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THE PERIOD MODULO PRODUCT OF CONSECUTIVE FIBONACCI NUMBERS

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Abstract: Let F_n be the nth Fibonacci number. The period modulo m, denoted by $s(m)$, is the smallest positive integer k for which $F_{n+k} \equiv F_n \pmod{m}$ for all $n \geq 0$. In this paper, we find the period modulo product of consecutive Fibonacci numbers. For instance, we prove that, for $n \geq 1$,

$$s(F_nF_{n+1}F_{n+2}F_{n+3}) = \begin{cases} n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod{3}, \\ \frac{2n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 0, 9 \pmod{12}, \\ \frac{n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 3, 6 \pmod{12}. \end{cases}$$

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These number are famous for possessing wonderful properties, see [2, 3, 5, 6, 15] for additional references and history. In particular, we will be concerned with the divisibility properties and the periodic nature of the Fibonacci sequence.

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Let m be a positive integer. The order of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is defined as the smallest positive integer k such that $m \mid F_k$ (some authors also call it the order of apparition, the rank of apparition, or Fibonacci entry point). The period modulo m of the Fibonacci sequence, denoted by $s(m)$, is defined as the smallest positive integer k such that $F_{n+k} \equiv F_n \pmod{m}$ for all $n \geq 0$. There are several results about $z(m)$ and $s(m)$ in the literature. For instance, Stanley [14] shows that if m is an integer greater than 3 then $s(F_m) = 2m$ if m is even and $s(F_m) = 4m$ if m is odd. It is also well known that $z(F_m) = m$ for $m \geq 3$ (see for example [4, p. 221]). For other classical results on $z(m)$ and $s(m)$, we refer the reader to [1, 4, 6, 12, 13, 14, 15, 16, 17].

Recently, D. Marques [7, 8, 9, 10, 11] has obtained a formula of $z(m)$ for various special numbers m. Particularly, he obtains the following [10] in 2012.

Theorem 1. *(Marques [10])*

(i) *For* $n \geq 3$,

$$z(F_n F_{n+1}) = n(n+1).$$

(ii) *For* $n \geq 2$,

$$z(F_n F_{n+1} F_{n+2}) = \begin{cases} n(n+1)(n+2), & \text{if } n \equiv 1 \pmod{2}, \\ \frac{n(n+1)(n+2)}{2}, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

(iii) *For* $n \geq 1$,

$$z(F_n F_{n+1} F_{n+2} F_{n+3}) = \begin{cases} \frac{n(n+1)(n+2)(n+3)}{2}, & \text{if } n \not\equiv 0 \pmod{3}, \\ \frac{n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 0, 9 \pmod{12}, \\ \frac{n(n+1)(n+2)(n+3)}{6}, & \text{if } n \equiv 3, 6 \pmod{12}. \end{cases}$$

The above theorem motivates us to study the period $s(n)$ when n is the product of consecutive Fibonacci numbers. Our main results are the formulas for $s(F_n F_{n+1})$, $s(F_n F_{n+1} F_{n+2})$, and $s(F_n F_{n+1} F_{n+2} F_{n+3})$. This task is a bit more difficult than the calculation of, for example, $z(F_n F_{n+1} F_{n+2} F_{n+3})$ in Theorem 1. This is because the latter only requires the smallest k such that $F_k \equiv 0 \pmod{F_n F_{n+1} F_{n+2} F_{n+3}}$ while our task is to find the smallest k such that $F_k \equiv 0 \pmod{F_n F_{n+1} F_{n+2} F_{n+3}}$ and $F_{k+1} \equiv 1 \pmod{F_n F_{n+1} F_{n+2} F_{n+3}}$ (see Lemma 2). But with the aid of Lemma 3, our task become easy.

We will give some lemmas in the next section. Then we will give the main results in the last section.

2. Preliminaries

We recall some facts on Fibonacci numbers for the convenience of the reader.

