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ON THE EXTENSION OF ARMENDARIZ RINGS RELATIVE TO A MONOID

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Abstract: For a monoid M, we introduce the concept of 3-M-Armendariz rings, which is a generalization of M-Armendariz rings, and investigate its properties. The results prove that the subrings of 3-M-Armendariz rings are 3-M-Armendariz rings. Every ring satisfying condition (P) is 3-M-Armendariz for any unique product monoid M. If a ring R is 3-M-Armendariz and satisfies condition (P), then $S_3(R)$ is 3-M-Armendariz. Sufficient and necessary conditions are given for a ring R to be 3-M-Armendariz.

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1. Introduction

Throughout this paper, R and M denote an associative ring, not necessary with identity and a monoid, respectively. Given a ring R, the polynomial ring over R is denoted by $R[x]$. The study of Armendariz rings was initiated by Armendariz [5] and Rege and Chhawchharia [8]. A ring R is called Armendariz if whenever

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polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$, for all $0 \leq i \leq n, 0 \leq j \leq m$. (The converse is always true.) Some properties of Armendariz rings have been studied in Rege and Chhawchharia [8], Anderson and Camillo [3], Kim and Lee [9], Huh et al. [1], and Lee and Wong [11]. Suiyi [14] introduced the notion of 3-Armendariz rings. A ring R is called a 3-Armendariz ring if whenever polynomials $f(x) = a_0+a_1x+\cdots+a_nx^n, g(x) = b_0+b_1x+\cdots+b_mx^m, h(x) = c_0+c_1x+\cdots+c_rx^r \in R[x]$, satisfy $f(x)g(x)h(x) = 0$, then $a_ib_jc_k = 0$, for all $0 \leq i \leq n, 0 \leq j \leq m, 0 \leq k \leq r$. Zhongkui [7], studied a generalization of Armendariz rings, which are called M-Armendariz rings, where M is a monoid. A ring R is called M-Armendariz if whenever elements $\alpha = a_1g_1+\cdots+a_ng_n, \beta = b_1h_1+\cdots+b_mh_m \in R[M]$, satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each i, j, where $g_i, h_j \in M$. A ring R is called reduced if it has no nonzero nilpotent elements. Reduced rings are Armendariz by Armendariz [5, lemma 1.1] and subrings of a Armendariz ring are also Armendariz. A ring is called abelian if every idempotent is central. Armendariz ring are abelian by Kim and Lee [9]. Subrings of a M-Armendariz ring are also M-Armendariz by Zhongkui [7]. Subrings of a 3-Armendariz ring are also 3-Armendariz by Suiyi [14].

Recall that a monoid M is called a u.p.-monoid (unique product monoid) if for any two non-empty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see Birkenmeier and Park [6], Passman [4]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has non unity element of finite order.

Motivated by results in Suiyi [14], Zhongkui [7], Rege and Chhawchharia [8] and Kim and Lee [9], we will investigate a generalization of M-Armendariz rings, which we call 3-M-Armendariz rings.

In the following, e will always stand for the identity of M.

2. 3-M-Armendariz Rings

Definition 2.1. *Let M be a monoid. A ring R is called a 3-M-Armendariz ring if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m$ and $\gamma = c_1l_1 + \cdots + c_rl_r \in R[M]$, satisfy $\alpha\beta\gamma = 0$, then $a_ib_jc_k = 0$ for each i, j and k, where $a_i, b_j, c_k \in R$ and $g_i, h_j, l_k \in M$.*

Every M-Armendariz ring is 3-M-Armendariz, but the converse is not true by the following example.

Example 2.2. *Let M a monoid with $|M| \geq 2$. Then $B_6^e(\mathbb{Z})$ is 3-M-Armendariz, but not M-Armendariz, where the ring $B_6^e(\mathbb{Z})$ is defined as following see([12]):*

$$B_{n=6}^e(\mathbb{Z}) = \sum_{i=1}^{k+1} \sum_{j=k+i-1}^{6} \mathbb{Z}E_{i,j}.$$

Proof. By Suiyi [14, Example 1], we have $(B_6^e(\mathbb{Z}))^3 = 0$. So $B_6^e(\mathbb{Z})$ is a 3-M-Armendariz ring. But $B_6^e(\mathbb{Z})$ is not M-Armendariz, because for any $g \neq e$, $(E_{13}e + (E_{13} - E_{14})g)(E_{46}e + (E_{36} + E_{46})g) = 0$ in $B_6^e(\mathbb{Z})[M]$, but $E_{13} \cdot (E_{36} + E_{46}) = E_{16} \neq 0$. □

Proposition 2.3. *Every subring of 3-M-Armendariz rings is 3-M-Armendariz.*

Proof. It is obvious. □

We introduce the following notation (see [14]).

Condition (P): For all $a, b, c \in R$, if $(abc)^2 = 0$, then $abc = 0$.

