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# STUDY OF MEROMORPHIC FUNCTIONS DEFINED BY THE CONVOLUTION OF LINEAR OPERATOR

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## 1. Introduction

A meromorphic function is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities). A simpler definition states that a meromorphic function $f(z)$ is a function of the form

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (see [12], p. 64). A

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meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain. A meromorphic function with an infinite number of poles is exemplified by $csc\frac{1}{z}$ on the punctured disk $U^* = \{z : 0 < |z| < 1\}$. An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. For example the Gamma function is meromorphic in the whole complex plane, see [12] and [13].

Let $A$ be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$ and for which

$$\Re\{h(z)\} > 0 \qquad (z \in U). \tag{1}$$

For functions $f$ and $g$ analytic in $U$, we say that $f$ is subordinate to $g$ and write

$$f \prec g \quad in \quad U \quad or \quad f(z) \prec g(z) \quad (z \in U)$$

if there exists an analytic function $w(z)$ in $U$ such that

$$|w(z)| \le |z| \qquad and \qquad f(z) = g(w(z)) \qquad (z \in U).$$

Furthermore, if the function $g$ is univalent in $U$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad and \quad f(U) = g(U), \qquad (z \in U).$$

Let $A$ be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$ and for which

$$\Re\{h(z)\} > 0 \qquad (z \in U^*). \tag{2}$$

For functions $f$ and $g$ analytic in $U$, we say that $f$ is subordinate to $g$ and write

$$f \prec g \quad in \quad U \quad or \quad f(z) \prec g(z) \quad (z \in U^*)$$

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Furthermore, if the function $g$ is univalent in $U$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad and \quad f(U) \subseteq g(U), \ (z \in U^*).$$

## 2. Preliminaries

Let $\Sigma$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{3}$$

which are analytic and univalent in the punctured unit disk $U^*$.
For functions $f_j(z)(j = 1; 2)$ defined by

$$f_j\left(z\right) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \tag{4}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$\left(f_1 * f_2\right)\left(z\right) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{5}$$

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by

$$\tilde{\phi}\left(\alpha, \beta; z\right) = \frac{1}{z} + \sum_{n=0}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| z^n, \tag{6}$$

for $\beta \neq 0, -1, -2, ...,$ and $\alpha \in \mathbb{C}/\left\{0\right\}$, where $(\lambda)n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that

$$\tilde{\phi}\left(\alpha, \beta; z\right) = \frac{1}{z} {}_2F_1\left(1, \alpha, \beta; z\right)$$

where

$${}_2F_1\left(b, \alpha, \beta; z\right) = \sum_{n=0}^{\infty} \frac{(b)_n \left(\alpha\right)_n}{\left(\beta\right)_n} \frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\tilde{\phi}(\alpha, \beta; z)$, using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L^*(\alpha, \beta)$ on $\Sigma$ by

$$L^*\left(\alpha, \beta\right) f\left(z\right) = \tilde{\phi}\left(\alpha, \beta; z\right) * f\left(z\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| a_n z^n. \tag{7}$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [2], [3], Liu [8], Liu and Srivastava [9], [10],[11], Cho and Kim [1] .

For a function $f \in L(\alpha,\beta) f(z)$ we define

$$I^0 \left( L(\alpha,\beta) f(z) \right) = L(\alpha,\beta) f(z),$$

and for $k = 1, 2, 3, ...,$

$$\begin{aligned} I^k \left( L(\alpha,\beta) f(z) \right) &= z \left( I^{k-1} L(\alpha,\beta) f(z) \right)' + \frac{2}{z} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| a_n z^n. \end{aligned} \tag{8}$$

where $I^k$ was studied by Ghanim and Darus in [4], [5], [6] and [7].

It follows from (7) that

$$z \left( L(\alpha,\beta) f(z) \right)' = \alpha L(\alpha+1,\beta) f(z) - (\alpha+1) L(\alpha,\beta) f(z). \tag{9}$$

Also, from (9) we get

$$z \left( I^k L(\alpha,\beta) f(z) \right)' = \alpha I^k L(\alpha+1,\beta) f(z) - (\alpha+1)\ I^k L(\alpha,\beta) f(z). \tag{10}$$

Throughout this paper, we assume that

$$m \in N,\ \beta \notin Z_0^-,\ \ \varepsilon_m^j = \exp\left( \frac{2\pi j}{m} \right) \tag{11}$$

and

$$f_m(\alpha,\beta;z) = \frac{1}{m} \sum_{j=0}^{m-1} \varepsilon_m^j \left( L(\alpha,\beta) f \right) \left( \varepsilon_m^j z \right), \qquad f \in \Sigma. \tag{12}$$

