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WAITING TIME IN THE M/M/m/($m+c$) QUEUE WITH IMPATIENT CUSTOMERS

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Abstract: The operation of call centers can be modeled by queues with impatient customers in order to dimension the appropriate number of agents given the call volume and the caller's satisfaction level for service. An exposition of theoretical results for the M/M/m/($m+c$) queueing model with impatient customers is presented with numerical illustration as a basic model of call centers. The patience time of each customer in the waiting room is assumed to be exponentially distributed. In particular, we derive the joint distribution for the waiting time of a customer and the probability that he is served and the probability that he abandons while waiting. The results can be used to calculate service level indicators for the operation of call centers.

AMS Subject Classification: 60K25, 68M20, 90B22
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1. Introduction

Queueing models have been widely used to model the performance of call centers with impatient customers, which means that waiting customers may leave before

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getting service. The multiserver queue M/M/m with impatient customers is called *Erlang-A* model, "A" for "abandonment", by Mandelbaum and Zeltyn [7] in contrast with the well-known *Erlang-C* model, M/M/m with only patient customers. See, for example, Gans et al. [2] and Koole and Mandelbaum [6] for survey of queueing models for call centers.

Our queueing system has m servers with Poisson arrival process of customers at rate λ and exponentially distributed service time with mean $1/\mu$. The capacity of the waiting room is finite, given by c. We assume that each customer in the waiting room leaves the queue (abandons) with probability $\theta\Delta t$ during a short interval $(t, t+\Delta t)$. In other words, the patience time of each customer is exponentially distributed with mean $1/\theta$. Once the service is started, he does not leave the system. Thus we call our model "M/M/$m/(m+c)$ queue with impatient customers." This system is depicted in Figure 1.

Analysis of an M/M/m queue ($c = \infty$) with impatient customers is available in several books such as [1, p. 95], [4, p. 64], [8, p. 109], and [9, p. 67]. This system has been used widely as a basic model of call centers [3], [7], [10], and [11]. Khudyakov et al. [5] studied the asymptotic behavior for an M/M/$m/(m+c)$ queue with impatient customers.

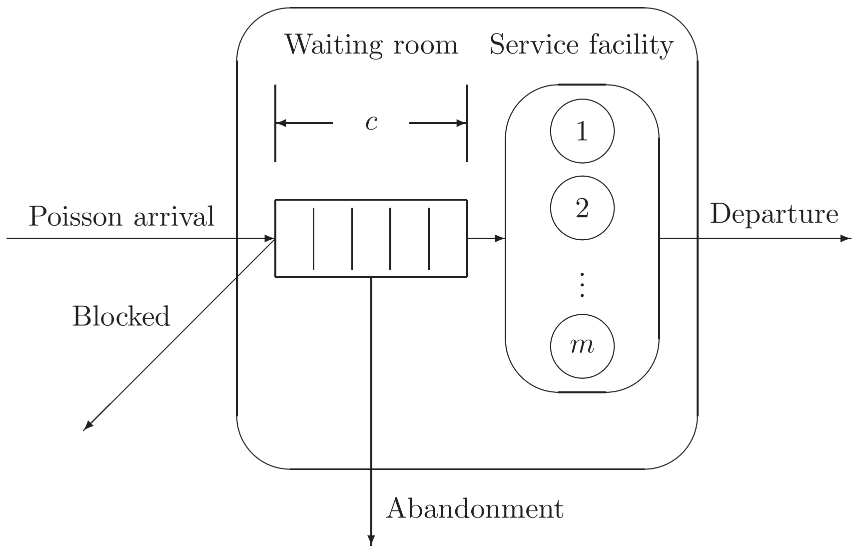

Figure 1: M/M/$m/(m+c)$ queueing system with impatient customers.

The purpose of this paper is to present a stochastic analysis of the M/M/$m/(m+c)$ queueing system with impatient customers in the steady state. We pay special attention to the waiting time of a customer who is served and the time to abandon of a customer who leaves the system before getting service. The number of customers in the system forms a birth-and-death process in a finite space for which the probability distribution is found explicitly. From that distribution we derive several performance measures of the system such as the probabilities of blocking, abandonment, and successful service for an arbitrary customer, the mean number of customers in the waiting room and in the service facility, the distribution and moments for the waiting time of accepted customers. We then analyze the joint distribution for the waiting time of a customer and the probability that he is served and the probability that he abandons while waiting. We show numerical illustration for our theoretical formulas. The results can be used to calculate service level indicators for the operation of call centers.

2. Number of Customers in the System

We first consider a birth-and-death process for the number of customers in the system. Let $N(t)$ be the number of customers present in the system at time t. The process $\{N(t); t \geq 0\}$ is a birth-and-death process with finite state space

$$\mathcal{S} = \{0, 1, 2, \ldots, m+c\}.$$

The birth and death rates are given by

$$\begin{aligned} \lambda_k &= \lambda && 0 \leq k \leq m+c-1, \\ \mu_k &= \begin{cases} k\mu & \\ m\mu + (k-m)\theta & \end{cases} && \begin{aligned} &1 \leq k \leq m-1, \\ &m \leq k \leq m+c. \end{aligned} \end{aligned}$$

The state transition rate diagram of this process is shown in Figure 2.

We consider the steady-state distribution defined by

$$P_k := \lim_{t \to \infty} P\{N(t) = k\} \qquad 0 \leq k \leq m+c.$$

The set of balance equations for $\{P_k; 0 \leq k \leq m+c\}$ is given by

$$\begin{aligned} &\lambda P_0 = \mu P_1, \\ &(\lambda + k\mu) P_k = \lambda P_{k-1} + (k+1)\mu P_{k+1} \qquad 1 \leq k \leq m-1, \\ &[\lambda + m\mu + (k-m)\theta]\, P_k = \lambda P_{k-1} + [m\mu + (k-m+1)\theta] P_{k+1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m \leq k \leq m+c-1, \\ &(m\mu + c\theta) P_{m+c} = \lambda P_{m+c-1}. \end{aligned}$$

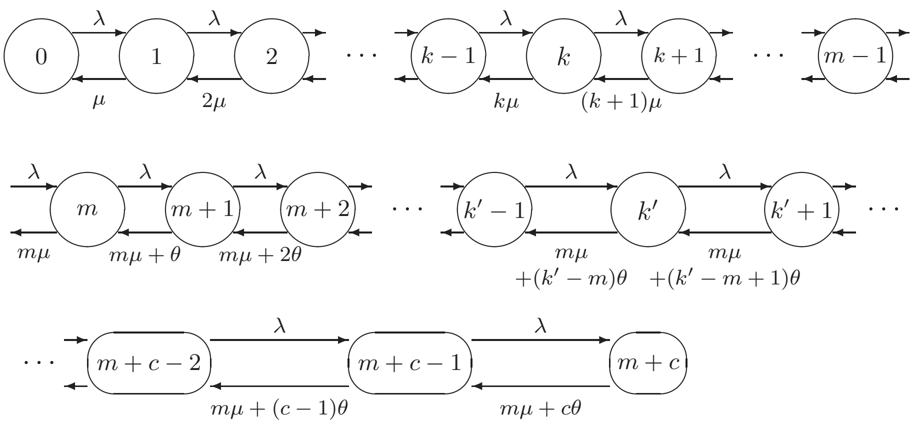

Figure 2: State transition rate diagram for the M/M/m/(m+c) queueing system with impatient customers in the waiting room.

