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V-TRANSFORMATION OF STRONG VARIETIES OF PARTIAL ALGEBRAS

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Abstract: In this paper, we define V-transformation and prove that V- transformation is a bijective if V- transformation is a composition of binary relations with inverse of itself then equal relation delte of C-terms.

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1. Introduction

Let $P^n(A) := \{f : A^n \multimap A\}$ be the set of all $n-ary$ partial operations defined on the set A and let $P(A) := \bigcup_{n=1}^{\infty} P^n(A)$ be the set of all partial operations on A. A *partial algebra* $\mathcal{A} = (A; (f_i^A)_{i\in I})$ of type $\tau = (n_i)_{i\in I}$ is a pair consisting of a set A and an indexed set $(f_i^A)_{i\in I}$ of partial operations where f_i^A is $n_i - ary$.

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Let $PAlg(\tau)$ be the class of all partial algebras of type τ. Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ and let $W_\tau(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations induced by n-ary terms on the partial algebra $\mathcal{A}$. For the definition of a term operation $t^{\mathcal{A}}$ induced by the term t on the partial algebra $\mathcal{A}$ (see [1]). Different from the total case, the set $W_\tau(X_n)^{\mathcal{A}}$ is in general a proper subset of $P^n(A)$. Therefore in [1] F. Börner introduced another concept of terms for partial algebras.

Let X be an alphabet and let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ, where each f_i has arity n_i and $X \cap \{f_i \mid i \in I\} = \phi$. We will need additional symbols $\varepsilon_j^k \notin X$, for every $k \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and $1 \leq j \leq k$. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element alphabet. The set of all n-ary terms of type τ over X_n is defined inductively as follows (see [1]):

(i) every $x_i \in X_n$ is an n-ary term of type τ;

(ii) if $w_1, \ldots, w_k$ are n-ary terms of type τ, then $\varepsilon_j^k(w_1, \ldots, w_k)$ is an n-ary term of type τ for all $1 \leq j \leq k$ and all $k \in \mathbb{N}^+$;

(iii) if $w_1, \ldots, w_n$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol, then $f_i(w_1, \ldots, w_{n_i})$ is an n-ary term of type τ.

Let $W_\tau^C(X_n)$ be the set of all n-ary terms of type τ. Then $W_\tau^C(X) := \bigcup_{n=1}^{\infty} W_\tau^C(X_n)$ denotes the set of all terms of this type.

On the sets $W_\tau^C(X_n)$ we may introduce the following superposition operations. Let $w_1, \ldots, w_m$ be n-ary terms and let t be an m-ary term. Then we define an n-ary term $\overline{S}_n^m(t, w_1, \ldots, w_m)$ inductively by the following steps:

(i) For $t = x_j$, $1 \leq j \leq m$ (m-ary variable), we define $\overline{S}_n^m(x_j, w_1, \ldots, w_m) = w_j$.

(ii) For $t = \varepsilon_j^k(s_1, \ldots, s_k)$ we set

$$\overline{S}_n^m(t, w_1, \ldots, w_m) = \varepsilon_j^k(\overline{S}_n^m(s_1, w_1, \ldots, w_m), \ldots, \overline{S}_n^m(s_k, w_1, \ldots, w_m)),$$

where $s_1, \ldots, s_k$ are m-ary, for all $k \in \mathbb{N}^+$ and $1 \leq j \leq k$.

(iii) For $t = f_i(s_1, \ldots, s_{n_i})$ we set

$$\overline{S}_n^m(t, w_1, \ldots, w_m) = f_i(\overline{S}_n^m(s_1, w_1, \ldots, w_m), \ldots, \overline{S}_n^m(s_{n_i}, w_1, \ldots, w_m)),$$

where $s_1, \ldots, s_{n_i}$ are m-ary.

This defines an operation

$$\overline{S}_n^m : W_\tau^C(X_m) \times (W_\tau^C(X_n))^m \longrightarrow W_\tau^C(X_n),$$

which describes the superposition of terms.

The *term clone* of type τ is the heterogeneous algebra

$$clone\ \tau^C := ((W_\tau^C(X_n)); \overline{S}_n^m, e_j^k)_{n,m,k \in \mathbb{N}^+, 1 \leq j \leq k},$$

where $e_j^k := x_j \in X_k$, $1 \leq j \leq k$.

Every n-ary term $w \in W_\tau^C(X_n)$ induces an n-ary term operation $w^{\mathcal{A}}$ of any partial algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ of type τ. For $a_1, \ldots, a_n \in A$, the value $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined in the following inductive way (see [1]) :

(i) If $w = x_i$ then $w^{\mathcal{A}} = x_i^{\mathcal{A}} = e_i^{n,A}$, where $e_i^{n,A}$ is as usual the n-ary total projection on the i-th component.