Lemma 2.4. [*13, Proposition 1*]. *If R is a reduced ring, then R satisfies the condition (P), but the converse is not true.*

Lemma 2.5. [*6, Lemma 1.1*] *Assume M is a u.p.-monoid. Then M is cancellative (i.e., for $g, h, x \in M$, if $gx = hx$ or $xg = xh$, then $g = h$).*

Theorem 2.6. *Let M be a u.p.-monoid and R a ring satisfying condition (P). Then R is 3-M-Armendariz.*

Proof. Let $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m$ and $\gamma = c_1l_1 + \cdots + c_rl_r \in R[M]$, be such that $\alpha\beta\gamma = 0$. We claim $a_ib_jc_k = 0$ for each i, j, k. We proceed by induction on n, m and r. Let $n = 1$. Then $\alpha = a_1g_1$. So

$$(a_1g_1)(b_1h_1 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r) = 0. \qquad (*)$$

If $m = 1$, then $a_1b_1c_1g_1h_1l_1 + \cdots + a_1b_1c_rg_1h_1l_r = 0$. By Lemma 2.5, $g_1h_1l_i \neq g_1h_1l_j$ for $i \neq j$. Thus, $a_1b_1c_k = 0$ for all k. The case $r = 1$, is proved by similar argument. Now suppose that $m > 1$ and $r > 1$. Since M is a *u.p.*-monoid, there exist j, k with $1 \leq j \leq m$ and $1 \leq k \leq r$ such that h_jl_k is uniquely presented by considering two subsets $A = \{h_1, \cdots, h_m\}$ and $B = \{l_1, \cdots, l_r\}$. Without loss of generality, we may assume that $j = 1, k = 1$. Thus, by Lemma 2.5 it follows that $g_1h_1l_1 \neq g_1h_jl_k$ for any $j \neq 1$ or $k \neq 1$. Hence we have $a_1b_1c_1 = 0$. Since R satisfies condition $(P), c_1a_1b_1 = 0$ follows. Now from $(*)$, it follows that

$$(c_1a_1g_1)(b_1h_1 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r) = 0.$$

Thus,

$$(c_1a_1g_1)(b_2h_2 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r) = 0.$$

By induction, we have $c_1a_1b_jc_k = 0$ for all j and k. Then $c_1a_1b_jc_1 = 0$ for all j. Thus, $(c_1a_1b_j)^2 = 0$, which implies that $c_1a_1b_j = 0$ for all j since R satisfies condition (P). So, again by condition (P), we have $a_1b_jc_1 = 0$ for all j. Thus, by $(*)$ it follows that

$$(a_1g_1)(b_1h_1 + \cdots + b_mh_m)(c_2l_2 + \cdots + c_rl_r) = 0.$$

By induction, we have $a_1b_jc_k = 0$ for all j and k with $1 \leq j \leq m, 2 \leq k \leq r$. Thus $a_1b_jc_k = 0$ for all j and k. If $m = 1$ or $r = 1$, then by analogy with the above argument, the result follows. Now suppose that $n > 1$. Consider two subsets $A = \{g_1, \cdots, g_n\}$ and $B = \{h_jl_k | 1 \leq j \leq m, 1 \leq k \leq r\}$. Since M is a $u.p.$-monoid, there exist i, j, k with $1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq r$, such that $g_ih_jl_k$ is uniquely presented. Without loss of generality, we may assume that $i = 1, j = 1, k = 1$. Thus, $a_1b_1c_1 = 0$, which implies $b_1c_1a_1 = 0$. From $\alpha\beta\gamma = 0$ it follows

$$b_1c_1(a_2g_2 + \cdots + a_ng_n)(b_1h_1 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r) = 0.$$

By induction, we have $b_1c_1a_ib_jc_k = 0$, for all i, j, k. Particularly, $b_1c_1a_ib_1c_1 = 0$, which implies that $a_ib_1c_1 = 0$ for all i. Thus, by condition(P), we have $c_1a_ib_1 = 0$ for all i. From $\alpha\beta\gamma = 0$, it follows that

$$\begin{aligned} 0 &= c_1(a_1g_1 + \cdots + a_ng_n)(b_1h_1 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r) \\ &= c_1(a_1g_1 + \cdots + a_ng_n)(b_2h_2 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r) \\ &\quad + c_1(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(b_1h_1)(c_1l_1 + c_2l_2 + \cdots + c_rl_r) \\ &= c_1(a_1g_1 + \cdots + a_ng_n)(b_2h_2 + \cdots + b_mh_m)(c_1l_1 + \cdots + c_rl_r). \end{aligned}$$

By induction, we have $c_1a_ib_jc_k = 0$ for all i, j, k. Particularly $c_1a_ib_jc_1 = 0$ for all i, j. Since R satisfies condition (P), it follows that $a_ib_jc_1 = 0$ for all i and j. Hence we have

$$(a_1g_1 + \cdots + a_ng_n)(b_1h_1 + \cdots + b_mh_m)(c_2l_2 + \cdots + c_rl_r) = 0.$$

By induction, it follows that $a_ib_jc_k = 0$, for all i, j, and k. $\square$

Let $(M, \leq)$ be an ordered monoid. If for any $g, g', h \in M, g < g'$ implies that $gh < g'h$ and $hg < hg'$, then $(M, \leq)$ is called a strictly ordered monoid.