The normalization condition is given by

$$\sum_{k=0}^{m+c} P_k = 1.$$

The above set of balance equations is equivalent with the following set of equations:

$$\begin{aligned} \lambda P_{k-1} &= k\mu P_k && 1 \le k \le m-1, \\ \lambda P_{k-1} &= [m\mu + (k-m)\theta]P_k && m \le k \le m+c. \end{aligned}$$

Therefore, for $1 \le k \le m$, we get

$$P_k = P_0 \frac{\rho^k}{k!},$$

where $\rho := \lambda/\mu$ is the traffic intensity. For $1 \le k \le c$, we get

$$\begin{aligned} P_{m+k} &= \frac{\lambda}{m\mu + k\theta} P_{m+k-1} = \frac{\lambda}{m\mu[1 + (k\theta/m\mu)]} P_{m+k-1} \\ &= \frac{\rho/m}{1 + k\xi/m} P_{m+k-1} = \frac{\rho/m}{1 + k\xi/m} \cdot \frac{\rho/m}{1 + (k-1)\xi/m} P_{m+k-2} \\ &= \cdots \\ &= \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1 + j\xi/m)} P_m = P_0 \frac{\rho^m}{m!} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1 + j\xi/m)}, \end{aligned}$$

where we have defined the ratio of the mean patience time to the mean service time by $\xi := \theta/\mu$. From the normalization condition, we determine

$$\frac{1}{P_0} = \sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!} \sum_{k=0}^{c} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)}.$$

Thus we have found $\{P_k; 0 \le k \le m+c\}$ explicitly.

In the numerical examples throughout this paper, we assume that

$$m = 5 \quad ; \quad c = 10 \quad ; \quad \mu = 1$$

In Figure 3, we plot $\{P_k; 0 \le k \le m+c\}$ with $\rho = 5$ for different values of $\xi = 0$, 0.5, 1.0, and 2.0. Note that the case $\xi = 0$ means that customers never abandon, i.e. a result for the standard M/M/m/($m+c$) queueing model.

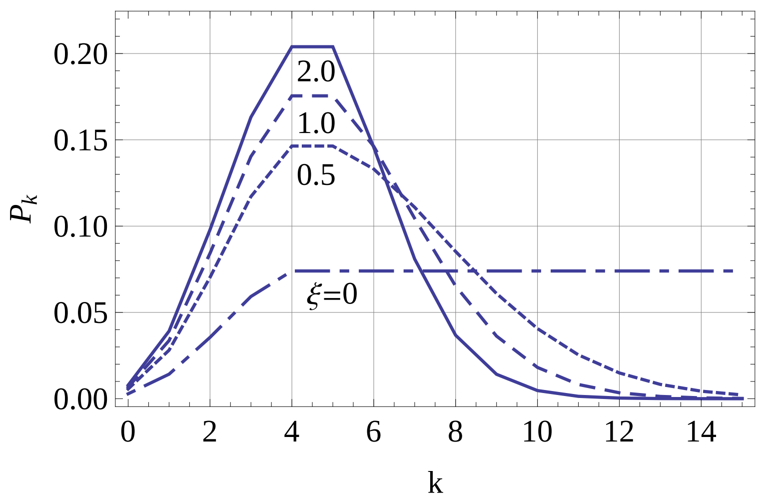

Figure 3: Probability distribution $\{P_k; 0 \le k \le m+c\}$ for the number of customers in the system at an arbitrary time ($\rho = 5$).

3. Performance Measures

Now that we have the probability distribution for the number of customers in the system, we can derive many performance measures of the system based on the mean values.

(1) Blocking probability

Those customers who arrive to the system when all the waiting room is full, i.e. there are $m+c$ customers in the system, are blocked. They are lost forever. Owing to the PASTA (Poisson arrivals see time averages) property, the probability distribution at the arrival time is the same as that at an arbitrary time. Therefore, we obtain the *customer blocking probability* as

$$P\{\mathcal{B}\} = P_{m+c} = P_0 \frac{\rho^m}{m!} \frac{(\rho/m)^c}{\prod_{j=1}^{c}(1+j\xi/m)}.$$

In Figure 4, we plot $P\{\mathcal{B}\}$ against ρ for different values of $\xi = 0$ (complete patience), 0.5, 1.0, and 2.0.

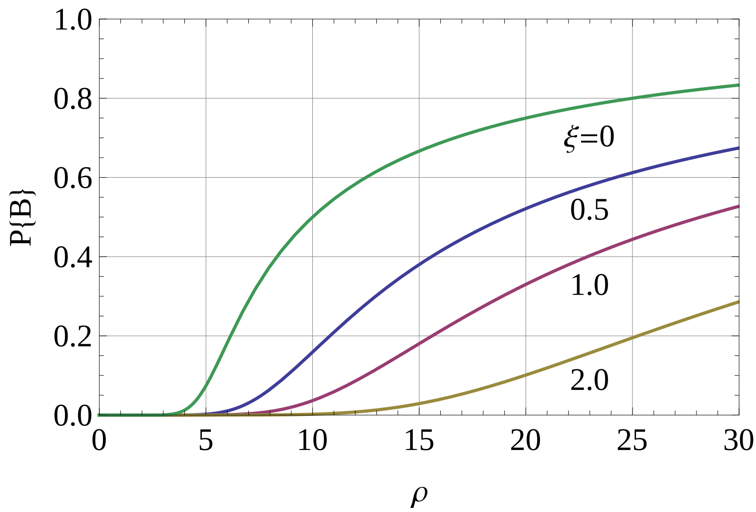

Figure 4: Blocking probability $P\{\mathcal{B}\}$.

(2) Mean number of customers in the waiting room

Let L be the number of customers in the waiting room at an arbitrary time. If there are $k+m$ customers in the system, there k customers in the waiting room. Therefore, we have

$$P\{L=0\} = \sum_{k=0}^{m} P_k = P_0 \sum_{k=0}^{m} \frac{\rho^k}{k!},$$

$$P\{L=k\} \;=\; P_{m+k} = P_0 \frac{\rho^m}{m!} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)} \qquad 1 \le k \le c.$$

In Figure 5, we plot the probability distribution $\{P\{L=k\}; 0 \le k \le c\}$ with $\rho = 5$ for different values of $\xi = 0$, 0.5, 1.0, and 2.0.

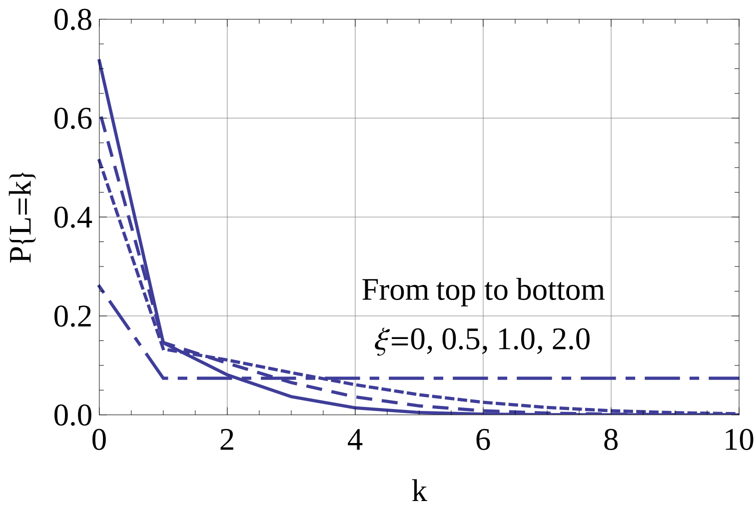

Figure 5: Probability distribution $\{P\{L=k\}; 0 \le k \le c\}$ for the number of customers in the waiting room at an arbitrary time ($\rho = 5$).