(ii) If $w = \varepsilon_j^k(w_1, \ldots, w_k)$ and we assume that $w_1^{\mathcal{A}}, \ldots, w_k^{\mathcal{A}}$ are the term operations induced by the terms $w_1, \ldots, w_k$ and that the $w_i^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined for $1 \leq i \leq k$, then $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $w^{\mathcal{A}}(a_1, \ldots, a_n) = w_j^{\mathcal{A}}(a_1, \ldots, a_n)$.

(iii) Now assume that $w = f_i(w_1, \ldots, w_{n_i})$ where f_i is an n_i-ary operation symbol, and assume that the $w_j^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined, with values $w_j^{\mathcal{A}}(a_1, \ldots, a_n) = b_j$ for $1 \leq j \leq n_i$. If $f_i^A(b_1, \ldots, b_{n_i})$ is defined, then $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and

$$w^{\mathcal{A}}(a_1, \ldots, a_n) = f_i^A(w_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, w_{n_i}^{\mathcal{A}}(a_1, \ldots, a_n)).$$

Definition 1. (see [4]) A pair $t_1 \approx t_2 \in W_\tau^C(X)^2$ is called a *strong identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{s}{\models} t_1 \approx t_2$) iff $t_1^{\mathcal{A}}$ is defined whenever $t_2^{\mathcal{A}}$ is defined and conversely and $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ on the common domain, i.e. the induced partial term operations $t_1^{\mathcal{A}}$ and $t_2^{\mathcal{A}}$ are equal.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W_\tau^C(X)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau^C(X)^2$ given by the following two operators:

$$Id^s : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau^C(X)^2)$$

and

$$Mod^s : \mathcal{P}(W_\tau^C(X)^2) \to \mathcal{P}(PAlg(\tau))$$

with

$$Id^sK := \{s \approx t \in W_\tau^C(X)^2 \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{s}{\models} s \approx t)\} \qquad \text{and}$$

$$Mod^s\Sigma := \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma\ (\mathcal{A} \underset{s}{\models} s \approx t)\}.$$

Clearly, the pair (Mod^s, Id^s) is a Galois connection between $PAlg(\tau)$ and $W_\tau^C(X)^2$. We have two closure operators Mod^sId^s and Id^sMod^s and their sets of fixed points.

Definition 2. Let $V \subseteq PAlg(\tau)$ be a class of partial algebras of type τ. The class V is called a *strong variety* of partial algebras if $V = Mod^sId^sV$.

Definition 3. (see [5]) Let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ and $W_\tau^C(X)$ be the set of all terms of this type. A mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau^C(X)$ which maps each n_i-ary fundamental operation f_i to a term of arity n_i is called a *hypersubstitution* of type τ.

Any hypersubstitution σ of type τ can be extended to map $\widehat{\sigma} : W_\tau^C(X) \longrightarrow W_\tau^C(X)$ defined for all terms, in the following way:

(i) $\widehat{\sigma}[x_i] = x_i$ for every $x_i \in X_n$,

(ii) $\widehat{\sigma}[\varepsilon_j^k(s_1, \ldots, s_k)] = \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_k])$, where $s_1, \ldots, s_k \in W_\tau^C(X_n)$,

(iii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \overline{S}_n^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, where $t_1, \ldots, t_{n_i} \in W_\tau^C(X_n)$.

Let $Var(t)$ be the set of all variables occurring in the term t.

Definition 4. (see [3]) The hypersubstitution σ is called *regular* if

$$Var(\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}, \quad i \in I.$$

Let $Hyp_R^C(\tau)$ be the set of all regular hypersubstitutions of type τ.

Lemma 5. (see [5]) *Let $\sigma_1, \sigma_2 \in Hyp_R^C(\tau)$. Then $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ } = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$, where $\circ$ is the usual composition of functions.*

Now we define a product of regular hypersubstitutions is the usual way, by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$.

Theorem 6. (see [5]) *The algebra $(Hyp_R^C(\tau); \circ_h, \sigma_{id})$ with $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ is a monoid.*

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of type τ and $\sigma \in Hyp_R(\tau)$. We let

$$\sigma(\mathcal{A}) := (A; (\sigma(f_i)^{\mathcal{A}})_{i\in I}),$$

which is called *derived algebra* of type τ, where $\sigma(f_i)^{\mathcal{A}}$ is the term operation induced by the term $\sigma(f_i)$ on the algebra $\mathcal{A}$.

2. Tree Transformations

Definition 7. Let σ be a regular hypersubstitution. Then

$$T_\sigma := \{(t, \widehat{\sigma}[t]) \mid t \in W_\tau^C(X)\}$$

is called the *tree transformation* defined by the regular hypersubstitution σ.

