{i=1}^{n-1}\{u_i,v_i\}$.

$$\text{Then } H_C(K_{n\times 2})=\begin{pmatrix} 1 & 1/2 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1/2 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1/2 & \ldots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1/2 & 1 & \ldots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1/2 & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1/2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 1/2 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1/2 & 0 \end{pmatrix}$$

Characteristic equation is
$\frac{1}{4^n}(\,2\rho+1)^{n-1}(\,2\rho-1)^{n-1}[\,4\rho^2-8(n-1)\rho-(12n-11)\,]=0$
Minimum covering Harary eigenvalues are $\rho=\frac{-1}{2}[(n-1)\text{times}]$, $\rho=\frac{1}{2}[(n-1)$ times], $\rho=\dfrac{(2n-2)\pm\sqrt{4n^2+4n-7}}{2}$ [one time each]

Minimum covering Harary energy, $HE_C(K_{n\times 2})$

$$=|\tfrac{-1}{2}|(n-1)+|\tfrac{1}{2}|(n-1)+\left|\frac{(2n-2)\,+\,\sqrt{4n^2+4n-7}}{2}\right|+\left|\frac{(2n-2)-\sqrt{4n^2+4n-7}}{2}\right|$$
$$=(n-1)+\sqrt{4n^2+4n-7}.$$ □

Theorem 3.2. *For $n\geq 3$, the minimum covering Harary energy of Star graph $K_{1,n-1}$ is equal to $\dfrac{(n+1)-\sqrt{n^2+10n+9}}{2}$.*

Proof. Consider the Star graph $K_{1,n-1}$ with vertex set $V=\{v_0,v_1,v_2,...,v_{n-1}\}$. The Minimum covering set $C=\{v_0\}$. Then

$$H_C(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & \frac{1}{2} & \dots & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 & \dots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{2} & \frac{1}{2} & \dots & 0 \end{pmatrix}_{n \times n}$$

Characteristic equation is $\dfrac{1}{2^n}(2\rho+1)^{n-1}[\,2\rho^2-(n+1)\rho-(n-1)]=0$

The minimum covering Harary eigenvalues are

$\rho = \frac{-1}{2}$ [$(n-1)$ times], $\rho = \dfrac{(n+1) \pm \sqrt{n^2+10n+9}}{4}$ [one time each].

Minimum covering Harary energy is, $HE_C(K_{1,n-1})$

$$= |\tfrac{-1}{2}|(n-1) + \left|\frac{(n+1)+\sqrt{n^2+10n+9}}{4}\right| + \left|\frac{(n+1)-\sqrt{n^2+10n+9}}{4}\right|$$

$$= \frac{(n+1)-\sqrt{n^2+10n+9}}{2}.$$ □

Definition 3.2. The Crown graph S_n^0 for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}$. $\therefore$ S_n^0 coincides with the Complete bipartite graph $K_{n,n}$ with horizontal edges removed.

Theorem 3.3. *The minimum covering Harary energy of the Crown graph* S_n^0 *is equal to* $\dfrac{(5n-5)+\sqrt{36n^2-48n+25}}{3}$

Proof. For the Crown graph S_n^0 with vertex set $V = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$. The minimum covering set is $C = \{u_1, u_2, ..., u_n\}$. Then

$$H_C(S_n^0) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} & \frac{1}{3} & 1 & 1 & \ldots & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} & \ldots & \frac{1}{2} & 1 & \frac{1}{3} & 1 & \ldots & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \ldots & \frac{1}{2} & 1 & 1 & \frac{1}{3} & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & 1 & 1 & 1 & 1 & \ldots & \frac{1}{3} \\ \frac{1}{3} & 1 & 1 & \ldots & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\ 1 & \frac{1}{3} & 1 & \ldots & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \ldots & \frac{1}{2} \\ 1 & 1 & \frac{1}{3} & \ldots & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \ldots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & 0 \end{pmatrix}_{(2n\times 2n)}$$

Characteristic equation is
$\frac{1}{36^n}(6\rho-5)^{n-1}(6\rho+5)^{n-1}[36\rho^2-36n\rho-(27n^2--48n+25)]=0.$
Minimum covering Harary eigenvalues are $\rho=\frac{5}{6}[(n-1)\text{times}]$,

$\rho=\frac{-5}{6}[(n-1)\text{ times}]$, $\rho=\dfrac{3n\pm\sqrt{36n^2-48n+25}}{6}$ [1 time each]