The mean number of customers in the waiting room is given by

$$E[L] = \sum_{k=1}^{c} k P_{m+k} = P_0 \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}.$$

When we plot $E[L]$ against ρ, we have the limit

$$\lim_{\rho\to 0} E[L] = 0 \quad ; \quad \lim_{\rho\to\infty} E[L] = c.$$

In Figure 6, we plot $E[L]$ against ρ for different values of $\xi = 0$, 0.5, 1.0, and 2.0. The mean number $E[L]$ decreases as ξ increases because more customers leave the system.

(3) Probability of abandonment in the waiting room

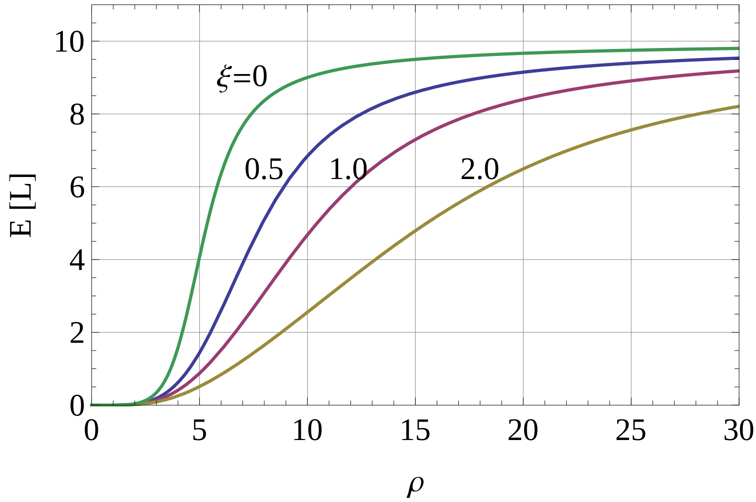

Figure 6: Mean number of customers in service $E[L]$.

Since each customer in the waiting room leaves the queue with rate θ, the fraction of arriving customers who abandon is given by

$$P\{\text{Ab}\} = \frac{1}{\lambda}\sum_{k=1}^{c}(k\theta)P_{m+k} = \frac{\theta}{\lambda}\cdot P_0\frac{\rho^m}{m!}\sum_{k=1}^{c}\frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}.$$

Therefore, we have the relation

$$\theta E[L] = \lambda P\{\text{Ab}\}.$$

Later we derive the relation

$$\lambda(1-P\{\mathcal{B}\}) = \theta E[L] + \mu E[S],$$

where S is the number of customers in the system. Then the probability that an arbitrary accepted customer abandons is given by

$$P\{\text{Ab}\,|\,\mathcal{NB}\} = \frac{P\{\text{Ab}\}}{1-P\{\mathcal{B}\}} = \frac{\theta E[L]}{\theta E[L]+\mu E[S]},$$

where the event $\mathcal{NB}$ is that an arriving customer is accepted. Since $\lim_{\rho\to\infty} E[S] = m$ (shown shortly), we have the limit

$$\lim_{\rho\to\infty} P\{\text{Ab}\,|\,\mathcal{NB}\} = \frac{\theta c}{\theta c+\mu m} = \frac{1}{1+m/(c\xi)}.$$

In Figure 7, we plot $P\{\text{Ab}|\mathcal{NB}\}$ against ρ for different values of $\xi = 0.5$, 1.0, and 2.0. Note that $P\{\text{Ab}|\mathcal{NB}\}$ increases as ξ increases.

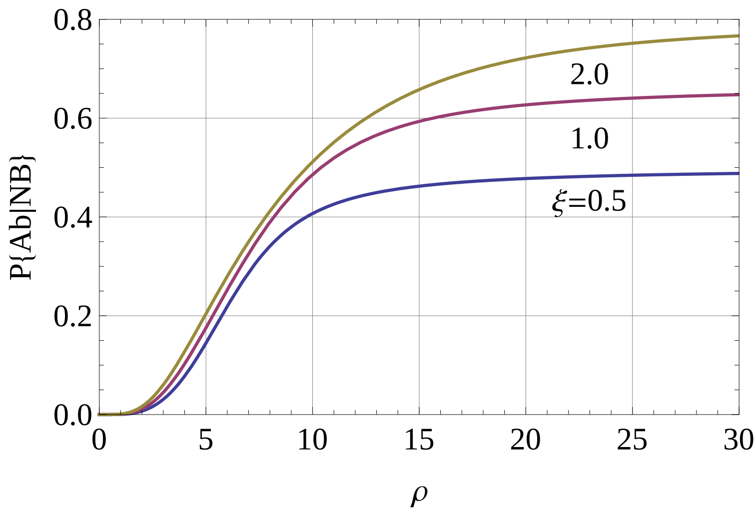

Figure 7: Probability of abandonment $P\{\text{Ab}|\mathcal{NB}\}$ for accepted customers.

(4) Mean number of customers in service

Let S be the number of customers in service at an arbitrary time. Then, we have

$$\begin{aligned} P\{S=k\} &= P_k = P_0 \frac{\rho^k}{k!} \qquad 1 \le k \le m-1, \\ P\{S=m\} &= \sum_{k=0}^{c} P_{m+k} = P_0 \frac{\rho^m}{m!} \sum_{k=0}^{c} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)}. \end{aligned}$$

In Figure 8, we plot the probability distribution $\{P\{S=k\}; 0 \le k \le m\}$ with $\rho = 5$ for different values of $\xi = 0$, 0.5, 1.0, and 2.0.

The mean number of customers in service equals the mean number of busy servers, which is given by

$$E[S] = \sum_{k=1}^{m} k P_k + m \sum_{k=1}^{c} P_{m+k}.$$

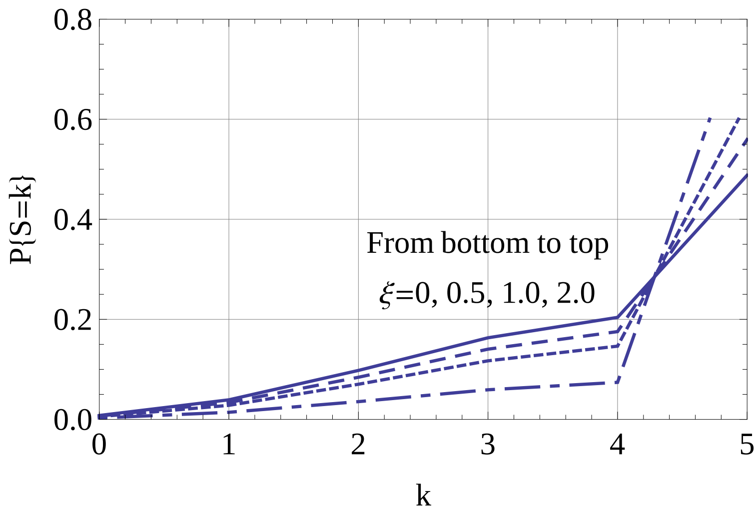

Figure 8: Probability distribution $\{P\{S=k\}; 0 \le k \le m\}$ for the number of customers in service at an arbitrary time ($\rho = 5$).

However, since $\lambda P_{k-1} = k\mu P_k$ for $1 \le k \le m$, we have

$$\mu \sum_{k=1}^{m} kP_k = \lambda \sum_{k=1}^{m} P_{k-1} = \lambda \sum_{k=0}^{m-1} P_k.$$

Therefore we get

$$E[S] = \rho \sum_{k=0}^{m-1} P_k + m \sum_{k=1}^{c} P_{m+k}.$$

When we plot $E[S]$ against ρ, we have the limit

$$\lim_{\rho\to 0} E[S] = 0 \quad ; \quad \lim_{\rho\to\infty} E[S] = m.$$

In Figure 9, we plot $E[S]$ against ρ for different values of $\xi = 0$, 0.5, 1.0, and 2.0. The mean number $E[S]$ in service seems not much sensitive to ξ.