Also, we define

$$f_{m,k}(\alpha,\beta;z) = I^k f_m(\alpha,\beta;z)$$

$$= \frac{1}{m} \sum_{j=0}^{m-1} \varepsilon_m^{j(k+1)} \left( I^k L(\alpha,\beta) f \right) \left( \varepsilon_m^j z \right), \quad k = 1, 2, 3.... \tag{13}$$

It is clear, for $k = 0$ and $m = 1$, we have

$$f_1(\alpha, \beta; z) = L(\alpha, \beta) f(z)$$

Making use of the linear operator $L(\alpha, \beta)$ and the principle of subordination between analytic functions, we introduce and investigate the following subclasses of the meromorphically analytic function class $\Sigma$:

$$\Sigma_{m,k}(\alpha, \beta; h), \quad \mathrm{M}_{m,k}(\alpha, \beta; h), \quad \mathrm{M}_{m,k}(\gamma; \alpha, \beta; h) \qquad (h \in A)$$

**Definition 1.** A function $f \in \Sigma$ is said to be in the class $\Sigma_{m,k}(\alpha, \beta; h)$ if it satisfies the following subordination condition:

$$\frac{-z\left(I^k L(\alpha, \beta) f\right)'(z)}{f_{m,k}(\alpha, \beta; z)} \prec h(z) \qquad (z \in U) \tag{14}$$

where $h \in A$ and $f_{m,k}(\alpha, \beta; z) \neq 0 \qquad (z \in U^*)$.

**Definition 2.** A function $f \in \Sigma$ is said to be in the class $\mathrm{M}_{m,k}(\alpha, \beta; h)$ if it satisfies the following subordination condition:

$$\frac{-z\left(I^k L(\alpha, \beta) f\right)'(z)}{g_{m,k}(\alpha, \beta; z)} \prec h(z) \qquad (z \in U) \tag{15}$$

for some $g \in \Sigma_{m,k}(\alpha, \beta; h)$ where $h \in A$ and $g_{m,k}(\alpha, \beta; z) \neq 0$ is defined as in (13).

**Definition 3.** A function $f \in \Sigma$ is said to be in the class $\mathrm{M}_{m,k}(\gamma; \alpha, \beta; h)$ if it satisfies the following subordination condition:

$$-\gamma \frac{z\left(I^k L(\alpha+1, \beta) f\right)'(z)}{g_{m,k}(\alpha+1, \beta; z)} - (1-\gamma) \frac{z\left(I^k L(\alpha, \beta) f\right)'(z)}{g_{m,k}(\alpha, \beta; z)} \prec h(z) \qquad (z \in U) \tag{16}$$

for some $\gamma$ $(\gamma \geq 0)$ and $g \in \Sigma_{m,k}(\alpha, \beta; h)$, where $h \in A$ and $g_{m,k}(\alpha+1, \beta; z) \neq 0$.

In order to prove our main results, we need the following lemmas.

**Lemma 4.** *[15] Let $a$ $(a \geq 0)$ and $\gamma$ be complex numbers and let $h(z)$ be analytic and convex univalent in $U$ with*

$$\Re\{a\, h(z) + \gamma\} > 0$$

*If $q(z)$ is analytic in $U$ with $q(0) = h(0)$, then the subordination:*

$$q(z) + \frac{zq'(z)}{aq(z) + \gamma} \prec h(z) \qquad (z \in U)$$

*implies that*

$$q(z) \prec h(z) \qquad (z \in U).$$

**Lemma 5.** *[14] Let $h(z)$ be analytic and convex univalent in $U$ and let $w(z)$ be analytic in $U$ with*

$$\Re\{w(z)\} \geq 0 \qquad (z \in U).$$

*If $q(z)$ is analytic in $U$ and $q(0) = h(0)$, then the subordination:*

$$q(z) + w(z)\, zq'(z) \prec h(z) \qquad (z \in U)$$

*implies that*

$$q(z) \prec h(z) \quad (z \in U).$$

**Lemma 6.** *Let $f \in \Sigma_{m,k}(\alpha, \beta; h)$. Then*

$$-\frac{z\left(f'_{m,k}(\alpha, \beta; z)\right)}{f_{m,k}(\alpha, \beta; z)} \prec h(z) \qquad (z \in U). \tag{17}$$