Minimum covering Harary energy, $HE_C(K_{n\times 2})$

$$= |\tfrac{5}{6}|(n-1)+|\tfrac{-5}{6}|(n-1)+\left|\frac{3n+\sqrt{36n^2-48n+25}}{6}\right|+\left|\frac{3n-\sqrt{36n^2-48n+25}}{6}\right|$$
$$=\frac{(5n-5)+\sqrt{36n^2-48n+25}}{3}.$$

□

Theorem 3.4. *The minimum covering Harary energy of a Complete bipartite graph $K_{m,n}$, $(m\le n)$ is*
$$\frac{(m+n)+\sqrt{(m^2+n^2+14mn+4m-4n+4)}}{2}.$$

Proof. For the Complete bipartite graph $K_{m,n}(m\le n)$ with vertex set $V=\{u_1,u_2,...,u_m,v_1,v_2,...,v_n\}$ the minimum covering set is $C=\{u_1,u_2,...,u_m\}$. Then

$$H_C(K_{m,n}) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & 1 & 1 & 1 & \dots & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} & \dots & \frac{1}{2} & 1 & 1 & 1 & \dots & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \dots & \frac{1}{2} & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ 1 & 1 & 1 & \dots & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & \frac{1}{2} \\ 1 & 1 & 1 & \dots & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & 0 \end{pmatrix}_{(m+n)\times(m+n)}$$

Characteristic equation is

$$\frac{(2\rho-1)^{m-1}(2\rho+1)^{n-1}[\,4\rho^2-2(m+n)\rho-(3mn+m-n+1)]}{2^{m+n}} = 0.$$

Minimum covering Harary eigenvalues are $\rho = \frac{1}{2}$ [$(m-1)$ times], $\rho = \frac{-1}{2}$ [$(n{-}1)$ times] and $\rho = \dfrac{(m+n) \pm \sqrt{m^2+n^2+14mn+4m-4n+4}}{4}$ [one time each].

Minimum covering Harary energy is, $HE_C(K_{m,n}) =$

$$|\tfrac{1}{2}|(m{-}1) + |\tfrac{-1}{3}|(n{-}1) + \left|\frac{(m+n)+\sqrt{(m^2+n^2+14mn+4m-4n+4)}}{4}\right|$$
$$+\left|\frac{(m+n)-\sqrt{(m^2+n^2+14mn+4m-4n+4)}}{4}\right|$$
$$= \frac{(m+n)+\sqrt{(m^2+n^2+14mn+4m-4n+4)}}{2}$$ □

Theorem 3.5. *For $n \geq 2$, the minimum covering Harary energy of Complete graph K_n is $\sqrt{(n+3)(n-1)}$.*

Proof. For complete graphs the minimum covering Harary matrix is same as minimum covering matrix [1], therefore the Minimum covering Harary energy is equal to Minimum covering energy.

□

4. Properties of Minimum Covering Harary Eigenvalues

Theorem 4.1. *Let G be a simple graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, edge set E and $C = \{u_1, u_2, ..., u_k\}$ be a minimum covering set. If $\rho_1, \rho_2, ..., \rho_n$ are the eigenvalues of minimum covering Harary matrix $H_C(G)$ then (i) $\sum_{i=1}^{n} \rho_i$ $= |C|$*

(ii) $\sum_{i=1}^{n} \rho_i^2 =| C | +2\sum_{i<j} \frac{1}{d(v_i, v_j)^2}$.

Proof. **(i)** We know that the sum of the eigenvalues of $H_C(G)$ is the trace of $H_C(G)$

$$\therefore \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} h_{ii} = |C| \ .$$

(ii) Similarly the sum of squares of the eigenvalues of $H_C(G)$ is trace of $[H_C(G)]^2$

$$\begin{aligned}
\therefore \sum_{i=1}^{n} \rho_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} h_{ji} \\
&= \sum_{i=1}^{n} (h_{ii})^2 + \sum_{i \neq j} h_{ij} h_{ji} \\
&= \sum_{i=1}^{n} (h_{ii})^2 + 2\sum_{i<j} (h_{ij})^2 \\
&=| C | +2\sum_{i<j} \frac{1}{d(v_i, v_j)^2}
\end{aligned}$$

□

Corollary 4.2. *Let G be a (n, m)simple graph with diameter 2 and $C = \{u_1, u_2, ..., u_k\}$ be a minimum covering set. If $\rho_1, \rho_2, ..., \rho_n$ are the eigenvalues of minimum covering Harary matrix $H_C(G)$ then $\sum_{i=1}^{n} \rho_i^2 = \frac{4|C| + n^2 - n + 6m}{4}$.*

Proof. We know that in $H_C(G)$ there are $2m$ elements with 1 and $n(n-1)-2m$ elements with $\frac{1}{2}$ and hence corollary follows from the above theorem. □

5. Bounds for Minimum Covering Harary Energy

Similar to McClelland's [17] bounds for energy of a graph, bounds for $HE_C(G)$ are given in the following theorem.