We denote by $T_{\sigma_1} \circ T_{\sigma_2}$ the composition of the tree transformations T_{σ_1} and T_{σ_2}. Since a tree transformation T_σ is a relation, we can consider inverses, domains and ranges of such transformations under the relational composition $\circ$. We define

$$T_{Hyp_R^C(\tau)} := \{T_\sigma \mid \sigma \in Hyp_R^C(\tau)\}$$

and prove that

Theorem 8. *$(T_{Hyp_R^C(\tau)}; \circ, T_{\sigma_{id}})$ is a monoid which is isomorphic to the monoid $Hyp_R^C(\tau)$ of all regular hypersubstitutions of type τ.*

Proof. We define a mapping $\varphi : Hyp_R^C(\tau) \to T_{Hyp_R^C(\tau)}$ by $\sigma \mapsto T_\sigma$. Clearly, φ is well-defined and surjective.

(i): We show that $T_{\sigma_1} \circ T_{\sigma_2} = T_{\sigma_1 \circ_h \sigma_2}$, i.e. $\varphi(\sigma_1 \circ_h \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$. Indeed, we have

$$\begin{aligned}
(t, t'') \in T_{\sigma_1} \circ T_{\sigma_2} \quad &\Leftrightarrow \quad \exists t'((t, t') \in T_{\sigma_2} \text{ and } (t', t'') \in T_{\sigma_1})\\
&\Leftrightarrow \quad t' = \widehat{\sigma}_2[t] \text{ and } t'' = \widehat{\sigma}_1[t']\\
&\Leftrightarrow \quad t'' = \widehat{\sigma}_1[\widehat{\sigma}_2[t]]\\
&\Leftrightarrow \quad t'' = (\sigma_1 \circ_h \sigma_2)^\wedge[t]\\
&\Leftrightarrow \quad (t, t'') \in T_{\sigma_1 \circ_h \sigma_2}.
\end{aligned}$$

This shows that $T_{Hyp_R^C(\tau)}$ is closed under composition and that φ preserves the operation.

(ii): We show that φ is one-to-one.

Assume that $T_{\sigma_1} = T_{\sigma_2}$. Then for all $t \in W_\tau^C(X)$ we have $\widehat{\sigma}_1[t] = \widehat{\sigma}_2[t]$. But this means that for all operation symbols f_i we also have

$$\widehat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})] = \sigma_1(f_i) = \sigma_2(f_i) = \widehat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]$$

and, therefore, $\sigma_1 = \sigma_2$.

Finally, since $T_{\sigma_1} \circ T_{\sigma_2} = T_{\sigma_1 \circ_h \sigma_2}$, the tree transformation $T_{\sigma_{id}}$ is an identity element with respect to the composition $\circ$. □

Theorem 8 allows us to describe properties of the relation T_σ by properties of the regular hypersubstitution σ and conversely.

Theorem 9. *Let $\sigma \in Hyp_\tau^C(X)$ be a regular hypersubstitution of type τ and let T_σ be the corresponding tree transformation. Then*

(i) T_σ is transitive iff σ is idempotent,

(ii) T_σ is reflexive iff $\sigma = \sigma_{id}$,

(iii) T_σ is symmetric iff $\sigma \circ_h \sigma = \sigma_{id}$.

Proof. (i): When σ is idempotent, we have $T_{\sigma\circ_h\sigma} = T_\sigma \circ T_\sigma = T_\sigma$ by Theorem 8, and T_σ is transitive. Conversely, when T_σ is transitive, we have $T_\sigma \circ T_\sigma \subseteq T_\sigma$, so that $T_{\sigma\circ_h\sigma} \subseteq T_\sigma$. Then

$$(t, (\sigma \circ_h \sigma)\hat{}[t]) \in T_{\sigma\circ_h\sigma} \Rightarrow (t, (\sigma \circ_h \sigma)\hat{}[t]) \in T_\sigma \Rightarrow (\sigma \circ_h \sigma)\hat{}[t] = \widehat{\sigma}[t],$$

for all $t \in W_\tau^C(X)$, and σ is idempotent.

(ii): Assume that T_σ is reflexive, so that $T_{\sigma_{id}} = \Delta_{W_\tau^C(X)} \subseteq T_\sigma$. Therefore $(t,t) \in T_\sigma$ for all $t \in W_\tau^C(X)$ and then $\widehat{\sigma}[t] = t$ for all $t \in W_\tau^C(X)$, making $\sigma = \sigma_{id}$.

If conversely $\sigma = \sigma_{id}$, then $T_{\sigma_{id}} = \{(t, \widehat{\sigma}_{id}[t]) \mid t \in W_\tau^C(X)\} = \{(t,t) \mid t \in W_\tau^C(X)\} = \Delta_{W_\tau^C(X)}$ and T_σ is reflexive.