*Proof.* Making use of (13), we have

$$f_{m,k}\left(\alpha, \beta; \varepsilon_m^j z\right) = \frac{1}{m}\sum_{n=0}^{m-1} \varepsilon_m^{n(k+1)} \left(I^k L(\alpha, \beta) f\right)\left(\varepsilon_m^{n+j} z\right)$$

$$= \frac{\varepsilon_m^{-j}}{m}\sum_{n=0}^{m-1} \varepsilon_m^{n(k+1)+j} \left(I^k L(\alpha, \beta) f\right)\left(\varepsilon_m^{n+j} z\right) = \varepsilon_m^{-j} f_{m,k}(\alpha, \beta; z),$$

$(j \in \{0, 1, ..., m-1\})$ and

$$f'_{m,k}(\alpha, \beta; z) = \frac{1}{m}\sum_{j=0}^{m-1} \varepsilon_m^{j(k+2)} \left(I^k L(\alpha, \beta) f\right)'\left(\varepsilon_m^j z\right)$$

Thus

$$-\frac{z\left(f'_{m,k}(\alpha, \beta; z)\right)}{f_{m,k}(\alpha, \beta; z)} = -\frac{1}{m}\sum_{j=0}^{m-1} \frac{\varepsilon_m^{j(k+2)} z\left(I^k L(\alpha, \beta) f\right)'\left(\varepsilon_m^j z\right)}{f_{m,k}(\alpha, \beta; z)}$$

$$= -\frac{1}{m}\sum_{j=0}^{m-1} \frac{\varepsilon_m^{jk} z\left(I^k L(\alpha, \beta) f\right)'\left(\varepsilon_m^j z\right)}{f_{m,k}\left(\alpha, \beta; \varepsilon_m^j z\right)} \qquad (z \in U). \tag{18}$$

Since $f \in \Sigma_{m,k}(\alpha, \beta; h)$ it follows that

$$-\frac{\varepsilon_m^j z \left(I^k L(\alpha, \beta) f\right)' \left(\varepsilon_m^j z\right)}{f_{m,k}\left(\alpha, \beta; \varepsilon_m^j z\right)} \prec h(z) \tag{19}$$

$(z \in U, j \in \{0, 1, 2, ..., k-1\})$. Since $h(z)$ is convex univalent in $U$, from (18) and (19) we conclude that (17) holds true. □

## 3. Main Results

**Theorem 7.** *Let $h \in A$ with*

$$\Re\{h(z)\} < 1 + \alpha \qquad (z \in U, \alpha > 0). \tag{20}$$

*If $f \in \Sigma_{m,k}(\alpha + 1, \beta; h)$, then $f \in \Sigma_{m,k}(\alpha, \beta; h)$, provided that*

$$f_{m,k}(\alpha, \beta; z) \neq 0 \qquad (z \in U^*).$$

*Proof.* By using (10) and (13), we have

$$(\alpha+1) f_{m,k}(\alpha, \beta; z) + z f'_{m,k}(\alpha, \beta; z) = \frac{\alpha}{m} \sum_{j=0}^{m-1} \varepsilon_m^{j(k+1)} \left(I^k L(\alpha+1, \beta) f\right)' \left(\varepsilon_m^j z\right)$$

$$= \alpha f_{m,k}(\alpha+1, \beta; z), \qquad (f \in \Sigma). \tag{21}$$

Let $f \in \Sigma_{m,k}(\alpha + 1, \beta; h)$ and suppose that

$$w(z) = -\frac{z\left(f'_{m,k}(\alpha, \beta; z)\right)}{f_{m,k}(\alpha, \beta; z)}. \tag{22}$$

Then $w(z)$ is analytic in $U$, with $w(0) = 1$, and it follows from (21) and (22) that

$$\alpha + 1 - w(z) = \alpha \frac{f_{m,k}(\alpha+1, \beta; z)}{f_{m,k}(\alpha, \beta; z)}. \tag{23}$$

Differentiating both sides of (23) with respect to $z$ logarithmically and using (22), we obtain

$$w(z)+\frac{z\,w'(z)}{\alpha+1-w(z)}=\frac{z\left(f'_{m,k}\left(\alpha+1,\beta;z\right)\right)}{f_{m,k}\left(\alpha,\beta;z\right)}. \tag{24}$$