Theorem 5.1. *Let G be a simple (n, m) graph. If C is the minimum covering set and $P = |H_C(G)|$ then*

$$\sqrt{| C | + 2\sum_{i<j}\frac{1}{d(v_i,v_j)^2} + n(n-1)P^{\frac{2}{n}}} \leq HE_C(G) \leq \sqrt{n\left(| C | + 2\sum_{i<j}\frac{1}{d(v_i,v_j)^2}\right)}$$

where $|C|$ is the cardinality of minimum covering set.

Proof.

Cauchy Schwarz inequality is $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$

If $a_i = 1, b_i = | \rho_i |$ then $\left(\sum_{i=1}^{n} | \rho_i |\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \rho_i^2\right)$

By using Theorem (4.1), we obtain

$$[HE_C(G)]^2 \leq n\left(| C | + 2\sum_{i<j}\frac{1}{d(v_i,v_j)^2}\right)$$

$$\Longrightarrow HE_C(G) \leq \sqrt{n\left(| C | + 2\sum_{i<j}\frac{1}{d(v_i,v_j)^2}\right)}$$

Since arithmetic mean is not smaller than geometric mean we have

$$\frac{1}{n(n-1)}\sum_{i\neq j} | \rho_i || \rho_j | \geq \left[\prod_{i\neq j} | \rho_i || \rho_j |\right]^{\frac{1}{n(n-1)}}$$

$$= \left[\prod_{i=1}^{n} | \rho_i |^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$= \left[\prod_{i=1}^{n} | \rho_i |\right]^{\frac{2}{n}}$$

$$= \left|\prod_{i=1}^{n} \rho_i\right|^{\frac{2}{n}}$$
$$= |H_C(G)|^{\frac{2}{n}} = P^{\frac{2}{n}}$$

$$\therefore \sum_{i\neq j} |\ \rho_i \ || \ \rho_j\ | \geq n(n-1)P^{\frac{2}{n}} \tag{1}$$

Now consider, $[HE_C(G)]^2 = \left(\sum_{i=1}^{n} |\ \rho_i\ |\right)^2$

$$= \sum_{i=1}^{n} |\ \rho_i\ |^2 + \sum_{i\neq j} |\ \rho_i\ ||\ \rho_j\ |$$

$$\therefore \quad [HE_C(G)]^2 \geq \left(|\ C\ | + 2\sum_{i<j} \frac{1}{d(v_i,v_j)^2}\right) + n(n-1)P^{\frac{2}{n}}$$

$$i.e.,\ HE_C(G) \geq \sqrt{|\ C\ | + 2\sum_{i<j} \frac{1}{d(v_i,v_j)^2} + n(n-1)P^{\frac{2}{n}}}$$

□

Theorem 5.2. *If $\rho_1(G)$ is the largest minimum covering Harary eigen value of $H_C(G)$, then $\rho_1(G) \geq \dfrac{2\sum_{i<j}\frac{1}{d(v_i,v_j)} + |C|}{n}$ where $|C|$ is the cardinality of minimum covering set.*

Proof. Let X be any nonzero vector. Then by [3], We have $\rho_1(H_C) = \max_{X\neq 0}\left\{\frac{X'H_CX}{X'X}\right\} \therefore \rho_1(H_C) \geq \frac{J'H_CJ}{J'J} = \dfrac{2\sum_{i<j}\frac{1}{d(v_i,v_j)} + |C|}{n}$ where J is a unit matrix. □

Lemma 5.1. *Let G be a graph of diameter 2 and $\rho_1(G)$ is the largest minimum covering Harary eigen value of $H_C(G)$, then $\rho_1(G) \geq \frac{2m+n^2-n+2|C|}{2n}$ where $|C|$ is the cardinality of minimum covering set.*