(iii): If T_σ is symmetric, then for all $t \in W_\tau^C(X)$ we have

$$(t, \widehat{\sigma}[t]) \in T_\sigma \Rightarrow (\widehat{\sigma}[t], t) \in T_\sigma.$$

Therefore $t = \widehat{\sigma}[\widehat{\sigma}[t]]$ and $\widehat{\sigma}_{id}[t] = (\sigma \circ_h \sigma)\hat{}[t]$ for all $t \in W_\tau^C(X)$, and we have $\sigma \circ_h \sigma = \sigma_{id}$.

If conversely $\sigma \circ_h \sigma = \sigma_{id}$ then we have $T_{\sigma\circ_h\sigma} = T_\sigma \circ T_\sigma = T_{\sigma_{id}}$. But this means $T_\sigma = (T_\sigma)^{-1}$, and T_σ is symmetric. □

A tree transformation is called *injective* if σ is injective, i.e., if from $\widehat{\sigma}[t] = \widehat{\sigma}[t']$ follows $t = t'$, and T_σ is called *surjective* if σ is surjective.

In general, the range of a tree transformation σ, i.e. the set

$$\widehat{\sigma}(W_\tau^C(X)) = \{t' \mid \exists t \in W_\tau^C(X)(t' = \widehat{\sigma}[t])\},$$

is a subset of $W_\tau^C(X)$. Therefore, we consider T_σ as a relation between $W_\tau^C(X)$ and $\widehat{\sigma}(W_\tau^C(X))$, so that $T_\sigma \subseteq W_\tau^C(X) \times \widehat{\sigma}(W_\tau^C(X))$. We notice that $T_\sigma \circ (T_\sigma)^{-1} = T_{\sigma_{id}} = \Delta_{W_\tau^C(X)}$ and $(T_\sigma)^{-1} \circ T_\sigma = \{(t,t') \mid \widehat{\sigma}[t] = \widehat{\sigma}[t']\} = ker\sigma$ (the kernel of σ). Then we have

Proposition 10. *Let $\sigma \in Hyp_R^C(\tau)$ be a regular hypersubstitution of type τ and let $T_\sigma = W_\tau^C(X) \times \widehat{\sigma}(W_\tau^C(X))$ be the corresponding tree transformation. Then T_σ is bijective iff $ker\sigma = \Delta_{W_\tau^C(X)} = T_{\sigma_{id}}$.*

Proof. T_σ is bijective iff $T_\sigma \circ (T_\sigma)^{-1} = (T_\sigma)^{-1} \circ T_\sigma = T_{\sigma_{id}} = \Delta_{W_\tau^C(X)}$. Now we use the previous remark. □

3. V-Transformations

Definition 11. Let V be a strong variety of partial algebras of type τ and $\sigma \in Hyp_R^C(\tau)$. The set

$$T_\sigma^V := \{(t, t') \mid t, t' \in W_\tau^C(X) \text{ and } \widehat{\sigma}[t] \approx t' \in Id^s V\}$$

is called the *V-transformation* defined by the regular hypersubstitution σ.

Definition 12. ([2]) Let V be a strong variety of partial algebras of type τ. Two regular hypersubstitutions $\sigma_1, \sigma_2 \in Hyp_R^C(\tau)$ are called *V-equivalent* iff $\sigma_1(f_i) \approx \sigma_2(f_i) \in Id^s V$ for all $i \in I$. In this case we write $\sigma_1 \sim_V \sigma_2$.

Theorem 13. ([2]) Let V be a strong variety of partial algebras of type τ, and let $\sigma_1, \sigma_2 \in Hyp_R^C(\tau)$. Then the following are equivalent:

(i) $\sigma_1 \sim_V \sigma_2$.

(ii) For all $t \in W_\tau^C(X)$ the equation $\widehat{\sigma}_1[t] \approx \widehat{\sigma}_2[t] \in Id^s V$.

(iii) For all $\mathcal{A} \in V$, $\sigma_1(\mathcal{A}) = \sigma_2(\mathcal{A})$.

Proposition 14. ([2]) Let V be a strong variety of partial algebras of type τ. If $\sigma_1 \sim_V \sigma_2$ and $\widehat{\sigma}_1[s] \approx \widehat{\sigma}_1[t] \in Id^s V$, then $\widehat{\sigma}_2[s] \approx \widehat{\sigma}_2[t] \in Id^s V$ when $\sigma_1, \sigma_2 \in Hyp_R^C(\tau)$ and $s, t \in W_\tau^C(X)$.