Let n and m be positive integers. The following results are well known and will be used throughout this article:

$$\text{For } n \geq 3,\ F_n \mid F_m \text{ if and only if } n \mid m. \tag{1}$$

$$\text{For } n \geq 1,\ F_{n-1}F_{n+1} - F_n^2 = (-1)^n \text{ (The Cassini's formula).} \tag{2}$$

$$\text{For } k, n \geq 1 \text{ and } r \geq 0,\ F_{kn+r} = \sum_{j=0}^{k} \binom{k}{j} F_n^j F_{n-1}^{k-j} F_{j+r} \tag{3}$$

$$\text{For } n \geq 1,\ F_{n+1}^2 \equiv F_{n-1}^2 \equiv (-1)^n \pmod{F_n}. \tag{4}$$

Identity (1) and (2) can be found, for example, in [3, 5, 6, 15]. Identity (3) and (4) might be less well known, so we will give a proof here for completeness.

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. It is well known that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. By solving the equation, $\alpha^n - \beta^n = (\alpha - \beta)F_n$, and $\alpha \cdot \alpha^n - \beta \cdot \beta^n = (\alpha - \beta)F_{n+1}$, for α^n and β^n, we obtain $\alpha^n = \alpha F_n + F_{n-1}$, $\beta^n = \beta F_n + F_{n-1}$. Let $k, n \geq 1$ and $r \geq 0$. Then

$$\begin{aligned}
F_{kn+r} &= \frac{\alpha^{kn+r} - \beta^{kn+r}}{\alpha - \beta} \\
&= \frac{1}{\alpha - \beta} \left[(\alpha F_n + F_{n-1})^k \alpha^r - (\beta F_n + F_{n-1})^k \beta^r \right] \\
&= \frac{1}{\alpha - \beta} \left[\sum_{j=0}^{k} \binom{k}{j} (\alpha F_n)^j F_{n-1}^{k-j} \alpha^r - \sum_{j=0}^{k} \binom{k}{j} (\beta F_n)^j F_{n-1}^{k-j} \beta^r \right] \\
&= \frac{1}{\alpha - \beta} \sum_{j=0}^{k} \left[\binom{k}{j} F_n^j F_{n-1}^{k-j} (\alpha^{j+r} - \beta^{j+r}) \right] \\
&= \sum_{j=0}^{k} \binom{k}{j} F_n^j F_{n-1}^{k-j} F_{j+r}.
\end{aligned}$$

This proves (3). Next we prove (4). Let $n \geq 1$. By the Cassini's formula, we have $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$. So $F_{n+1}^2 = F_n F_{n+2} - (-1)^{n+1} \equiv (-1)^{n+2} \equiv (-1)^n \pmod{F_n}$ and $F_{n-1}^2 = (F_{n+1} - F_n)^2 \equiv F_{n+1}^2 \equiv (-1)^n \pmod{F_n}$. This proves (4).

The next result is actually an equivalent definition for the period of the Fibonacci sequence modulo m. It appeared for example in [12, 16, 17].

Lemma 2. *Let m be a positive integer. Then $s(m)$ is equal to the smallest positive integer k such that $F_k \equiv 0 \pmod{m}$ and $F_{k+1} \equiv 1 \pmod{m}$.*

Proof. Let m be a positive integer and let k be the smallest positive integer such that $F_k \equiv 0 \pmod{m}$ and $F_{k+1} \equiv 1 \pmod{m}$. We will prove that $F_{n+k} \equiv F_n \pmod{m}$ for all $n \geq 0$ by strong induction on n. It is clearly true for $n = 0$ and $n = 1$. Assume that it is valid for any integer j such that $0 \leq j \leq n$. Thus

$$F_{(n+1)+k} = F_{n+k+1} = F_{n+k} + F_{n+k-1} \equiv F_n + F_{n-1} \equiv F_{n+1} \pmod{m}.$$

Therefore $F_{n+k} \equiv F_n \pmod{m}$ for all $n \geq 0$. Suppose that r is a positive integer such that $F_{n+r} \equiv F_n \pmod{m}$ for all $n \geq 0$. Then $F_r \equiv F_0 \equiv 0 \pmod{m}$ and $F_{r+1} \equiv F_1 \equiv 1 \pmod{m}$. Thus $k \leq r$. Therefore k is the smallest positive integer such that $F_{n+k} \equiv F_n \pmod{m}$ for all $n \geq 0$. We conclude that $s(m) = k$. □

It is shown in [4] that $z(n) \mid s(n)$ for every $n \geq 1$. So it is natural to define the quantity $t(n) = \frac{s(n)}{z(n)}$. So $t(n)$ is an integer for all $n \geq 1$. A useful result on $t(n)$ is obtained by Vinson[16] as follows.