From (24) and Lemma 6 (with a replaced by $\alpha+1$) we find that

$$w(z)+\frac{z\,w'(z)}{\alpha+1-w(z)}\prec h \qquad (z\in U)\,. \tag{25}$$

Now, in view of (20) and (25), and application of Lemma 4 yields

$$w(z)\prec h(z) \qquad (z\in U)\,. \tag{26}$$

Set

$$q\left(z\right)=\frac{-z\left(I^{k}L\left(\alpha,\beta\right)f\right)'\left(z\right)}{f_{m,k}\left(\alpha,\beta;z\right)}. \tag{27}$$

Then $q(z)$ is analytic in $U$, with $q(0)=1$, and it follows from (10) and (27) that

$$f_{m,k}\left(\alpha,\beta;z\right)q\left(z\right)=-\alpha I^{k}L\left(\alpha+1,\beta\right)f\left(z\right)+\left(1+\alpha\right)I^{k}L\left(\alpha,\beta\right)f\left(z\right). \tag{28}$$

Differentiating both sides of (28) with respect to $z$ and using (27), we get

$$zq'\left(z\right)+\left(\alpha+1+\frac{z\left(f'_{m,k}\left(\alpha,\beta;z\right)\right)}{f_{m,k}\left(\alpha,\beta;z\right)}\right)q\left(z\right)=-\frac{\alpha z\left(I^{k}L\left(\alpha+1,\beta\right)f\right)'\left(z\right)}{f_{m,k}\left(\alpha,\beta;z\right)} \tag{29}$$

Furthermore, we find from (21), (22) and (29) that

$$q\left(z\right)+\frac{zq'\left(z\right)}{\alpha+1-w(z)}=-\frac{z\left(I^{k}L\left(\alpha+1,\beta\right)f\right)'\left(z\right)}{f_{m,k}\left(\alpha+1,\beta;z\right)}\prec h(z) \qquad (z\in U)\,, \tag{30}$$

since $f\in\Sigma_{m,k}\left(\alpha+1,\beta;h\right)$. By (20) and (26), we see that

$$\Re\left\{\alpha+1-w(z)\right\}>0.$$

Therefore, we deduce from (30) and Lemma 5 that

$$q\left(z\right)\prec h(z) \qquad (z\in U)\,,$$

which implies that $f\in\Sigma_{m,k}\left(\alpha,\beta;h\right)$ and the proof of Theorem 7 is thus completed. □

**Theorem 8.** *Let $h \in A$ with*

$$\Re\{h(z)\} < 1+\alpha \qquad (z \in U, \alpha > 0). \tag{31}$$

*If $f \in \mathrm{M}_{m,k}(\alpha+1,\beta;h)$, with respect to $g \in \Sigma_{m,k}(\alpha+1,\beta;h)$, then $f \in \mathrm{M}_{m,k}(\alpha,\beta;h)$ provided that $g_{m,k}(\alpha,\beta;z) \neq 0 \qquad (z \in U^*)$.*

*Proof.* According to the hypotheses of Theorem 8, we have

$$\frac{-z\left(I^k L(\alpha+1,\beta) f\right)'(z)}{g_{m,k}(\alpha+1,\beta;z)} \prec h(z) \qquad (z \in U) \tag{32}$$

with $g \in \Sigma_{m,k}(\alpha+1,\beta;h)$. Furthermore, it follows from Theorem 7 that $g \in \Sigma_{m,k}(\alpha,\beta;h)$ and Lemma 6 yields

$$\Omega(z) = -\frac{z g'_{m,k}(\alpha,\beta;z)}{g_{m,k}(\alpha,\beta;z)} \prec h(z) \qquad (z \in U). \tag{33}$$

Suppose that

$$q(z) = -\frac{z\left(I^k L(\alpha,\beta) f\right)'(z)}{g_{m,k}(\alpha,\beta;z)}. \tag{34}$$

By using (10), (34) can be written as follows:

$$g_{m,k}(\alpha,\beta;z)\, q(z) = -\alpha I^k L(\alpha+1,\beta) f(z) + (1+\alpha) I^k L(\alpha,\beta) f(z). \tag{35}$$

Differentiating both sides of (35) with respect to $z$ and using (21) (with $f$ replaced by $g$), we find that

$$q(z) + \frac{z q'(z)}{\alpha+1-\Omega(z)} = -\frac{z\left(I^k L(\alpha+1,\beta) f\right)'(z)}{g_{m,k}(\alpha+1,\beta;z)} \qquad (z \in U). \tag{36}$$