Proposition 15. *If $\sigma_1 \sim_V \sigma_2$ then $T_{\sigma_1}^V = T_{\sigma_2}^V$.*

Proof. We have only to show the inclusion $T_{\sigma_1}^V \subseteq T_{\sigma_2}^V$. Let $\sigma_1 \sim_V \sigma_2$ and $(t, t') \in T_{\sigma_1}^V$. Then by Theorem 13(ii) we have $\widehat{\sigma}_1[t] \approx \widehat{\sigma}_2[t] \in Id^s V$ for all terms $t \in W_\tau^C(X)$ and $\widehat{\sigma}_1[t] \approx t' \in Id^s V$. Therefore $(t, t') \in T_{\sigma_2}^V$ and $T_{\sigma_1}^V \subseteq T_{\sigma_2}^V$. □

We are interested in all hypersubstitutions from $Hyp_R^C(\tau)$ which preserve all strong identities from V. Such hypersubstitutions are called V-proper.

Definition 16. ([2]) Let V be a strong variety of partial algebras of type τ. A regular hypersubstitution $\sigma \in Hyp_R^C(\tau)$ is called a *V-proper hypersubstitution* if for every $s \approx t \in Id^s V$, we have $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in Id^s V$.

We use $P(V)$ for the set of all V-proper hypersubstitutions of type τ.

Proposition 17. ([2]) The algebra $(P(V); \circ_h, \sigma_{id})$ is a submonoid of $(Hyp_R^C(\tau); \circ_h, \sigma_{id})$.

Now we compare the usual product of binary relations with the product of regular hypersubstitutions.

Proposition 18. *If V is a strong variety of partial algebras of type τ, $\sigma_1 \in P(V)$, and $\sigma_2 \in W_\tau^C(X)$, then*

$$T_{\sigma_1}^V \circ T_{\sigma_2}^V = T_{\sigma_1 \circ_h \sigma_2}^V.$$

Proof. Indeed, we have $(t, t') \in T_{\sigma_1}^V \circ T_{\sigma_2}^V$ iff there is a term t'' such that $(t, t'') \in T_{\sigma_2}^V$ and $(t'', t') \in T_{\sigma_1}^V$ iff $\widehat{\sigma}_2[t] \approx t'' \in Id^s V$ and $\widehat{\sigma}_1[t''] \approx t' \in Id^s V$. Since $\sigma_1 \in P(V)$, we conclude that $\widehat{\sigma}_1[\widehat{\sigma}_2[t]] \approx \widehat{\sigma}_1[t''] \approx t' \in Id^s V$ and therefore $(\sigma_1 \circ_h \sigma_2)\widehat{\ }[t] \approx t' \in Id^s V$ and thus $(t, t') \in T_{\sigma_1 \circ_h \sigma_2}^V$. This shows, $T_{\sigma_1}^V \circ T_{\sigma_2}^V \subseteq T_{\sigma_1 \circ_h \sigma_2}^V$.

If conversely, $(t, t') \in T_{\sigma_1 \circ_h \sigma_2}^V$ then $(\sigma_1 \circ_h \sigma_2)\widehat{\ }[t] \approx t' \in Id^s V$ and then $\widehat{\sigma}_1[\widehat{\sigma}_2[t]] \approx t' \in Id^s V$ with $t'' \approx \widehat{\sigma}_2[t] \in Id^s V$ we have $\widehat{\sigma}_1[t''] \approx t' \in Id^s V$ (because that $\sigma_1 \in P(V)$) and then $(t, t'') \in T_{\sigma_2}^V$, $(t'', t') \in T_{\sigma_1}^V$ and therefore $(t, t') \in T_{\sigma_1}^V \circ T_{\sigma_2}^V$. This shows the inclusion $T_{\sigma_1 \circ_h \sigma_2}^V \subseteq T_{\sigma_1}^V \circ T_{\sigma_2}^V$ and altogether we have equality. □

On the set $\mathcal{T}_{P(V)} := \{T_\sigma^V \mid \sigma \in P(V)\}$ we may take the relational product as a binary relation. Indeed, if $\sigma_1, \sigma_2 \in P(V)$ then by Proposition 17 from $T_{\sigma_1}^V, T_{\sigma_2}^V \in \mathcal{T}_{P(V)}$ we obtain $T_{\sigma_1}^V \circ T_{\sigma_2}^V \in \mathcal{T}_{P(V)}$. Since $\sigma_{id} \in P(V)$, the V-transformation $T_{\sigma_{id}}^V$ serves as identity element and we obtain a monoid $(\mathcal{T}_{P(V)}; \circ, T_{\sigma_{id}}^V)$.