(5) Throughput and the probability of successful service

The *throughput* of the system, denoted by λ', is defined as the mean number of customers served per unit time. On the other hand, the probability

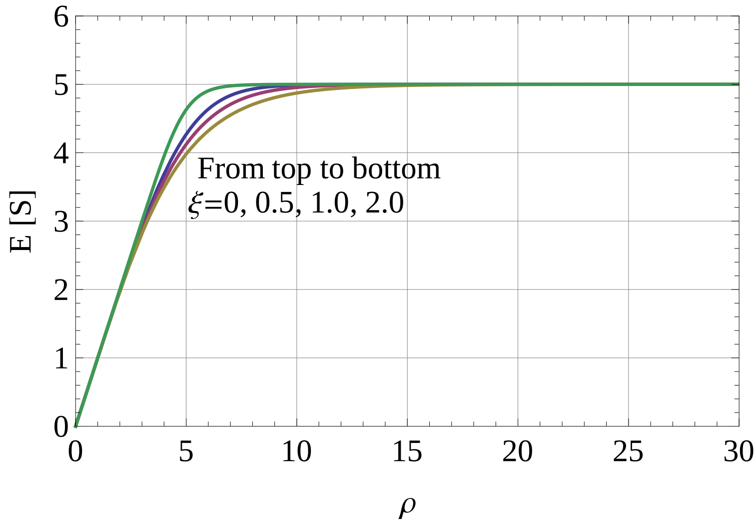

Figure 9: Mean number of customers in service $E[S]$.

of successful service, denoted by $P\{\mathrm{Sr}\}$, is given by the fraction of arriving customers who are served successfully. Therefore we have the relation

$$\lambda' = \lambda P\{\mathrm{Sr}\} = \mu E[S].$$

Hence we get

$$P\{\mathrm{Sr}\} = \frac{E[S]}{\rho} = \sum_{k=0}^{m-1} P_k + \frac{m}{\rho}\sum_{k=1}^{c} P_{m+k}.$$

The probability that an arbitrary accepted customer gets served is given by

$$P\{\mathrm{Sr}\,|\,\mathcal{NB}\} = \frac{P\{\mathrm{Sr}\}}{1 - P\{\mathcal{B}\}} = \frac{\mu E[S]}{\theta E[L] + \mu E[S]},$$

which has the limit

$$\lim_{\rho\to\infty} P\{\mathrm{Sr}\,|\,\mathcal{NB}\} = \frac{\mu m}{\theta c + \mu m} = \frac{1}{1 + c\xi/m}.$$

In Figure 10, we plot $P\{\mathrm{Sr}\,|\,\mathcal{NB}\}$ against ρ for different values of $\xi = 0.5$, 1.0, and 2.0. Note that $P\{\mathrm{Sr}\,|\,\mathcal{NB}\}$ decreases as ξ increases.

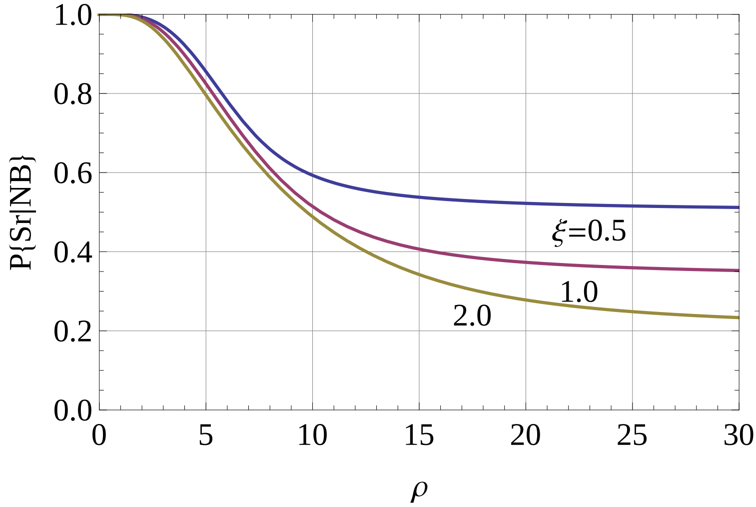

Figure 10: Probability of successful service $P\{\text{Sr} \mid \mathcal{NB}\}$ for accepted customers.

(6) Ratio of blocked, abandoned and served customers

An arriving customer has three kinds of destiny. He is blocked at the arrival time with probability $P\{\mathcal{B}\}$, he is accepted but abandons with probability $P\{\text{Ab}\}$, or he is served successfully with probability $P\{\text{Sr}\}$:

$$P\{\mathcal{B}\} + P\{\text{Ab}\} + P\{\text{Sr}\} = 1.$$

Multiplying both sides by λ, we get the relation

$$\lambda P\{\mathcal{B}\} + \theta E[L] + \mu E[S] = \lambda.$$

On the left-hand side, $\lambda P\{\mathcal{B}\}$ is the mean number of customers who are blocked upon arrival per unit time, $\theta E[L]$ is the mean number of customers who leave the waiting room per unit time, and $\lambda' = \mu E[S]$ is the mean number of customers who are served successfully per unit time.

In Figure 11, we plot the ratio of $P\{\mathcal{B}\}$, $P\{\text{Ab}\}$, and $P\{\text{Sr}\}$ against ρ for $\xi = 0.5$.

Using the normalization condition $\sum_{k=0}^{m+c} P_k = 1$, we get

$$P\{\mathcal{B}\} + P\{\text{Ab}\} \;=\; 1 - P\{\text{Sr}\} = 1 - \sum_{k=0}^{m-1} P_k - \frac{m}{\rho} \sum_{k=1}^{c} P_{m+k}$$

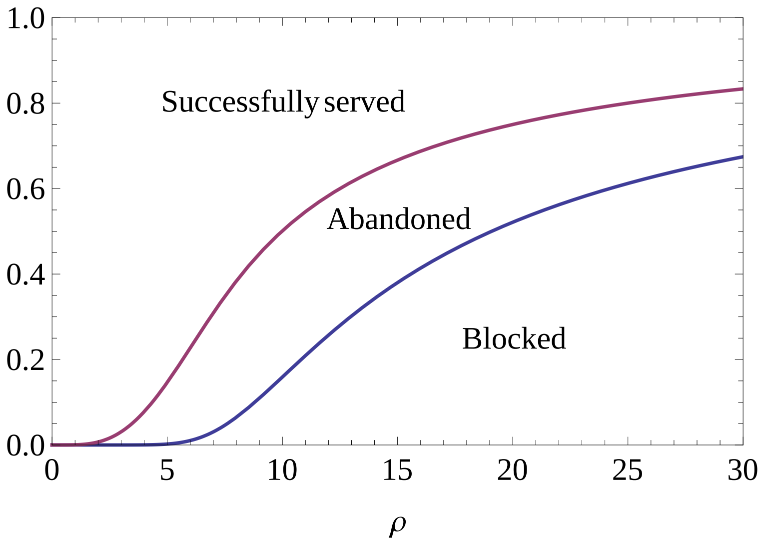

Figure 11: Ratio of blocked, abandoned and served customers ($\xi = 0.5$).

$$
\begin{aligned}
&= 1 - \sum_{k=0}^{m-1} P_k - \frac{m}{\rho}\left(1 - \sum_{k=0}^{m} P_k\right) \\
&= \left(1 - \frac{m}{\rho}\right)\left(1 - \sum_{k=0}^{m-1} P_k\right) + \frac{m}{\rho} P_m \\
&= \left(1 - \frac{m}{\rho}\right)\left(1 - \sum_{k=0}^{m-1} P_k\right) + P_{m-1},
\end{aligned}
$$

which is the probability that an arbitrary customer is lost either by being blocked upon arrival or by abandonment in the waiting room.