Lemma 3. *(Vinson [16]) The following statement holds.*

(i) $t(m) = 4$ *if* $m > 2$ *and* $z(m)$ *is odd,*

(ii) $t(m) = 1$ *if* $8 \nmid m$ *and* $2 \mid z(p)$ *but* $4 \nmid z(p)$ *for every odd prime* p *dividing* m, *and*

(iii) $t(m) = 2$ *for other* m.

3. Main Results

In this section, we give the proof of the main theorems. As mentioned before, Lemma 2 gives an equivalent definition of $s(n)$ and we will use Lemma 2 without further referring.

Theorem 4. $s(F_1F_2) = s(1) = 1, s(F_2F_3) = s(2) = 3$, *and* $s(F_nF_{n+1}) = 2n(n+1)$ *for every* $n \geq 3$.

Proof. It is straightforward to verify that $s(F_1F_2) = s(1) = 1, s(F_2F_3) = s(2) = 3$. So we let $n \geq 3$. By Theorem 1, we have $z(F_nF_{n+1}) = n(n+1)$, which is even. Then by Lemma 3, we see that $t(F_nF_{n+1}) = 1$ or 2. Therefore

$$s(F_nF_{n+1}) = n(n+1) \text{ or } s(F_nF_{n+1}) = 2n(n+1). \tag{5}$$

First we consider $F_{n(n+1)+1} \pmod{F_n}$. By (3), we have

$$F_{n(n+1)+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} F_n^j F_{n-1}^{n+1-j} F_{j+1} \equiv F_{n-1}^{n+1} \pmod{F_n}.$$

We consider the following two cases:

Case 1: If n is even, then by (4), we have $F_{n-1}^{n+1} \equiv (F_{n-1}^2)^{\frac{n}{2}} F_{n-1} \equiv (-1)^{n(\frac{n}{2})} F_{n-1} \equiv F_{n-1} \not\equiv 1 \pmod{F_n}$. This is because $2 \leq F_{n-1} < F_n$.

Case 2: Assume that n is odd. Then by (4), $F_{n-1}^{n+1} \equiv (F_{n-1}^2)^{\frac{n+1}{2}} \equiv (-1)^{n(\frac{n+1}{2})} \equiv (-1)^{\frac{n+1}{2}} \pmod{F_n}$. Since $(-1)^{\frac{n+1}{2}} = 1$ if and only if $4 \mid n+1$, we see that $F_{n-1}^{n+1} \equiv 1 \pmod{F_n}$ if and only if $n \equiv 3 \pmod 4$. We conclude that

$$F_{n(n+1)+1} \equiv 1 \pmod{F_n} \text{ if and only if } n \equiv 3 \pmod 4. \tag{6}$$

Next we consider $F_{n(n+1)+1} \pmod{F_{n+1}}$. By (3), we have

$$F_{n(n+1)+1} = \sum_{j=0}^{n} \binom{n}{j} F_{n+1}^j F_n^{n-j} F_{j+1} \equiv F_n^n \pmod{F_{n+1}}.$$

Similar to the proof of (6), we apply (4) to obtain the following:

If n is odd, then $F_n^n \equiv (F_n^2)^{\frac{n-1}{2}} F_n \equiv (-1)^{(n+1)(\frac{n-1}{2})} F_n \equiv F_n \not\equiv 1 \pmod{F_{n+1}}$.