Proposition 19. *The monoid $(\mathcal{T}_{P(V)}; \circ, T_{\sigma_{id}}^V)$ is a homomorphic image of $(P(V); \circ_h, \sigma_{id})$.*

Proof. A homomorphism $\varphi : P(V) \to \mathcal{T}_{P(V)}$ is defined by

$$\varphi(\sigma) := T_\sigma^V.$$

Indeed, we have

$$\varphi(\sigma_1 \circ_h \sigma_2) = T_{\sigma_1 \circ_h \sigma_2}^V = T_{\sigma_1}^V \circ T_{\sigma_2}^V = \varphi(\sigma_1) \circ \varphi(\sigma_2)$$

and $\varphi(\sigma_{id}) = T_{\sigma_{id}}^V$. □

Note that φ is not one-to-one because of Proposition 15 and the kernel of φ agrees with the relation $\sim_V$.

Definition 20. ([2]) A regular hypersubstitution $\sigma \in Hyp_R^C(\tau)$ is called an *inner hypersubstitution* of a strong variety V of partial algebras of type τ if for every $i \in I$,

$$\widehat{\sigma}[f_i(x_1, \ldots, x_{n_i})] \approx f_i(x_1, \ldots, x_{n_i}) \in Id^s V.$$

Let $P_0(V)$ be the set of all inner hypersubstitutions of V.

By definition $P_0(V)$ is the equivalence class $[\sigma_{id}]_{\sim_V}$.

Note further $T^V_{\sigma_{id}}$ is not the only identity element of $(\mathcal{T}_{P(V)}; \circ, T^V_{\sigma_{id}})$ since all T^V_σ with $\sigma \sim_V \sigma_{id}$, i.e. where σ are inner hypersubstitutions are equal to $T^V_{\sigma_{id}}$ (Proposition 15).

Proposition 21. ([2]) The algebra $(P_0(V); \circ_h, \sigma_{id})$ is a submonoid of $(P(V); \circ_h, \sigma_{id})$.

Proposition 22. *Let V be a strong variety of partial algebras of type τ and let $\sigma \in Hyp_R^C(\tau)$. Then T^V_σ is reflexive iff $\sigma \in P_0(V)$.*

Proof. If T^V_σ is reflexive, then $\widehat{\sigma}[t] \approx t \in Id^s V$ for all $t \in W^C_\tau(X)$. This is valid also for $t = f_i(x_1, \ldots, t_{n_i}), i \in I$ and then $\widehat{\sigma}[f_i(x_1, \ldots, t_{n_i})] \approx f_i(x_1, \ldots, t_{n_i}) \in Id^s V$, i.e. $\widehat{\sigma}[f_i(x_1, \ldots, t_{n_i})] \approx \widehat{\sigma}_{id}[f_i(x_1, \ldots, t_{n_i})] \in Id^s V$ and $\sigma \sim_V \sigma_{id}$. Therefore $\sigma \in P_0(V)$. If conversely, $\sigma \in P_0(V)$ then $\sigma \sim_V \sigma_{id}$ and by Proposition 21 $\widehat{\sigma}[t] \approx t \in Id^s V$ for $t \in W^C_\tau(X)$, but this means $(t, t) \in T^V_\sigma$ and T^V_σ is reflexive. □

The next, we introduce the concept of a semantical kernel.

Definition 23. Let $\sigma \in Hyp_R^C(\tau)$ and let V be a strong variety of partial algebras of type τ. The set

$$Ker_V \sigma := \{(t, t') \mid t, t' \in W^C_\tau(X) \text{ and } \widehat{\sigma}[t] \approx \widehat{\sigma}[t'] \in Id^s V\}$$

will be called the *kernel of σ with respect to V* or the *semantical kernel* of σ.

Proposition 24. *If $\sigma_1 \sim_V \sigma_2$ then $Ker_V \sigma_1 = Ker_V \sigma_2$.*

Proof. By Proposition 14. □

Proposition 25. *Let V be a strong variety of partial algebras of type τ and let $\sigma \in Hyp_R^C(\tau)$. Then*

$$ker_V \sigma \subseteq ker_V(\rho \circ_h \sigma)$$

for all $\rho \in P(V)$.

Proof. For any $(t,t') \in ker_V\sigma$ we have $\widehat{\sigma}[t] \approx \widehat{\sigma}[t'] \in Id^sV$. Since ρ is a V-proper hypersubstitution, this implies that $\widehat{\rho}[\widehat{\sigma}[t]] \approx \widehat{\rho}[\widehat{\sigma}[t']] \in Id^sV$, and so $(t,t') \in ker_V(\rho \circ_h \sigma)$. □

Proposition 26. *Let V be a strong variety of partial algebras of type τ and let $\sigma \in Hyp_R^C(\tau)$. Then $(T_\sigma^V)^{-1} \circ T_\sigma^V = Ker_V\sigma$.*