(6) Number of Customers in the System Seen by an Accepted Arrival

Let $\hat{N}$ be the number of customers in the system seen by an arbitrary arriving customer who is accepted. The distribution of $\hat{N}$ is given by

$$\hat{P}_k := P\{\hat{N} = k\} = \frac{P_k}{1 - P\{\mathcal{B}\}} \qquad 0 \le k \le m + c - 1.$$

This result can be written explicitly as

$$\hat{P}_k \quad = \quad \hat{P}_0 \frac{\rho^k}{k!} \qquad\qquad 1 \le k \le m,$$

$$\hat{P}_{m+k} = \hat{P}_0 \frac{\rho^m}{m!} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)} \qquad 1 \le k \le c-1,$$

where

$$\frac{1}{\hat{P}_0} = \frac{1-P\{\mathcal{B}\}}{P_0} = \sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)}.$$

Therefore, if we write the dependence on m and c explicitly in the distribution as $P_k[m;c]$ and $\hat{P}_k[m;c]$, we can write

$$\hat{P}_k[m;c] = P_k[m;c-1] \qquad 0 \le k \le m+c-1.$$

(7) Probability of wait

We denote by W the waiting time of accepted customers who are either served or abandon in the waiting room. Later we consider separately the waiting time for served customers and the time that abandoned customers stay in the waiting room.

An arriving customer who is accepted must wait if all servers are busy, i.e. if there are m or more customers in the system immediately before the arrival. Therefore, the probability of wait for an arbitrary accepted customer is given by

$$P\{W>0\} = \sum_{k=0}^{c-1} \hat{P}_{m+k} = \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)}.$$

In Figure 12, we plot $P\{W>0\}$ against ρ for different values of $\xi = 0$, 0.5, 1.0, and 2.0. We observe that $P\{W>0\}$ decreases as ξ increases.

The probability of no wait is given by

$$P\{W=0\} = 1 - P\{W>0\} = \sum_{k=0}^{m-1} \hat{P}_k = \hat{P}_0 \sum_{k=0}^{m-1} \frac{\rho^k}{k!}.$$

(8) Mean waiting time

It follows from Little's law applied to accepted customers that

$$E[W] = \frac{E[L]}{\lambda(1-P\{\mathcal{B}\})} = \frac{\hat{P}_0}{\lambda} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}.$$

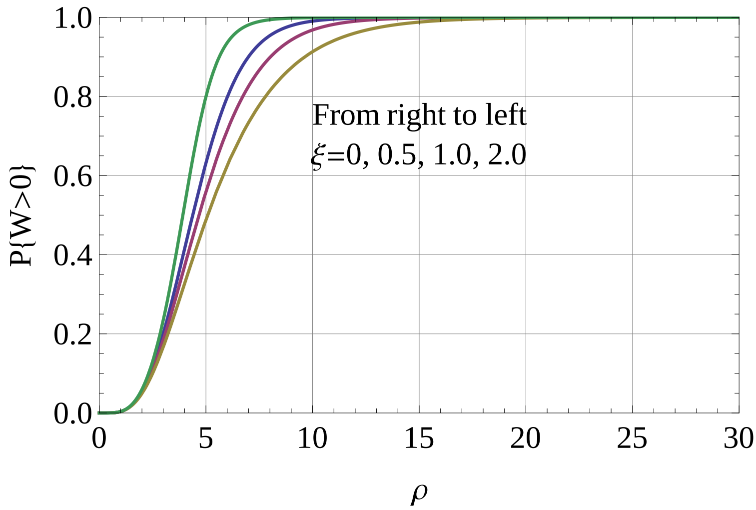

Figure 12: Probability of wait $P\{W > 0\}$.

Therefore, we have the relation

$$P\{\text{Ab}\,|\,\mathcal{NB}\} = \theta E[W],$$

which leads to the limit

$$\lim_{\rho\to\infty} E[W] = \frac{c}{\theta c + \mu m}.$$

In Figure 13, we plot $E[W]$ against ρ with $\mu = 1$ for different values of $\xi = 0$, 0.5, 1.0, and 2.0. We observe that $E[W]$ decreases as ξ increases because less customers are in the system.

4. Distribution of the Waiting Time

We now proceed to study the distribution of the waiting time W for accepted customers, including both customers who are served and those who abandon by assuming the first-come first-served (FCFS) service discipline:

$$P\{W > t\} = \sum_{k=0}^{c-1} P\{W > t\,|\,\hat{N} = m + k\}\hat{P}_{m+k} \qquad t \geq 0.$$

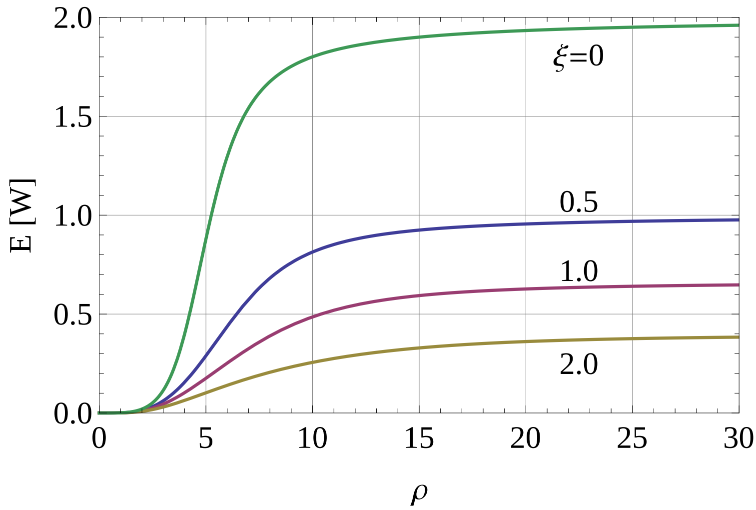

Figure 13: Mean waiting time $E[W]$ ($\mu = 1$).

Let us find $P\{W > t \,|\, \hat{N} = m + k\}$, the distribution of the waiting time for a customer who arrives to find k other customers in the waiting room at time 0.

Assume that there are k other customers in the waiting room. Then we denote by $g_j^k(t)$, $0 \le j \le k$, the probability that j out of those k customers have left the waiting room, either by the service completion in the service facility or by the abandonment of waiting, by time t. For $k = 0$, we have the probability of no service completion by time t:

$$g_0^0(t) = e^{-m\mu t} \qquad t \ge 0.$$

For $k \ge 1$, we consider the events during $(t, t + \Delta t]$ to get

$$\begin{aligned} g_0^k(t + \Delta t) &= g_0^k(t)[1 - (m\mu + k\theta)\Delta t] + o(\Delta t), \\ g_j^k(t + \Delta t) &= g_{j-1}^k(t)\{[m\mu + (k - j + 1)\theta]\Delta t + o(\Delta t)\} \\ &+ g_j^k(t)\{1 - [m\mu + (k - j)\theta]\Delta t + o(\Delta t)\} + o(\Delta t) \\ &\qquad 1 \le j \le k. \end{aligned}$$

The first equation refers the case in which no customers have left beforetime t and neither abandonment nor service completion occurs during $(t, t+\Delta t]$. The second equation refers to the first case in which $j-1$ customers have left by time t (so there are $k - j + 1$ customers in the waiting room at time t) and either abandonment or service completion occurs during $(t, t + \Delta t]$ and the second case in which j customers have left by time t (so there are $k - j$ customers in the waiting room at time t) and neither abandonment nor service completion occurs during $(t, t + \Delta t]$.