Corollary 2.7. *Let M be a strictly totally ordered monoid and R a ring satisfying condition (P). Then R is 3-M-Armendariz.*

Corollary 2.8. *If a ring R satisfies condition (P), then R is 3-$\mathbb{Z}$-Armendariz, that is, for any $\alpha = a_{-m}x^{-m} + a_{-(m-1)}x^{-(m-1)} + \cdots + a_p x^p, \beta = b_{-n}x^{-n} + b_{-(n-1)}x^{-(n-1)} + \cdots + b_q x^q$ and $\gamma = c_{-t}x^{-t} + c_{-(t-1)}x^{-(t-1)} + \cdots + c_s x^s \in R[x, x^{-1}]$, if $\alpha\beta\gamma = 0$, then $a_i b_j c_k = 0$ for $-m \leq i \leq p, -n \leq j \leq q$ and $-t \leq k \leq s$.*

It was shown in Zhongkui [7], Proposition 1.4, that if I is a reduced ideal of R such that R/I is M-Armendariz, then R is M-Armendariz. Here we have the following result for 3-M-Armendariz property.

Theorem 2.9. *Let M be a strictly totally ordered monoid and I an ideal of R. If I is reduced and R/I is 3-M-Armendariz, then R is 3-M-Armendariz.*

Proof. Let $\alpha, \beta, \gamma \in R[M]$ be such that $\alpha\beta\gamma = 0$. We write $\alpha = a_1 g_1 + \cdots + a_n g_n, \beta = b_1 h_1 + \cdots + b_m h_m$ and $\gamma = c_1 l_1 + \cdots + c_r l_r \in R[M]$, with

$$g_1 < g_2 < \cdots < g_n, h_1 < h_2 < \cdots < h_m, l_1 < l_2 < \cdots < l_r.$$

We will use transfinite induction on the strictly totally ordered set $(M, \leq)$ to show that $a_i b_j c_k = 0$, for any i, j and k. Note that in $(R/I)[M], (\overline{a}_1 g_1 + \overline{a}_2 g_2 + \cdots + \overline{a}_n g_n)(\overline{b}_1 h_1 + \overline{b}_2 h_2 + \cdots + \overline{b}_m h_m)(\overline{c}_1 l_1 + \overline{c}_2 l_2 + \cdots + \overline{c}_r l_r) = 0$. Thus, we have $a_i b_j c_k \in I$, for all i, j and k, with $1 \leq i \leq n, 1 \leq j \leq m$ and $1 \leq k \leq r$, since R/I is 3-M-Armendariz. If there exist $1 \leq i \leq n, 1 \leq j \leq m$ and $1 \leq k \leq r$, such that $g_i h_j l_k = g_1 h_1 l_1$, then $g_1 \leq g_i, h_1 \leq h_j$ and $l_1 \leq l_k$. If $g_1 < g_i$, then $g_1 h_1 l_1 < g_i h_1 l_1 \leq g_i h_j l_k = g_1 h_1 l_1$, a contradiction, thus $g_1 = g_i$. Similarly, $h_1 = h_j$ and $l_1 = l_k$. Hence $a_1 b_1 c_1 = 0$. Now suppose that $w \in M$ is such that for any g_i, h_j and l_k, if $g_i h_j l_k < w$, then $a_i b_j c_k = 0$. We will show that $a_i b_j c_k = 0$, for any g_i, h_j and l_k, with $g_i h_j l_k = w$. Set $X = \{(g_i, h_j, l_k) \mid g_i h_j l_k = w\}$. Then X is a finite set. We write X as $\{(g_{i_t}, h_{j_t}, l_{k_t}) \mid t = 1, 2, \cdots, u\}$ such that

$$g_{i_1} < g_{i_2} < \cdots < g_{i_u}.$$

We claim that

$$h_{j_u} l_{k_u} < \cdots < h_{j_2} l_{k_2} < h_{j_1} l_{k_1}.$$

In fact, if $h_{j_1} l_{k_1} < h_{j_2} l_{k_2}$, then

$$w = g_{i_1} h_{j_1} l_{k_1} < g_{i_1} h_{j_2} l_{k_2} < g_{i_2} h_{j_2} l_{k_2} = w,$$