If n is even, then $F_n^n \equiv (F_n^2)^{\frac{n}{2}} \equiv (-1)^{(n+1)(\frac{n}{2})} \equiv (-1)^{\frac{n}{2}} \pmod{F_{n+1}}$, which is congruent to $1 \pmod{F_{n+1}}$ if and only if $4 \mid n$. Therefore we obtain

$$F_{n(n+1)+1} \equiv 1 \pmod{F_{n+1}} \text{ if and only if } n \equiv 0 \pmod 4. \tag{7}$$

Since $\gcd(F_n, F_{n+1}) = 1$, we have

$$F_k \equiv 1 \pmod{F_n F_{n+1}} \text{ if and only if } F_k \equiv 1 \pmod{F_n} \text{ and } F_k \equiv 1 \pmod{F_{n+1}}. \tag{8}$$

From (6), (7) and (8), we see that there is no $n \in \mathbb{N}$ satisfying $F_{n(n+1)+1} \equiv 1 \pmod{F_n F_{n+1}}$. Hence $s(F_n F_{n+1}) \neq n(n+1)$. Now by (5), we can conclude that $s(F_n F_{n+1}) = 2n(n+1)$. This completes the proof. □

Remark 1. We can also directly calculate $F_{2n(n+1)+1} \pmod{F_n F_{n+1}}$ by applying Identity (3) and (4) as follows.

$$F_{2n(n+1)+1} = \sum_{j=0}^{2(n+1)} \binom{2(n+1)}{j} F_n^j F_{n-1}^{2(n+1)-j} F_{j+1}$$

$$\equiv F_{n-1}^{2(n+1)} \equiv (F_{n-1}^2)^{n+1} \equiv (-1)^{n(n+1)} \equiv 1 \pmod{F_n}$$

$$F_{2n(n+1)+1} = \sum_{j=0}^{2n} \binom{2n}{j} F_{n+1}^j F_n^{2n-j} F_{j+1} \equiv F_n^{2n} \equiv (F_n^2)^n$$
$$\equiv (-1)^{(n+1)n} \equiv 1 \pmod{F_{n+1}}.$$

Since $\gcd(F_n, F_{n+1}) = 1$, we have $F_{2n(n+1)+1} \equiv 1 \pmod{F_n F_{n+1}}$, as required.

Note. also that, from this point on, we will apply (4) without referring to it.

Theorem 5. *$s(F_1F_2F_3) = s(2) = 3$ and for $n \geq 2$, we have*

$$s(F_nF_{n+1}F_{n+2}) = \begin{cases} 2n(n+1)(n+2), & \text{if } n \equiv 1 \pmod 2, \\ n(n+1)(n+2), & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

.

Proof. It easy to verify that $s(F_1F_2F_3) = s(2) = 3$. So we let $n \geq 2$. We split the proof into two cases:

Case 1: Assume that n is even. Then $n+2$ is even and $2 \mid \frac{n(n+1)(n+2)}{2}$. Therefore, by Theorem 1, $z(F_nF_{n+1}F_{n+2})$ is even. So, by Lemma 3, $t(F_nF_{n+1}F_{n+2}) = 1$ or 2. We will show that $t(F_nF_{n+1}F_{n+2}) = 2$. By Lemma 3, it suffices to find a prime $p \mid F_nF_{n+1}F_{n+2}$ such that $4 \mid z(p)$. Since n is even, $4 \mid n$ or $4 \mid n+2$. Then, by (1), $F_4 \mid F_n$ or $F_4 \mid F_{n+2}$ and therefore $F_4 \mid F_nF_{n+1}F_{n+2}$. That is $3 \mid F_nF_{n+1}F_{n+2}$. Since $z(3) = 4$, we have $t(F_nF_{n+1}F_{n+2}) = 2$, as desired. In conclusion, if n is even

$$s(F_nF_{n+1}F_{n+2}) = 2z(F_nF_{n+1}F_{n+2}) = n(n+1)(n+2).$$

Case 2: Assume that n is odd. By Theorem 1, $z(F_nF_{n+1}F_{n+2}) = n(n+1)(n+2)$ which is an even number. So, by Lemma 3, we obtain $t(F_nF_{n+1}F_{n+2}) = 1$ or 2. Thus

$$s(F_nF_{n+1}F_{n+2}) = n(n+1)(n+2) \text{ or } s(F_nF_{n+1}F_{n+2}) = 2n(n+1)(n+2). \quad (9)$$