Proof. Let G be a connected graph of diameter 2 and d_i denotes the degree of vertex v_i. Clearly i-th row of H_C consists of d_i number of 1's and $(n-d_i-1)$ number of $\frac{1}{2}$'s. By using Rayleigh's principle, for $J=[1,1,1,\cdots,1]$ we have

$$\rho_1(H_C) \geq \frac{J'H_CJ}{J'J} = \frac{\sum_{i=1}^{n}\left[d_i \times 1 + (n-d_i-1)\frac{1}{2}\right] + |C|}{n}$$

$$= \frac{\sum_{i=1}^{n}\frac{(d_i+n-1)}{2} + |C|}{n} = \frac{2m+n^2-n+2|C|}{2n}.$$

□

Similar to Koolen and Moulton's [16] upper bound for energy of a graph, upper bound for $HE_C(G)$ is given in the following theorem.

Theorem 5.3. *If G is a (m,n) graph with diameter 2 and $\frac{2m+n^2-n+2|C|}{2n} \geq 1$ then $HE_C(G) \leq \frac{2m+n^2-n+2|C|}{2n}$ $+\sqrt{(n-1)\left[\frac{4|C|+n^2-n+6m}{4} - \left(\frac{2m+n^2-n+2|C|}{2n}\right)^2\right]}$.*

Proof.

$$\text{Cauchy-Schwartz inequality is}\left[\sum_{i=2}^{n} a_i b_i\right]^2 \leq \left(\sum_{i=2}^{n} a_i^2\right)\left(\sum_{i=2}^{n} b_i^2\right)$$

$$\text{Put } a_i = 1, b_i = \mid \rho_i \mid \text{ then}\left(\sum_{i=2}^{n} \mid \rho_i \mid\right)^2 \leq \sum_{i=2}^{n} 1 \sum_{i=2}^{n} \rho_i^2$$

From corollary (4.2) we have,

$$[HE_C(G) - \rho_1]^2 \leq (n-1)\left(\frac{4|C|+n^2-n+6m}{4} - \rho_1^2\right)$$

$$\Rightarrow HE_C(G) \leq \rho_1 + \sqrt{(n-1)\left(\frac{4|C|+n^2-n+6m}{4} - \rho_1^2\right)}$$

Let $f(x) = x + \sqrt{(n-1)\left(\frac{4|C|+n^2-n+6m}{4} - x^2\right)}$.

For decreasing function

$$f'(x) \leq 0 \Rightarrow 1 - \frac{x(n-1)}{\sqrt{(n-1)\left(\frac{4|C|+n^2-n+6m}{4} - x^2\right)}} \leq 0$$

$$\Rightarrow x \geq \sqrt{\frac{4|C| + n^2 - n + 6m}{4n}} \; \therefore \; f(x) \text{ is}$$

decreasing in $\Big[\sqrt{\frac{4|C| + n^2 - n + 6m}{4n}}, \sqrt{\frac{4|C| + n^2 - n + 6m}{4}}\Big]$. Clearly,

$$\sqrt{\frac{2m + n^2 - n + 2|C|}{2n}} \in \Big[\sqrt{\frac{4|C| + n^2 - n + 6m}{4n}}, \sqrt{\frac{4|C| + n^2 - n + 6m}{4}}\Big].$$

Since $\frac{2m + n^2 - n + 2|C|}{2n} \geq 1$, and by using Lemma (5.1), we have

$$\sqrt{\frac{2m + n^2 - n + 2|C|}{2n}} \leq \frac{2m + n^2 - n + 2|C|}{2n} \leq \rho_1$$

$$\therefore f(\rho_1) \leq f\Big(\frac{2m + n^2 - n + 2|C|}{2n}\Big)$$

$$HE_C(G) \leq f(\rho_1) \leq f\Big(\frac{2m + n^2 - n + 2|C|}{2n}\Big)$$

$$HE_C(G) \leq f\Big(\frac{2m + n^2 - n + 2|C|}{2n}\Big).$$

Therefore

$$HE_C(G) \leq \frac{2m + n^2 - n + 2|C|}{2n} + \sqrt{(n-1)\Big[\frac{4|C| + n^2 - n + 6m}{4} - \Big(\frac{2m + n^2 - n + 2|C|}{2n}\Big)^2\Big]}.$$

□

Bapat and S.pati [4] proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum covering energy is given in the following theorem.

Lemma 5.2. *Let G be a graph with a minimum covering set C. If the minimum covering Harary energy $HE_C(G)$ is a rational number, then $HE_C(G) \equiv |C| \pmod 2$.*

Proof. Proof is similar to Theorem 3.7 of [1]. □

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