a contradiction. If $h_{j_1}l_{k_1} = h_{j_2}l_{k_2}$, then from $g_{i_1}h_{j_1}l_{k_1} = w = g_{i_2}h_{j_2}l_{k_2}$ it follows that $g_{i_1} = g_{i_2}$, a contradiction again. Thus, $h_{j_2}l_{k_2} < h_{j_1}l_{k_1}$. Similarly we have the claim. For any $t \geq 2, g_{i_1}h_{j_t}l_{k_t} < g_{i_t}h_{j_t}l_{k_t} = w$, and thus, by induction hypothesis, we have $a_{i_1}b_{j_t}c_{k_t} = 0$. Since $b_{j_t}c_{k_t}Ia_{i_1} \subseteq I, (b_{j_t}c_{k_t}Ia_{i_1})^2 = 0$, and I is reduced, we have $b_{j_t}c_{k_t}Ia_{i_1} = 0$ for any $t \geq 2$. Thus, for any $t \geq 2$,

$$(a_{i_t}b_{j_t}c_{k_t})(a_{i_1}b_{j_1}c_{k_1})^2 \subseteq (a_{i_t}b_{j_t}c_{k_t})I(a_{i_1}b_{j_1}c_{k_1}) = a_{i_t}(b_{j_t}c_{k_t}Ia_{i_1})b_{j_1}c_{k_1} = 0.$$

Now, from

$$\sum_{(g_i,h_j,l_k)\in X}(a_ib_jc_k) = \sum_{t=1}^{u} a_{i_t}b_{j_t}c_{k_t} = 0,$$

it follows that

$$(\sum_{t=1}^{u} a_{i_t}b_{j_t}c_{k_t})(a_{i_1}b_{j_1}c_{k_1})^2 = (a_{i_1}b_{j_1}c_{k_1})^3 = 0.$$

Since $a_{i_1}b_{j_1}c_{k_1} \in I$ and I is reduced, we have $a_{i_1}b_{j_1}c_{k_1} = 0$. Thus, $\sum_{t=2}^{u} a_{i_t}b_{j_t}c_{k_t} = 0$. Multiplying $(a_{i_2}b_{j_2}c_{k_2})^2$ on $\sum_{t=2}^{u} a_{i_t}b_{j_t}c_{k_t} = 0$, from the right-hand side, we obtain $a_{i_2}b_{j_2}c_{k_2} = 0$, by the same way as the above. Continuing this process, we can prove $a_{i_t}b_{j_t}c_{k_t} = 0$, for $t = 1, 2, \cdots, u$. Thus, $a_ib_jc_k = 0$ for any i, j and k where $g_ih_jl_k = w$. Now the result follows. □

Recall that a monoid M is called torsion-free if the following property holds: if $g, h \in M$ and $k \geq 1$ are such that $g^k = h^k$, then $g = h$.

Corollary 2.10. *Let M be a commutative, cancellative, and torsion-free monoid. If one of the following conditions holds, then R is 3-M-Armendariz.*

1. *R satisfies condition (P).*
2. *R/I is 3-M-Armendariz for some ideal I of R, and I is reduced.*

Proof. If M is commutative, cancellative, and torsion-free, then, by Ribenboim [10], there exists a compatible strict total order $\leq$ on M. Now the results follow from Corollary 2.7 and Theorem 2.9. □

Proposition 2.11. *Suppose that R is 3-M-Armendariz, $n \geq 3$. If α_1, α_2, $\cdots, \alpha_n \in R[M]$ are such that, $\alpha_1\alpha_2\cdots\alpha_n = 0$, then $a_1a_2\cdots a_n = 0$, where a_i is a coefficient of α_i.*

Proof. It follows easily from the definition. □

Let R be a ring. Define a ring $S_3(R)$ as follows:

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}.$$

Theorem 2.12. *Let M be a monoid with $|M| \geq 2$, and R be 3-M-Armendariz and satisfies condition (P). Then $S_3(R)$ is 3-M-Armendariz.*

Proof. Let R be 3-M-Armendariz and satisfies condition (P). We show that $S_3(R)$ is 3-M-Armendariz. Now we complete the proof by adapting the proof of Kim and Lee [9, Proposition 2]. It is easy to see that there exists an isomorphism of rings $S(R)[M] \to S(R[M])$ define by:

$$\sum_{i=1}^{n} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} g_i \to \begin{pmatrix} \sum_{i=1}^n a_i g_i & \sum_{i=1}^n b_i g_i & \sum_{i=1}^n c_i g_i \\ 0 & \sum_{i=1}^n a_i g_i & \sum_{i=1}^n d_i g_i \\ 0 & 0 & \sum_{i=1}^n a_i g_i \end{pmatrix}.$$