Proof. We have

$$\begin{aligned}(t,t'') \in (T_\sigma^V)^{-1} \circ T_\sigma^V &\Leftrightarrow \exists t'((t,t') \in T_\sigma^V \text{ and } (t',t'') \in (T_\sigma^V)^{-1})\\ &\Leftrightarrow \exists t'((t,t') \in T_\sigma^V \text{ and } (t'',t') \in T_\sigma^V)\\ &\Leftrightarrow \exists t'(\widehat{\sigma}[t] \approx t' \in Id^sV \text{ and } \widehat{\sigma}[t''] \approx t' \in Id^sV)\\ &\Leftrightarrow \widehat{\sigma}[t] \approx \widehat{\sigma}[t''] \in Id^sV\\ &\Leftrightarrow (t,t'') \in ker_V\sigma.\end{aligned}$$

□

Proposition 27. *For a V-proper hypersubstitution σ the following are equivalent:*

(i) T_σ^V *is transitive.*

(ii) $\sigma \circ_h \sigma \sim_V \sigma$.

(iii) $T_\sigma^V \subseteq ker_V\sigma$.

Proof. (i)⇒(ii): If T_σ^V is transitive, then $T_\sigma^V \circ T_\sigma^V \subseteq T_\sigma^V$ and therefore $T_\sigma^V \circ T_\sigma^V = T_{\sigma\circ_h\sigma}^V \subseteq T_\sigma^V$. This means, if $(t,t') \in T_{\sigma\circ_h\sigma}^V$, i.e.$(\sigma\circ_h\sigma)\widehat{}[t] \approx t' \in Id^sV$ then $(t,t') \in T_\sigma^V$ i.e. $\widehat{\sigma}[t] \approx t' \in Id^sV$. But $(\sigma \circ_h \sigma)\widehat{}[t] \approx \widehat{\sigma}[t] \in Id^sV$ for all $t \in W_\tau^C(X)$ and there $\sigma \circ_h \sigma \sim_V \sigma$ by Theorem 13.

(ii)⇒(i): If $\sigma \circ_h \sigma \sim_V \sigma$ then by Proposition 15 $T_{\sigma\circ_h\sigma}^V = T_\sigma^V$ and Proposition 18, $T_\sigma^V \circ T_\sigma^V = T_{\sigma\circ_h\sigma}^V = T_\sigma^V$ and T_σ^V is transitive.

(ii)⇒(iii): Assume that $\sigma \circ_h \sigma \sim_V \sigma$ and that $(t,t') \in T_\sigma^V$, i.e. $\widehat{\sigma}[t] \approx t' \in Id^sV$. Then $\widehat{\sigma}[\widehat{\sigma}[t]] \approx \widehat{\sigma}[t'] \in Id^sV$ since σ is V-proper and we have $(\sigma \circ_h \sigma)\widehat{}[t] \approx \widehat{\sigma}[t'] \in Id^sV$. From $(\sigma \circ_h \sigma)\widehat{}[t] \approx \widehat{\sigma}[t'] \in Id^sV$ we obtain $\widehat{\sigma}[t] \approx \widehat{\sigma}[t'] \in Id^sV$ and then $(t,t') \in ker_V\sigma$. This show $T_\sigma^V \subseteq ker_V\sigma$.

(iii)⇒(ii): If $T_\sigma^V \subseteq ker_V\sigma$ then $(t,t') \in T_\sigma^V$, i.e. $\widehat{\sigma}[t] \approx t' \in Id^sV$ and since σ is V-proper we get $\widehat{\sigma}[\widehat{\sigma}[t]] \approx \widehat{\sigma}[t'] \in Id^sV$. Since $(t,t') \in ker_V\sigma$ we have $\widehat{\sigma}[t] \approx \widehat{\sigma}[t'] \in Id^sV$ and then $\widehat{\sigma}[\widehat{\sigma}[t]] \approx \widehat{\sigma}[t] \in Id^sV$, i.e. $\sigma \circ_h \sigma \sim_V \sigma$. □

Proposition 28. *Let V be a strong variety of partial algebras of type τ. Then:*

(i) T_σ^V *is surjective iff* $T_\sigma^V \circ (T_\sigma^V)^{-1} = Id^sV$.

(ii) T_σ^V *is injective iff* $(T_\sigma^V)^{-1} \circ T_\sigma^V = ker_V\sigma = \triangle_{W_\tau^C(X)}$.

(iii) T_σ^V *is bijective iff* $T_\sigma^V \circ (T_\sigma^V)^{-1} = (T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$.