Thus we have the set of differential-difference equations

$$\begin{aligned} \frac{dg_0^k(t)}{dt} &= -(m\mu + k\theta)g_0^k(t), \\ \frac{dg_j^k(t)}{dt} &= [m\mu + (k - j + 1)\theta]g_{j-1}^k(t) - [m\mu + (k - j)\theta]g_j^k(t) \\ & \qquad\qquad\qquad\qquad\qquad\qquad 1 \leq j \leq k. \end{aligned}$$

The initial condition is given by

$$g_0^k(0) = 1 \quad ; \quad g_j^k(0) = 0 \qquad 1 \leq j \leq k.$$

Let us find $\{g_j^k(t); 0 \leq j \leq k\}$ for the fixed value of $k \geq 1$. First, the probability that no one leaves before t is clearly given by

$$g_0^k(t) = e^{-(m\mu + k\theta)t} \qquad t \geq 0.$$

Instead of $\{g_j^k(t); 0 \leq j \leq k\}$, we work on $\{G_j^k(t); 0 \leq j \leq k\}$ defined by

$$g_j^k(t) := e^{-[m\mu + (k-j)\theta]t} G_j^k(t) \qquad 0 \leq j \leq k.$$

Then we have the equations for $\{G_j^k(t); 0 \leq j \leq k\}$:

$$G_0^k(t) \equiv 1 \quad ; \quad \frac{dG_j^k(t)}{dt} = [m\mu + (k - j + 1)\theta]e^{-\theta t}G_{j-1}^k(t) \quad 1 \leq j \leq k.$$

Using the initial condition $G_j^k(0) = 0$ for $1 \leq j \leq k$, we get

$$G_j^k(t) = [m\mu + (k - j + 1)\theta] \int_0^t e^{-\theta x} G_{j-1}^k(x)dx \qquad 1 \leq j \leq k.$$

By recursive calculation, we have

$$\begin{aligned} G_1^k(t) &= (m\mu + k\theta) \int_0^t e^{-\theta x} G_0^k(x)dx = (m\mu + k\theta) \int_0^t e^{-\theta x} dx \\ &= \frac{m\mu + k\theta}{\theta}(1 - e^{-\theta t}), \end{aligned}$$

$$
\begin{aligned}
G_2^k(t) &= [m\mu + (k-1)\theta] \int_0^t e^{-\theta x} G_1^k(x) dx \\
&= [m\mu + (k-1)\theta] \frac{m\mu + k\theta}{\theta} \int_0^t e^{-\theta x}(1 - e^{-\theta x}) dx \\
&= \frac{(m\mu + k\theta)[m\mu + (k-1)\theta]}{2\theta^2} (1 - e^{-\theta t})^2,
\end{aligned}
$$

and so on, and obtain

$$
\begin{aligned}
G_j^k(t) &= \frac{(m\mu + k\theta)[m\mu + (k-1)\theta] \cdots [m\mu + (k-j+1)\theta]}{j!\theta^j} \\
&\times \ (1 - e^{-\theta t})^j \qquad\qquad 1 \le j \le k,
\end{aligned}
$$

which can be confirmed by mathematical induction on j. Finally for $j = k$, we obtain

$$
\begin{aligned}
g_k^k(t) &= \frac{(m\mu + \theta)(m\mu + 2\theta) \cdots (m\mu + k\theta)}{k!\theta^k} (1 - e^{-\theta t})^k e^{-m\mu t} \\
&= \frac{(m/\xi)^k}{k!} \left[\prod_{j=1}^k \left(1 + \frac{j\xi}{m} \right) \right] (1 - e^{-\theta t})^k e^{-m\mu t} \qquad k \ge 1,
\end{aligned}
$$

which is the probability that all the k customers who were in the waiting room at time 0 have left by time t; therefore, a customer who arrived at time 0 is at the forefront of the waiting room at time t if he has not abandoned before t. This can be written as

$$
g_k^k(t) = \frac{(m/\xi)^k}{k!} \left[\prod_{j=0}^k \left(1 + \frac{j\xi}{m} \right) \right] (1 - e^{-\theta t})^k e^{-m\mu t} \qquad k \ge 0
$$

so that it includes the case $k = 0$.

Now at time t, if one of m services finishes during $(t, t + \Delta t]$ (which occurs with probability $m\mu\Delta t$), the waiting time of the customer who arrived at time 0 is between t and $t + \Delta t$ provided that he has not left the queue by that time:

$$
P\{t < W \le t + \Delta t \,|\, \hat{N} = m + k \,|\, \text{Not left by time } t\} = g_k^k(t) m\mu\Delta t.
$$

The probability that the customer who arrived at time 0 has not left by time t is given by $e^{-\theta t}$. Therefore, the conditional distribution of the waiting time is given by

$$
P\{W > t \,|\, \hat{N} = m + k\} = e^{-\theta t} \int_t^\infty g_k^k(x) m\mu dx
$$

$$= \frac{(m/\xi)^k}{k!}\left[\prod_{j=0}^{k}\left(1+\frac{j\xi}{m}\right)\right] m\mu e^{-\theta t}\int_t^{\infty}(1-e^{-\theta x})^k e^{-m\mu x}dx$$

$$= \frac{(m/\xi)^k}{k!}\left[\prod_{j=0}^{k}\left(1+\frac{j\xi}{m}\right)\right] e^{-(m\mu+\theta)t}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j e^{-j\theta t}}{1+j\xi/m} \quad k \geq 0.$$

Substituting this expression into the first formula, we obtain

$$P\{W>t\} = \hat{P}_0\frac{\rho^m}{m!}e^{-(m\mu+\theta)t}\sum_{k=0}^{c-1}\frac{(\lambda/\theta)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j e^{-j\theta t}}{1+j\xi/m}. \qquad (1)$$

This is the complementary distribution function for the waiting time of an accepted customer who either gets served or abandons before service. The corresponding pdf is given by

$$f_W(t) = -\frac{d}{dt}P\{W>t\}$$
$$= \hat{P}_0\frac{\rho^m}{m!}m\mu e^{-(m\mu+\theta)t}\sum_{k=0}^{c-1}\frac{(\lambda/\theta)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{1+(j+1)\xi/m}{1+j\xi/m}(-1)^j e^{-j\theta t}.$$

Equation (1) is not appropriate for $\xi = 0$. In this case, from

$$g_k^k(t) = \frac{(m\mu t)^k}{k!}e^{-m\mu t} \quad ; \quad P\{W>t\,|\,\hat{N}=m+k\} = e^{-m\mu t}\sum_{j=0}^{k}\frac{(m\mu t)^j}{j!}$$

for $k \geq 0$, we get

$$P\{W>t\} = \hat{P}_0\frac{\rho^m}{m!}\frac{e^{-m\mu t}}{1-\rho/m}\sum_{k=0}^{c-1}\left[1-\left(\frac{\rho}{m}\right)^{c-k}\right]\frac{(\lambda t)^k}{k!},$$

$$f_W(t) = \hat{P}_0\frac{\rho^m}{m!}m\mu e^{-m\mu t}\sum_{k=0}^{c-1}\frac{(\lambda t)^k}{k!},$$

where

$$\frac{1}{\hat{P}_0} = \sum_{k=0}^{m-1}\frac{\rho^k}{k!}+\frac{\rho^m}{m!}\left[1-\left(\frac{\rho}{m}\right)^c\right]\Big/\left(1-\frac{\rho}{m}\right).$$

In Figures 14 and 15, we plot $P\{W>t\}$ and $f_W(t)$, respectively, with $\rho = 5, \mu = 1$ for different values of $\xi = 0$, 0.5, 1.0, and 2.0.