Suppose that $\alpha = A_1 g_1 + A_2 g_2 + \cdots + A_n g_n, \beta = B_1 h_1 + B_2 h_2 + \cdots + B_m h_m$ and $\gamma = C_1 l_1 + C_2 l_2 + \cdots + C_r l_r \in S_3(R)[M]$ are such that $\alpha\beta\gamma = 0$, where $A_i, B_j, C_k \in S_3(R)$. We claim $A_i B_j C_k = 0$ for each i, j and k. Assume that

$$A_i = \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix}, B_j = \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix}, C_k = \begin{pmatrix} a''_k & b''_k & c''_k \\ 0 & a''_k & d''_k \\ 0 & 0 & a''_k \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \sum_{i=1}^n a_i g_i & \sum_{i=1}^n b_i g_i & \sum_{i=1}^n c_i g_i \\ 0 & \sum_{i=1}^n a_i g_i & \sum_{i=1}^n d_i g_i \\ 0 & 0 & \sum_{i=1}^n a_i g_i \end{pmatrix} \times \begin{pmatrix} \sum_{j=1}^m a'_j h_j & \sum_{j=1}^m b'_j h_j & \sum_{j=1}^m c'_j h_j \\ 0 & \sum_{j=1}^m a'_j h_j & \sum_{j=1}^m d'_j h_j \\ 0 & 0 & \sum_{j=1}^m a'_j h_j \end{pmatrix} \times \begin{pmatrix} \sum_{k=1}^r a''_k l_k & \sum_{k=1}^r b''_k l_k & \sum_{k=1}^r c''_k l_k \\ 0 & \sum_{k=1}^r a''_k l_k & \sum_{k=1}^r d''_k l_k \\ 0 & 0 & \sum_{k=1}^r a''_k l_k \end{pmatrix} = 0.$$

Thus,

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0, \tag{1}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} b''_k l_k) + (\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} a''_k l_k) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0, \tag{2}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} c''_k l_k) + (\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} d''_k l_k) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k) + (\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j)(\sum_{k=1}^{r} a''_k l_k) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k) + (\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0, \tag{3}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k) + (\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k) + (\sum_{i=1}^{n} d_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0. \tag{4}$$

Since R is 3-M-Armendariz we have $a_i a'_j a''_k = 0$ for all i, j and k. Thus, $a''_k a_i a'_j = a'_j a''_k a_i = 0$, since R satisfies condition (P). Hence

$$(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j) = (\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i) = 0$$

for all i, j and k. If we multiply the equation (2) on left side by $(\sum_{k=1}^{r} a''_k l_k)$ then

$$(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} a''_k l_k) + (\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0. \tag{5}$$

Multiplying (5) on left side by $(\sum_{j=1}^{m} a'_j h_j)$ then we have

$$(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0. \tag{6}$$

Since R is 3-M-Armendariz ring and satisfies condition (P), by Proposition 2.11 it follows that $a'_j a''_k b_i a'_j a''_k = 0$. Thus, $b_i a'_j a''_k = 0$ for all i, j and k. Now (5) becomes

$$(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.$$

Thus, $a_i b'_j a''_k = 0$ for all i, j and k, since R is 3-M-Armendariz and satisfies condition (P). Hence from (2) it follows

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} b''_k l_k) = 0.$$

Thus, $a_i a'_j b''_k = 0$ for all i, j and k. Now multiplying (4) on left side by $(\sum_{k=1}^{r} a''_k l_k)$ yields

$$\begin{aligned}&(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{j=1}^{m} a_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k)\\&+(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} d_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.\end{aligned} \tag{7}$$

Multiplying (7) on left side by $(\sum_{j=1}^{m} a'_j h_j)$ then we have

$$(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} d_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.$$

Thus, $d_i a'_j a''_k = 0$ for all i, j and k, since R satisfies condition (P). Similarly, we have

$$(\sum_{k=1}^{r} a''_k l_k)(\sum_{j=1}^{m} a_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.$$

Thus, $a_i d'_j a''_k = 0$ for all i, j and k. Now from (4) it follows

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k) = 0.$$

Thus, $a_i a'_j d''_k = 0$ for all i, j and k. Now we multiply (3) on left side by $(\sum_{k=1}^{r} a''_k l_k)$ then

$$\begin{aligned}&(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j)(\sum_{k=1}^{r} a''_k l_k)\\&+(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k)\\&+(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.\end{aligned} \tag{8}$$

Multiplying (8) on left side by $(\sum_{j=1}^{m} a'_j h_j)$ yileds

$$(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.$$

Thus, $c_i a'_j a''_k = 0$ for all i, j and k. Thus (8) becomes

$$\begin{aligned}&(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j)(\sum_{k=1}^{r} a''_k l_k)\\&+(\textstyle\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.\end{aligned} \tag{9}$$

Multiplying (9) on left side by $(\sum_{j=1}^{m} d'_j h_j)$ yields

$$(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.$$

Thus, we have $b_i d'_j a''_k = 0$ for all i, j and k. Now from (8) it follows

$$(\sum_{k=1}^{r} a''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j)(\sum_{k=1}^{r} a''_k l_k) = 0.$$

Thus, $a_i c'_j a''_k = 0$ for all i, j and k. Now (3) becomes

$$\begin{aligned}&(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} c''_k l_k) + (\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} d''_k l_k)\\ &+(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k) = 0.\end{aligned} \tag{10}$$