Combining (32) and (36), we obtain

$$q(z) + \frac{z q'(z)}{\alpha+1-\Omega(z)} \prec h(z) \qquad (z \in U) \tag{37}$$

Consequently, in view of (31), (33) and (37), we deduce from Lemma 5 that

$$q(z) \prec h(z) \qquad (z \in U),$$

which shows that $f \in \mathrm{M}_{m,k}(\alpha,\beta;h)$ with respect to $g \in \Sigma_{m,k}(\alpha,\beta;h)$. □

**Theorem 9.** *Let $h \in A$ with*

$$\Re\{h(z)\} < 1+\alpha \qquad (z \in U, \alpha > 0). \tag{38}$$

*Then*

$$\mathrm{M}_{m,k}(\gamma_1;\alpha,\beta;h) \subset \mathrm{M}_{m,k}(\gamma_2;\alpha,\beta;h) \qquad (0 \le \gamma_1 < \gamma_2).$$

*Proof.* For $f \in \mathrm{M}_{m,k}(\gamma_2;\alpha,\beta;h)$, there exists a function $g \in \Sigma_{m,k}(\alpha,\beta;h)$ satisfying the following condition:

$$g_{m,k}(\alpha+1,\beta;z) \neq 0 \qquad (z \in U^*).$$

such that

$$-\gamma_2 \frac{z\left(I^k L(\alpha+1,\beta) f\right)'(z)}{g_{m,k}(\alpha+1,\beta;z)} - (1-\gamma_2)\frac{z\left(I^k L(\alpha,\beta) f\right)'(z)}{g_{m,k}(\alpha,\beta;z)} \prec h(z) \qquad (z \in U). \tag{39}$$

Put

$$q(z) = \frac{z\left(I^k L(\alpha,\beta) f\right)'(z)}{g_{m,k}(\alpha,\beta;z)} \qquad (z \in U).$$

Since $g \in \Sigma_{m,k}(\alpha,\beta;h)$ it follows from (33) to (36) (used in the proof of Theorem 8) and (39) that

$$q(z) + \frac{\gamma_2 z q'(z)}{\alpha+1-\Omega(z)}$$

$$= -\gamma_2 \frac{z\left(I^k L(\alpha+1,\beta) f\right)'(z)}{g_{m,k}(\alpha+1,\beta;z)} - (1-\gamma_2)\frac{z\left(I^k L(\alpha,\beta) f\right)'(z)}{g_{m,k}(\alpha,\beta;z)} \prec h(z) \tag{40}$$

In light of (33) and (38), we thus observe that

$$\frac{1}{\gamma_2}\Re\{\alpha+1-\Omega(z)\} > 0 \qquad (z \in U).$$

Hence, by (40) and Lemma 5, we have

$$q(z) \prec h(z) \qquad (z \in U). \tag{41}$$

Since $(0 \le \gamma_1 < \gamma_2)$ and since $h(z)$ is convex univalent in $U$, we deduce from (39) and (41) that

$$-\gamma_1 \frac{z\left(I^k L(\alpha+1,\beta) f\right)'(z)}{g_{m,k}(\alpha+1,\beta;z)} - (1-\gamma_1)\frac{z\left(I^k L(\alpha,\beta) f\right)'(z)}{g_{m,k}(\alpha,\beta;z)}$$

$$= \frac{\gamma_1}{\gamma_2}\left(-\gamma_2 \frac{z\left(I^k L\left(\alpha+1,\beta\right)f\right)'(z)}{g_{m,k}\left(\alpha+1,\beta;z\right)} - \left(1-\gamma_2\right)\frac{z\left(I^k L\left(\alpha,\beta\right)f\right)'(z)}{g_{m,k}\left(\alpha,\beta;z\right)}\right)$$

$$+\left(1-\frac{\gamma_1}{\gamma_2}\right)q\left(z\right) \prec h(z) \qquad \left(z\in U\right). \tag{42}$$

Thus $f \in \mathrm{M}_{m,k}\left(\gamma_1;\alpha,\beta;h\right)$ and the proof of Theorem 9 is completed. □

## 4. Convolution Properties

Let $\mathcal{A}$ be the class of functions of the form:

$$f\left(z\right) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{43}$$

which are analytic in $U$. A function $f \in \mathcal{A}$ is said to be starlike of order $\delta$ in $U$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'\left(z\right)}{f\left(z\right)}\right) > \delta, \qquad \left(z\in U\right) \tag{44}$$

for some $\delta\left(\delta<1\right)$. We denote this class by $S^*\left(\delta\right)$. A function $f \in \mathcal{A}$ is said to be prestarlike of order $\delta\left(\delta<1\right)$ in $U$ if

$$\frac{z}{\left(1-z\right)^{2(1-\delta)}} * f\left(z\right) \in S^*\left(\delta\right) \tag{45}$$