Proof. (i) Assume that T_σ^V is surjective. To show the equality $T_\sigma^V \circ (T_\sigma^V)^{-1} = Id^sV$. Assume that $t \approx t' \in Id^sV$. Since T_σ^V is surjective, to t' there is a term t'' such that $(t'', t') \in T_\sigma^V$, i.e. such that $\widehat{\sigma}[t''] \approx t' \in Id^sV$. Then we have also $\widehat{\sigma}[t''] \approx t \in Id^sV$ and $(t'', t) \in T_\sigma^V$, i.e. $(t, t'') \in (T_\sigma^V)^{-1}$. Altogether, this gives $(t, t') \in T_\sigma^V \circ (T_\sigma^V)^{-1}$ and thus $Id^sV \subseteq T_\sigma^V \circ (T_\sigma^V)^{-1}$. The next, we assume that $(t, t') \in T_\sigma^V \circ (T_\sigma^V)^{-1}$ then there exist t'' such that $(t, t'') \in (T_\sigma^V)^{-1}$ and $(t'', t') \in T_\sigma^V$. We have $(t'', t) \in T_\sigma^V$ and $(t'', t') \in T_\sigma^V$ i.e. $\widehat{\sigma}[t''] \approx t \in Id^sV$ and $\widehat{\sigma}[t''] \approx t' \in Id^sV$. So, $t \approx t' \in Id^sV$. Conversely, we assume that $T_\sigma^V \circ (T_\sigma^V)^{-1} = Id^sV$. Let $t \in W_\tau^C(X)$ be an arbitrary term of $W_\tau^C(X)$. We have to show that there is a term $t' \in W_\tau^C(X)$ with $\widehat{\sigma}[t'] \approx t \in Id^sV$. From $t \approx t \in Id^sV = T_\sigma^V \circ (T_\sigma^V)^{-1}$ we obtain the existence of a term $t' \in W_\tau^C(X)$ such that $(t', t) \in T_\sigma^V$, but this means $\widehat{\sigma}[t'] \approx t \in Id^sV$ and this shows surjectivity.

(ii) $(T_\sigma^V)^{-1} \circ T_\sigma^V = Ker_V\sigma$ is clear. Assume that T_σ^V is injective and let $(t, t') \in (T_\sigma^V)^{-1} \circ T_\sigma^V$. Then we have $(t, t'') \in T_\sigma^V$ and $(t', t'') \in T_\sigma^V$ there follows $t = t'$. We get $(t, t') \in \triangle_{W_\tau^C(X)}$ and thus $(T_\sigma^V)^{-1} \circ T_\sigma^V \subseteq \triangle_{W_\tau^C(X)}$. If conversely $(t, t') \in \triangle_{W_\tau^C(X)}$ then $t = t'$ and $\widehat{\sigma}[t] = \widehat{\sigma}[t'] =: t''$. Then we have also $\widehat{\sigma}[t] \approx t'' \in Id^sV$ and $\widehat{\sigma}[t'] \approx t'' \in Id^sV$ and $(t, t'') \in T_\sigma^V$, $(t', t'') \in T_\sigma^V$ and then $(t, t') \in (T_\sigma^V)^{-1} \circ T_\sigma^V$, i.e. $\triangle_{W_\tau^C(X)} \subseteq (T_\sigma^V)^{-1} \circ T_\sigma^V$. Altogether, this gives $(T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$. Assume now that $(T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$ and that $(t, t'') \in T_\sigma^V$ and $(t', t'') \in T_\sigma^V$. Then $(t, t') \in (T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$, i.e. $t = t'$ and therefore T_σ^V is injective.

(iii) Assume that T_σ^V is bijective. Since σ is V-proper, we have by (i) and (ii):

$$T_\sigma^V \circ (T_\sigma^V)^{-1} = Id^sV \subseteq ker_V\sigma = (T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$$

and therefore $Id^sV = \triangle_{W_\tau^C(X)}$ and $T_\sigma^V \circ (T_\sigma^V)^{-1} = (T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$.

If conversely, $T_\sigma^V \circ (T_\sigma^V)^{-1} = (T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau^C(X)}$, then by (ii) T_σ^V is injective. Therfore, $T_\sigma^V = \{(t, t') \mid \widehat{\sigma}[t] \approx t' \in Id^sV\} = Id^sV = \triangle_{W_\tau^C(X)}$. □

References

[1] F. Börner, *Varieties of Partial Algebras*, Beiträge zur Algebra und Geometrie, Vol. **37** (1996), No. 2, 259-287.

[2] S. Busaman, K. Denecke, *Unsolid and Fluid Strong Varieties of Partial Algebras*, preprint 2006, submitted to Int. Journal of Mathematics and Mathematical Sciences.

[3] J. Płonka, *On hyperidentities of some varieties*, General Algebra and Discrete Mathematics, Heldermann-Verlag, Berlin (1995), 199-214.

[4] B. Staruch, B. Staruch, *Strong regular varieties of partial algebras*, Algebra Universalis, **31** (1994), 157-176.

[5] D. Welke, *Hyperidentitäten partieller Algebren*, Ph.D.Thesis, Universität Potsdam, Potsdam (1996).