We consider $F_{n(n+1)(n+2)+1} \pmod{F_{n+1}}$. By (3), we have

$$F_{n(n+1)(n+2)+1} = \sum_{j=0}^{n(n+2)} \binom{n(n+2)}{j} F_{n+1}^j F_n^{n(n+2)-j} F_{j+1} \equiv F_n^{n(n+2)} \pmod{F_{n+1}}.$$

Since n is odd, $n(n+2)$ is odd. So, by (4), we have

$$F_n^{n(n+2)} \equiv (F_n^2)^{\frac{n(n+2)-1}{2}} F_n \equiv (-1)^{(n+1)\frac{n(n+2)-1}{2}} F_n \equiv F_n \not\equiv 1 \pmod{F_{n+1}}.$$

Thus $F_{n(n+1)(n+2)+1} \not\equiv 1 \pmod{F_n F_{n+1} F_{n+2}}$. Therefore

$$s(F_n F_{n+1} F_{n+2}) \neq n(n+1)(n+2).$$

Now by (9), we conclude that $s(F_n F_{n+1} F_{n+2}) = 2n(n+1)(n+2)$. This completes the proof. □

Remark 2. We can also calculate $F_{2n(n+1)(n+2)+1} \pmod{F_n F_{n+1} F_{n+2}}$ by applying (3) as follows

$$\begin{aligned}
F_{2n(n+1)(n+2)+1} &= \sum_{j=0}^{2(n+1)(n+2)} \binom{2(n+1)(n+2)}{j} F_n^j F_{n-1}^{2(n+1)(n+2)-j} F_{j+1} \\
&\equiv F_{n-1}^{2(n+1)(n+2)} \equiv (F_{n-1}^2)^{(n+1)(n+2)} \equiv (-1)^{n(n+1)(n+2)} \equiv 1 \pmod{F_n}. \\
F_{2n(n+1)(n+2)+1} &= \sum_{j=0}^{2n(n+2)} \binom{2n(n+2)}{j} F_{n+1}^j F_n^{2n(n+2)-j} F_{j+1} \\
&\equiv F_n^{2n(n+2)} \equiv (F_n^2)^{n(n+2)} \equiv (-1)^{(n+1)(n)(n+2)} \equiv 1 \pmod{F_{n+1}}. \\
F_{2n(n+1)(n+2)+1} &= \sum_{j=0}^{2n(n+1)} \binom{2n(n+1)}{j} F_{n+2}^j F_{n+1}^{2n(n+1)-j} F_{j+1} \\
&\equiv F_{n+1}^{2n(n+1)} \equiv (F_{n+1}^2)^{n(n+1)} \equiv (-1)^{(n+2)(n)(n+1)} \equiv 1 \pmod{F_{n+2}}.
\end{aligned}$$

Since the number F_n, F_{n+1}, F_{n+2} are pairwise relatively prime, we have

$$F_{2n(n+1)(n+2)+1} \equiv 1 \pmod{F_n F_{n+1} F_{n+2}},$$

as required.

Theorem 6. *For $n \geq 1$,*

$$s(F_n F_{n+1} F_{n+2} F_{n+3}) = \begin{cases} n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod{3}, \\ \frac{2n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 0, 9 \pmod{12}, \\ \frac{n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 3, 6 \pmod{12}. \end{cases}$$

Proof. Let $n \geq 3$. We split the proof into three cases:

Case 1: Assume that $n \not\equiv 0 \pmod 3$. By Theorem 1, $z(F_nF_{n+1}F_{n+2}F_{n+3}) = \frac{n(n+1)(n+2)(n+3)}{2}$. Since $4 \mid n(n+1)(n+2)(n+3)$, $2 \mid \frac{n(n+1)(n+2)(n+3)}{2}$. So, by Lemma 3, we have $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 1$ or 2. We will show that $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 2$. By Lemma 3, it suffices to find a prime $p \mid F_nF_{n+1}F_{n+2}F_{n+3}$ such that $4 \mid z(p)$, or to show that $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. Since $n \not\equiv 0 \pmod 3$, we have $n \equiv 1, 2, 4, 5 \pmod 6$.