Multiplying (10) on left side by $(\sum_{k=1}^{r} d''_k l_k)$ yields

$$\begin{aligned}&(\sum_{k=1}^{r} d''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} d''_k l_k)\\ &+(\sum_{k=1}^{r} d''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k) = 0.\end{aligned} \tag{11}$$

Multiplying (11) on left side by $(\sum_{j=1}^{m} a'_j h_j)$ then

$$(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k)(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} d''_k l_k) = 0.$$

Thus, $b_i a'_j d''_k = 0$ for all i, j and k. Thus (11) becomes

$$(\sum_{k=1}^{r} d''_k l_k)(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j)(\sum_{k=1}^{r} d''_k l_k) = 0.$$

Thus, we have $a_i b'_j d''_k = 0$ for all i, j and k. Now from (10) it follows

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j)(\sum_{k=1}^{r} c''_k l_k) = 0.$$

Thus, $a_i a'_j c''_k = 0$ for all i, j and k. Now it is easy to see that $A_i B_j C_k = 0$ for all i, j and k.

Then $S_3(R)$ is 3-M-Armendariz. □

Remark 2.13. *Let R be a ring and let*

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & a \end{pmatrix} a, a_{ij} \in R \right\}.$$

Where n is a positive integer. Then Theorem 2.12 suggests that $S_n(R)$ may be also 3-M-Armendariz for $n \geq 4$ if R is 3-M-Armendariz and satisfies condition (P). But the examples appeared in Kim and Lee [9, Example 3], and Hong et al. [2, Example 18], eliminate the possibility.

Example 2.14. *Let M be a monoid with $|M| \geq 2$ and R a ring with identity. Take $e \neq g \in M$. Let $\alpha = E_{13}e + (E_{13} - E_{12})g, \beta = E_{24}e + (E_{24} + E_{34})g, \gamma = I_4 e$, be in $S_n(R)[M]$, where E_{ij}'s are the matrix units in $S_n(R)(n \geq 4)$. Then $\alpha\beta\gamma = 0$, but $E_{13}(E_{24} + E_{34})I_4 \neq 0$. Thus, $S_n(R)$ is not 3-M-Armendariz $(n \geq 4)$.*

Corollary 2.15. *Let M be a monoid and R 3-M-Armendariz ring. If R satisfies condition (P), then the trivial extension $T(R, R)$ is 3-M-Armendariz.*

Proof. Note that $T(R, R)$ is isomorphic to the ring

$$\left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}.$$

Now the result follows from Theorem 2.12 and from the fact that every subring of a 3-M-Armendariz ring is 3-M-Armendariz. □

Recall that an element u of a ring R is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

Proposition 2.16. *Let R be a ring and Δ be a multiplicative monoid in R consisting of central regular elements. Then R is 3-M-Armendariz if and only if $\Delta^{-1}R$ is also 3-M-Armendariz.*

Proof. Let R be a 3-M-Armendariz ring, and $S = \Delta^{-1}R$. Put $\alpha\beta\gamma = 0$, where $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$ and $\gamma = \sum_{k=1}^{r} c_k l_k \in S[M]$. We may assume that $a_i = \varepsilon_i u^{-1}, b_j = \eta_j v^{-1}$ and $c_k = \mu_k w^{-1}$ with $\varepsilon_i, \eta_j, \mu_k$ are in R for all i, j and k, and $u, v, w \in \Delta$. Then we have

$$0 = \alpha\beta\gamma = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} a_i b_j c_k g_i h_j l_k$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} \varepsilon_i \eta_j \mu_k u^{-1} v^{-1} w^{-1} g_i h_j l_k$$

$$= (\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \varepsilon_i \eta_j \mu_k g_i h_j l_k)(uvw)^{-1}.$$

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \varepsilon_i \eta_j \mu_k g_i h_j l_k = 0$$

in $R[M]$. Since R is 3-M-Armendariz, $\varepsilon_i \eta_j \mu_k = 0$, for all i, j and k and so

$$a_i b_j c_k = \varepsilon_i u^{-1} \eta_j v^{-1} \mu_k w^{-1} = \varepsilon_i \eta_j \mu_k (uvw)^{-1} = 0,$$

for all i, j, k. Thus, S is 3-M-Armendariz. The converse follows from Proposition 2.3. □

Proposition 2.17. *Let M be a monoid, R_1 and R_2 rings. Then the following conditions are equivalent:*