Let us confirm and explore some more results from Eq. (1).

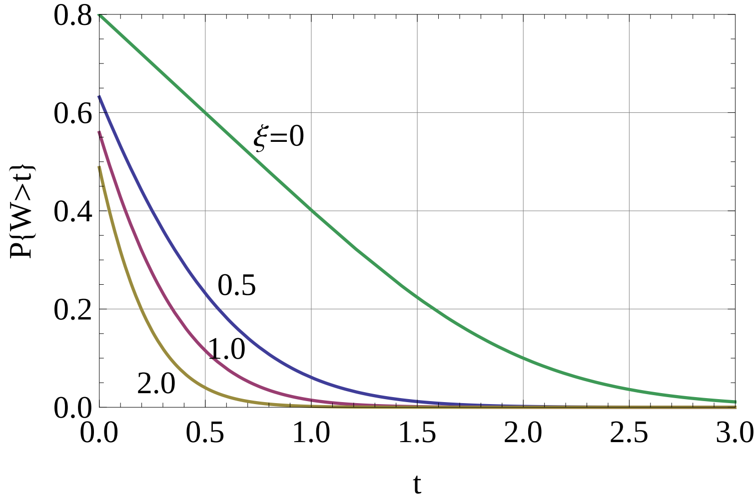

Figure 14: Complementary distribution function for the waiting time $P\{W > t\}$ $(\rho = 5, \mu = 1)$.

- Probability of wait

  From Eq. (1), the probability that an arriving customer who is accepted but forced to wait is given by

$$P\{W > 0\} = \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{1 + j\xi/m}.$$

  The identity

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{x+j} = k! \prod_{j=0}^{k} \frac{1}{x+j} \qquad k \geq 0$$

  leads to

$$P\{W > 0\} = \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1 + j\xi/m)},$$

  which agrees with the above result.

  The complementary distribution function for the waiting time of those customers who are accepted but wait is given by

$$P\{W > t \mid W > 0\} = \frac{P\{W > t, W > 0\}}{P\{W > 0\}} = \frac{P\{W > t\}}{P\{W > 0\}}$$

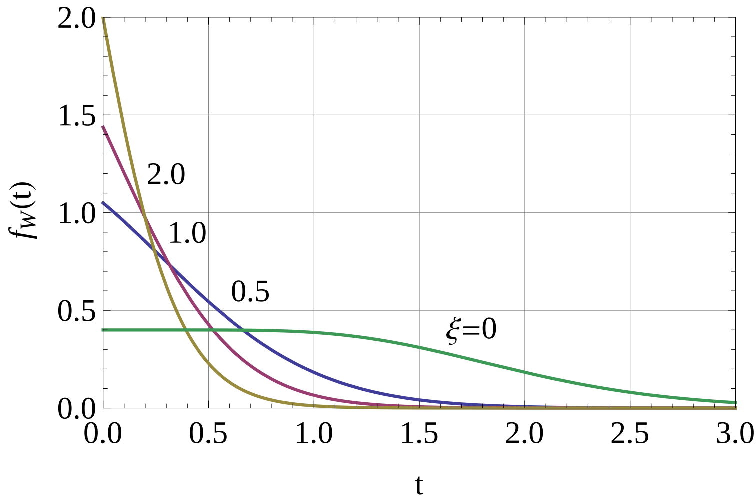

Figure 15: Probability density function for the waiting time $f_W(t)$ ($\rho = 5, \mu = 1$).

$$= \frac{e^{-(m\mu+\theta)t} \displaystyle\sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1 + j\xi/m}}{\displaystyle\sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1 + j\xi/m)}} \qquad t \geq 0.$$

- Distribution function for the waiting time

  For those customers who are accepted, we have the relation

$$P\{W = 0\} + P\{0 < W \leq t\}$$
$$= P\{W \leq t\} = 1 - P\{W > t\} \qquad t > 0.$$

  Thus we get the distribution function for the waiting time

$$P\{0 < W \leq t\} = P\{W > 0\} - P\{W > t\}$$
$$= \hat{P}_0 \frac{\rho^m}{m!} \left[\sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1 + j\xi/m)} \right.$$
$$\left. - e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1 + j\xi/m} \right] \qquad t > 0$$

and the conditional distribution function

$$P\{0 < W \le t\,|\,W > 0\} = 1 - P\{W > t\,|\,W > 0\} \qquad t > 0.$$

- Mean waiting time

  Let us also confirm the mean waiting time. From Eq. (1), the mean waiting time is given by

$$E[W] = \int_0^\infty P\{W > t\}dt$$
$$= \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \frac{1}{m\mu} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{(1 + j\xi/m)[1 + (j+1)\xi/m]}.$$

  Another identity

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{(x+j)(x+j+1)} = (k+1)! \prod_{j=0}^{k+1} \frac{1}{x+j} \qquad k \ge 0$$

  yields

$$E[W] = \frac{\hat{P}_0}{\lambda} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1 + j\xi/m)},$$

  which agrees with the previous result.

  We are interested in the mean remaining waiting time after a customer has waited time t (≥ 0). Since

$$\int_t^\infty x f_W(x)dx = tP\{W > t\} + \int_t^\infty P\{W > x\}dx,$$

  we get the mean remaining waiting time

$$E[W\,|\,W > t] = \frac{\int_t^\infty x f_W(x)dx}{P\{W > t\}} = t + \frac{\int_t^\infty P\{W > x\}dx}{P\{W > t\}}$$

$$= t + \frac{\displaystyle\sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{(1 + j\xi/m)[1 + (j+1)\xi/m]}}{\displaystyle m\mu \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1 + j\xi/m}}.$$

In particular, the mean waiting time of a customer who waits is given by

$$E[W\,|\,W>0]=\frac{E[W]}{P\{W>0\}}=\frac{\dfrac{1}{\lambda}\displaystyle\sum_{k=1}^{c}\frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}}{\displaystyle\sum_{k=0}^{c-1}\frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)}}.$$

For $\xi=0$, we have

$$E[W\,|\,W>t]=t+\frac{\displaystyle\sum_{k=0}^{c-1}\left(\frac{\rho}{m}\right)^k\sum_{j=0}^{k}(k-j+1)\frac{(m\mu t)^j}{j!}}{m\mu\displaystyle\sum_{k=0}^{c-1}\left(\frac{\rho}{m}\right)^k\sum_{j=0}^{k}\frac{(m\mu t)^j}{j!}}.$$

We have the limits

$$\lim_{\rho\to 0}E[W\,|\,W>t]=\frac{1}{\theta+m\mu},\quad \lim_{\rho\to\infty}E[W\,|\,W>t]=\frac{c}{\theta c+\mu m}.$$

In Figure 16, we plot the mean remaining waiting time $E[W\,|\,W>t]-t$ after a customer has waited t time units. It decreases as t increases. In Figure 17, we plot the mean waiting time $E[W\mid W>0]$ of a customer who waits.