(i) Assume that $n \equiv 4, 5 \pmod 6$. Then $6 \mid n+2$ or $6 \mid n+1$. By (1), we have $F_6 \mid F_{n+2}$ or $F_6 \mid F_{n+1}$ and therefore $F_6 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. That is $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$, as desired.

(ii) Assume that $n \equiv 1, 2 \pmod 6$. Then $n \equiv 1, 2, 7, 8 \pmod{12}$. So $4 \mid n + k$ for some $k \in \{0, 1, 2, 3\}$. By (1), we have $F_4 \mid F_{n+k}$ for some $k \in \{0, 1, 2, 3\}$. That is $3 \mid F_{n+k}$ for some $k \in \{0, 1, 2, 3\}$ and so $3 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. Thus 3 is an odd prime dividing $F_nF_{n+1}F_{n+2}F_{n+3}$ and $4 \mid z(3)$, as required.

Therefore $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 2$. We conclude that

$$s(F_nF_{n+1}F_{n+2}F_{n+3}) = 2z(F_nF_{n+1}F_{n+2}F_{n+3}) = n(n+1)(n+2)(n+3).$$

Case 2: Assume that $n \equiv 0, 9 \pmod{12}$. By Theorem 1, $z(F_nF_{n+1}F_{n+2}F_{n+3}) = \frac{n(n+1)(n+2)(n+3)}{3}$. Since $n \equiv 0, 9 \pmod{12}$, we have $6 \mid n$ or $6 \mid n+3$. So $2 \mid \frac{n(n+1)(n+2)(n+3)}{3}$. Therefore $z(F_nF_{n+1}F_{n+2}F_{n+3})$ is even. By Lemma 3, we have $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 1$ or 2. We will show that $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 2$. By Lemma 3, it suffices to show that $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. Since $6 \mid n$ or $6 \mid n+3$, we have $F_6 \mid F_n$ or $F_6 \mid F_{n+3}$. So $F_6 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. That is $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$, as desired. Therefore $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 2$ and hence

$$s(F_nF_{n+1}F_{n+2}F_{n+3}) = 2z(F_nF_{n+1}F_{n+2}F_{n+3}) = \frac{2n(n+1)(n+2)(n+3)}{3}.$$

Case 3: Assume that $n \equiv 3, 6 \pmod{12}$. Then, by Theorem 1, we have

$$z(F_nF_{n+1}F_{n+2}F_{n+3}) = \frac{n(n+1)(n+2)(n+3)}{6}.$$

We will show that $t(F_nF_{n+1}F_{n+2}F_{n+3}) = 2$. By Lemma 3, it suffices to show that

$$z(F_nF_{n+1}F_{n+2}F_{n+3}) \text{ is even and } 8 \mid F_nF_{n+1}F_{n+2}F_{n+3}.$$

(i) Assume that $n \equiv 3 \pmod{12}$. Then $6 \mid n+3$ and $2 \mid n+1$. So $2 \mid \frac{n(n+1)(n+2)(n+3)}{6}$. Thus $z(F_nF_{n+1}F_{n+2}F_{n+3})$ is even. Since $6 \mid n+3$, $F_6 \mid F_{n+3}$. So $F_6 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. That is $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$.

(ii) Assume that $n \equiv 6 \pmod{12}$. Then $6 \mid n$ and $2 \mid n+2$. So $2 \mid \frac{n(n+1)(n+2)(n+3)}{6}$. Thus $z(F_nF_{n+1}F_{n+2}F_{n+3})$ is even. Since $6 \mid n$, $F_6 \mid F_n$. So $F_6 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. That is $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$.

In any case, $z(F_nF_{n+1}F_{n+2}F_{n+3})$ is even and $8 \mid F_nF_{n+1}F_{n+2}F_{n+3}$. Therefore

$$t(F_nF_{n+1}F_{n+2}F_{n+3}) = 2.$$

We conclude that

$$s(F_nF_{n+1}F_{n+2}F_{n+3}) = 2z(F_nF_{n+1}F_{n+2}F_{n+3}) = \frac{n(n+1)(n+2)(n+3)}{3}.$$

This completes the proof. □

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