1. *$R_1 \times R_2$ is 3-M-Armendariz;*
2. *R_1 and R_2 are 3-M-Armendariz.*

Proof. (2)⇒(1). Let R_1 and R_2 are 3-M-Armendariz. Suppose that $\alpha = a_1 g_1 + \cdots + a_n g_n, \beta = b_1 h_1 + \cdots + b_m h_m$ and $\gamma = c_1 l_1 + \cdots + c_r l_r \in (R_1 \times R_2)[M]$, such that $\alpha\beta\gamma = 0$, where a_i, b_j, c_k are in $(R_1 \times R_2)$, and g_i, h_j, l_k are in M for all i, j and k. Then

$$\sum_{g\in M}(\sum_{g_i h_j l_k = g}(a_i b_j c_k))g = 0.$$

Thus,

$$\sum_{g_i h_j l_k = g} a_i b_j c_k = 0, \forall g \in M.$$

Set $a_i = (a_i^1, a_i^2), b_j = (b_j^1, b_j^2), c_k = (c_k^1, c_k^2)$, where $a_i^1, b_j^1, c_k^1 \in R_1$ and $a_i^2, b_j^2, c_k^2 \in R_2$ for all i, j and k. Then

$$\sum_{g_i h_j l_k = g} a_i^1 b_j^1 c_k^1 = 0, \sum_{g_i h_j l_k = g} a_i^2 b_j^2 c_k^2 = 0, \forall g \in M.$$

Thus,

$$(\sum_i a_i^1 g_i)(\sum_j b_j^1 h_j)(\sum_k c_k^1 l_k) = \sum_{g\in M}(\sum_{g_i h_j l_k = g} a_i^1 b_j^1 c_k^1)g = 0.$$

Similarly, $(\sum_i a_i^2 g_i)(\sum_j b_j^2 h_j)(\sum_k c_k^2 l_k) = 0$. Since R_1 and R_2 are 3-M-Armendariz, we have $a_i^1 b_j^1 c_k^1 = 0$ and $a_i^2 b_j^2 c_k^2 = 0$ for all i, j and k. Thus, $a_i b_j c_k = 0$ for all i, j and k. This shows that $R_1 \times R_2$ is 3-M-Armendariz.

(1)⇒(2). This is clear since $R_1 \cong R_1 \times \{0\}$ and $R_2 \cong \{0\} \times R_2$. □

Corollary 2.18. *Let M be a monoid and $R_i, i \in I$, be rings. Then the following statements are equivalent:*

1. *$\prod_{i \in I} R_i$ is 3-M-Armendariz;*
2. *$\bigoplus_{i \in I} R_i$ is 3-M-Armendariz;*
3. *R_i is 3-M-Armendariz for each $i \in I$.*

Proof. By the same method as above, we have (1)⇔(3) and (2)⇔(3). □

3. Monoid Rings

Anderson and Camillo [3, Theorem 2], have shown that a ring R is Armendariz if and only if $R[x]$ is Armendariz, and Zhongkui [7, proposition 2.1], have shown that if R is a reduced and M-Armendariz ring, then $R[M]$ is N-Armendariz, where M is a monoid and N a *u.p.*-monoid. For 3-M-Armendariz, we have the following results.

Proposition 3.1. *Let M be a monoid and N a u.p.-monoid. If R satisfies condition (P) and is 3-M-Armendariz, then $R[M]$ is 3-N-Armendariz.*

Proof. Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m$ and $\gamma = c_1l_1 + \cdots + c_kl_k \in R[M]$, such that $(\alpha\beta\gamma)^2 = 0$. Then $(a_ib_jc_k)^2 = 0$, for all i, j and k, since R is 3-M-Armendariz. Thus, $(a_ib_jc_k) = 0$, for all i, j and k, since R satisfies condition (P). Hence $\alpha\beta\gamma = 0$. This shows that $R[M]$ satisfies condition (P). Now the result follows from Theorem 2.6. □

Proposition 3.2. *Let M be a monoid and N a u.p.-monoid. If R satisfies condition (P), and is 3-M-Armendariz, then $R[N]$ is 3-M-Armendariz.*

Proof. There exists an isomorphism of rings $R[N][M] \cong R[M][N]$ defined by

$$\sum_p (\sum_i a_{ip} n_i) m_p \to \sum_i (\sum_p a_{ip} m_p) n_i.$$

Now suppose that $\alpha_i, \beta_j, \gamma_k \in R[N]$ are such that $(\sum_i \alpha_i g_i)(\sum_j \beta_j h_j)(\sum_k \gamma_k l_k) = 0$, where $g_i, h_j, l_k \in M$. We will show that $\alpha_i\beta_j\gamma_k = 0$, for all i, j and k. Assume that $\alpha_i = \sum_p a_{ip}n_p, \beta_j = \sum_q b_{jq}n'_q$ and $\gamma_k = \sum_s c_{ks}n''_s$, where $n_p, n'_q, n''_s \in N$ for all p, q and s. Then

$$(\sum_i(\sum_p a_{ip}n_p)g_i)(\sum_j(\sum_q b_{jq}n'_q)h_j)(\sum_k(\sum_s c_{ks}n''_s)l_k) = 0.$$

Thus, in $R[M][N]$ we have

$$(\sum_p(\sum_i a_{ip}g_i)n_p)(\sum_q(\sum_j b_{jq}h_j)n'_q)(\sum_s(\sum_k c_{ks}l_k)n''_s) = 0.$$