We denote this class by $\boldsymbol{S}\left(\delta\right)$ (see [16]). It is clear that a function $f \in \mathcal{A}$ is in the class $\boldsymbol{S}\left(0\right)$ if and only if $f(z)$ is convex univalent in $U$ and that

$$\boldsymbol{S}\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$$

**Lemma 10.** *[16] Let $\delta < 1, f \in \boldsymbol{S}\left(\delta\right)$ and $g \in S^*\left(\delta\right)$. Then, for any analytic function $F(z)$ in $U$,*

$$\frac{f*\left(gF\right)}{f*g}\left(U\right) \subset \overline{co}\left(F\left(U\right)\right), \tag{46}$$

*where the symbol $*$ means the Hadamard product (or convolution) of two analytic functions in $U$ and $\overline{co}\left(F\left(U\right)\right)$ stands for the convex hull of $F\left(U\right)$.*

**Theorem 11.** *Let $h \in A$ with*

$$\Re\{h(z)\} < 2-\delta \qquad (z \in U; \delta < 0). \tag{47}$$

*If $f \in \Sigma_{m,k}(\alpha,\beta;h)$, $g \in \Sigma$ and*

$$z^2 g(z) \in \boldsymbol{S}(\delta) \qquad (\delta < 1) \tag{48}$$

*then*

$$f * g \in \Sigma_{m,k}(\alpha,\beta;h).$$

*Proof.* Let $f \in \Sigma_{m,k}(\alpha,\beta;h)$ and suppose that

$$w(z) = z^2 f_{m,k}(\alpha,\beta;z) \tag{49}$$

Then

$$F(z) = -\frac{z\left(I^k L(\alpha,\beta) f\right)'(z)}{f_{m,k}(\alpha,\beta;z)} \prec h(z) \qquad (z \in U), \tag{50}$$

and

$$\frac{zw'(z)}{w(z)} = 2 + \frac{z\left(f'_{m,k}(\alpha,\beta;z)\right)(z)}{f_{m,k}(\alpha,\beta;z)} \prec 2-h(z) \qquad (z \in U, w \in \mathcal{A}), \tag{51}$$

where we have used Lemma 6. In view of (47) and (51), we see that

$$\Re\left(\frac{zw'(z)}{w(z)}\right) > \delta \qquad (z \in U), \tag{52}$$

that is,

$$w \in S^*(\delta) \quad (\delta < 1).$$

For $g \in \Sigma$, it is easy to verify that

$$z^2\left(I^k L(\alpha,\beta)(f*g)\right)\left(\varepsilon_m^j z\right) = \left(z^2 g(z)\right) * z^2\left(I^k L(\alpha,\beta) f\right)\left(\varepsilon_m^j z\right) \tag{53}$$

$(j \in \{0,1,2,...,m-1\})$ and

$$z^3\left(I^k L(\alpha,\beta)(f*g)\right)'(z) = \left(z^2 g(z)\right) * \left(z^3\left(I^k L(\alpha,\beta) f\right)'(z)\right). \tag{54}$$

Making use of (49), (50), (53) and (54), we find that

$$-\frac{z\left(I^k L(\alpha,\beta)(f*g)\right)'(z)}{\frac{1}{m}\sum\limits_{j=0}^{m-1}\varepsilon_m^{j(k+1)}\left(I^k L(\alpha,\beta)(f*g)\right)\left(\varepsilon_m^j z\right)}$$

$$= -\frac{\left(z^2 g(z)\right) * z^3 \left(I^k L(\alpha, \beta) f\right)'(z)}{(z^2 g(z)) * (z^2 f_{m,k}(\alpha, \beta; z))} = -\frac{\left(z^2 g(z)\right) * (w(z) F(z))}{(z^2 g(z)) * w(z)} \qquad (z \in U). \tag{55}$$

Since $h(z)$ is convex univalent in $U$, it follows from (48), (50), (52), (55) and Lemma 10 that

$$-\frac{\left(z^2 g(z)\right) * (w(z) F(z))}{(z^2 g(z)) * w(z)} \prec h(z) \qquad (z \in U).$$

Hence $f * g \in \Sigma_{m,k}(\alpha, \beta; h)$. □

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