By Proposition 3.1, $R[M]$ is 3-N-Armendariz. Thus

$$(\sum_i a_{ip}g_i)(\sum_j b_{jq}h_j)(\sum_k c_{ks}l_k) = 0$$

for all p, q and s. So $a_{ip}b_{jq}c_{ks} = 0$ for all i, j, k, p, q, s, since R is 3-M-Armendariz. Hence

$$\alpha_i\beta_j\gamma_k = (\sum_p a_{ip}n_p)(\sum_q b_{jq}n'_q)(\sum_s c_{ks}n''_s) = 0,$$

for all p, q and s. □

Theorem 3.3. *Let M be a monoid and N a u.p.-monoid. If R satisfies condition (P), and is 3-M-Armendariz, then R is 3-$(M \times N)$-Armendariz.*

Proof. Suppose that $\sum_{i=1}^{s} a_i(m_i, n_i)$ is in $R[M \times N]$. Without loss of generality, we assume that $\{n_1, n_2, \cdots, n_s\}$=$\{n_1, n_2, \cdots, n_t\}$ with $n_i \neq n_j$ when $1 \leq i \neq j \leq t$. For any $1 \leq p \leq t$, denote $A_p = \{i \mid 1 \leq i \leq s, n_i = n_p\}$. Then $\sum_{p=1}^{t}\sum_{i\in A_p}(a_im_i)n_p \in R[M][N]$. Note that $m_i \neq m_{i'}$ for any $i, i' \in A_p$ with $i \neq i'$. Now it is easy to see that there exists an isomorphism of rings $R[M \times N] \to R[M][N]$ define by

$$\sum_{i=1}^{s} a_i(m_i, n_i) \to \sum_{p=1}^{t}\sum_{i\in A_p}(a_im_i)n_p.$$

Suppose that

$$(\sum_{i=1}^{s} a_i(m_i, n_i))(\sum_{j=1}^{s'} b_j(m'_j, n'_j))(\sum_{k=1}^{s''} c_k(m''_k, n''_k)) = 0$$

in $R[M \times N]$. Then from the above isomorphism it follows that

$$(\sum_{p=1}^{t}(\sum_{i\in A_p} a_i m_i)n_p)(\sum_{q=1}^{t'}(\sum_{j\in B_q} b_j m'_j)n'_q)(\sum_{r=1}^{t''}(\sum_{k\in C_r} c_k m''_k)n''_r) = 0.$$

By Proposition 3.1 $R[M]$ is 3-N-Armendariz. Thus we have

$$(\sum_{i\in A_p} a_i m_i)(\sum_{j\in B_q} b_j m'_j)(\sum_{k\in C_r} c_k m''_k) = 0$$

for any p, q and r. Hence $a_i b_j c_k = 0$, for any $i \in A_p, j \in B_q$ and any $k \in C_r$, since R is 3-M-Armendariz. Thus, $a_i b_j c_k = 0$ for all i, j, k, such that $1 \leq i \leq s, 1 \leq j \leq s', 1 \leq k \leq s''$. This means that R is 3-$(M \times N)$-Armendariz. □

Let $M_i, i \in I$, be monoids. Denote $\coprod_{i\in I} M_i = \{(g_i)_{i\in I} \mid$ there exist only finite i's such that $g_i \neq e_i$, the identity of $M_i\}$. Then $\coprod_{i\in I} M_i$ is a monoid with the operation $(g_i)_{i\in I}(g'_i)_{i\in I} = (g_i g'_i)_{i\in I}$.

Corollary 3.4. *Let $M_i, i \in I$ be u.p.-monoids and a ring R satisfies condition (P). If R is 3-$M_{i_0}-$Armendariz for some $i_0 \in I$, then R is 3-$\coprod_{i\in I} M_i$-Armendariz.*

Proof. Let $\alpha = \sum_i a_i g_i, \beta = \sum_j b_j h_j, \gamma = \sum_k c_k l_k \in R[\coprod_{i\in I} M_i]$ such that $\alpha\beta\gamma = 0$. Then $\alpha, \beta, \gamma \in R[M_1 \times M_2 \times \cdots \times M_n]$, for some finite subset $\{M_1, M_2, \cdots, M_n\} \subseteq \{M_i | i \in I\}$. Thus $\alpha, \beta, \gamma \in R[M_{i_0} \times M_1 \times M_2 \times \cdots \times M_n]$. The ring R, by Theorem 3.3 and by induction, is 3-$(M_{i_0} \times M_1 \times M_2 \times \cdots \times M_n)$-Armendariz, so $a_i b_j c_k = 0$ for all i, j and k. Hence R is 3-$\coprod_{i\in I} M_i$-Armendariz. □

Corollary 3.5. *Let M be a monoid and a ring R satisfies condition (P). If R is 3-M-Armendariz, then $R[x]$ and $R[x, x^{-1}]$ are 3-M-Armendariz.*

Proof. Note that $R[x] \cong R[\mathbb{N} \cup \{0\}]$ and $R[x, x^{-1}] \cong R[\mathbb{Z}]$. □

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