- Higher moments of the waiting time

  We can get the second and third moments of W in a similar fashion as follows.

$$E[W^2]=\frac{2\hat{P}_0}{m\lambda\mu}\frac{\rho^m}{m!}\sum_{k=1}^{c}\frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}\sum_{j=1}^{k}\frac{j}{1+j\xi/m},$$

$$E[W^3]=\frac{3\hat{P}_0}{\lambda(m\mu)\theta}\frac{\rho^m}{m!}\sum_{k=1}^{c}\frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}$$
$$\times\left[\sum_{j=1}^{k}\frac{2j}{1+j\xi/m}-\left(\sum_{j=1}^{k}\frac{1}{1+j\xi/m}\right)^2-\sum_{j=1}^{k}\frac{1}{(1+j\xi/m)^2}\right].$$

- Laplace transform

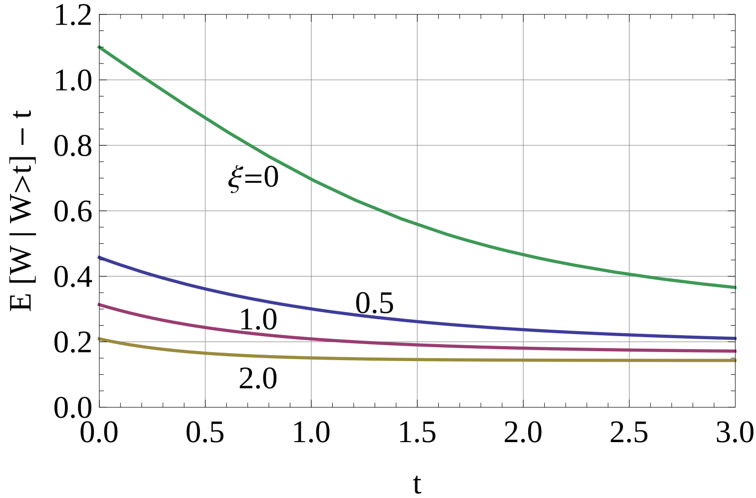

Figure 16: Mean remaining waiting time $E[W \,|\, W > t]{-}t$ ($\rho = 5, \mu = 1$).

The Laplace transform of Eq. (1) is given by

$$\int_0^\infty e^{-st} P\{W > t\} dt$$
$$= \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{(1 + j\xi/m)[s + m\mu + (j+1)\xi\mu]}.$$

After some algebra, we get

$$\int_0^\infty e^{-st} P\{W > t\} dt = \hat{P}_0 \frac{\rho^m}{m!} \frac{1}{s+\theta}$$
$$\times \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1{+}j\xi/m)} \left[1 - \frac{m\mu}{m\mu{+}(k{+}1)\theta} \prod_{j=1}^{k+1} \frac{m\mu{+}j\theta}{s{+}m\mu{+}j\theta} \right]. \tag{2}$$

5. Conditional Waiting Times

Let us derive separately the distribution for the waiting time of an accepted customer who is served and who abandons stays in the waiting room. The simultaneous consideration of the waiting time and the event of service or abandonment

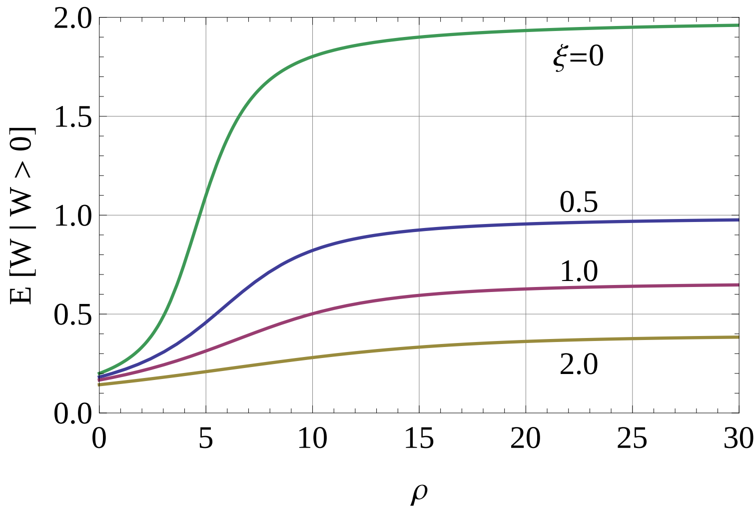

Figure 17: Mean waiting time of a customer who waits $E[W \mid W > 0]$ ($\mu = 1$).

for each call is important in the operations management of call centers. Mandelbaum and Zeltyn [7] propose the following service measures for call centers, given two parameters (large) T and (small) τ:

- $P\{W \le T, \mathrm{Sr}\}$: fraction of well-served,
- $P\{W > T, \mathrm{Sr}\}$:

  fraction of served, with a potential of improvement,
- $P\{W > \tau, \mathrm{Ab}\}$: fraction of poorly-served,
- $P\{W \le \tau, \mathrm{Ab}\}$:

  fraction of those whose service-level is undetermined,

We derive explicit formulas to calculate these measures below.

5.1. State Transition of an Accepted Customer

Following the approach due to Whitt [10], we consider the situation of an accepted customer in which there are k other customers in front of him in the waiting room, where $0 \le k \le c-1$, as shown in Figure 18. The next event

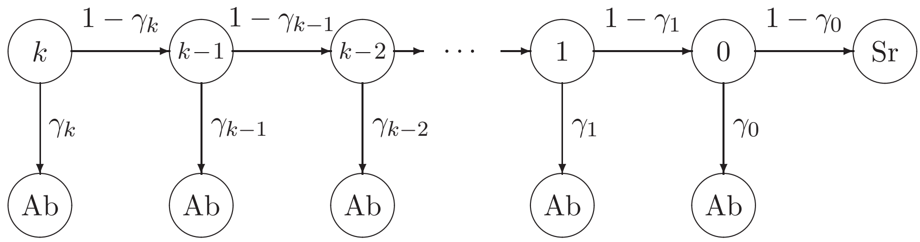

Figure 18: State transition diagram for an accepted customer until either service or abandonment.

that happens to him is either the advancement by one position in the waiting room (either by service completion or by the abandonment of one of k other customers in front of him) with probability $1-\gamma_k$ or his own abandonment with probability γ_k, where

$$\gamma_k := \frac{\theta}{m\mu + (k+1)\theta} \qquad 0 \le k \le c-1.$$

The time until this event occurs is exponentially distributed with mean m_k and variance $(m_k)^2$, where

$$m_k := \frac{1}{m\mu + (k+1)\theta} \qquad 0 \le k \le c-1.$$

The Laplace transform of the pdf for this distribution is given by

$$w_k^*(s) := \frac{m\mu + (k+1)\theta}{s + m\mu + (k+1)\theta} \qquad 0 \le k \le c-1.$$

A sequence of such events occurs until the customer either abandons or gets served eventually, as shown in the diagram in Figure 18. Note that the interevent times in this sequence are mutually independent.

5.2. Waiting Time of a Served Customer

Referring to Figure 18, we see that the probability of successful service for an accepted customer who finds k other customers in front of him in the waiting room is given by

$$P_k\{\mathrm{Sr}\} \;=\; (1-\gamma_k)(1-\gamma_{k-1})\cdots(1-\gamma_0)$$

$$= \prod_{j=0}^{k}(1-\gamma_j) = \frac{m\mu}{m\mu+(k+1)\theta} \qquad 0 \le k \le c-1.$$

Therefore, the probability of successful service for an accepted customer is given by

$$\begin{aligned} P\{\mathrm{Sr}|\mathcal{NB}\} &= \sum_{k=0}^{m-1}\hat{P}_k + \sum_{k=0}^{c-1}\hat{P}_{m+k}P_k\{\mathrm{Sr}\} = \frac{P\{\mathrm{Sr}\}}{1-P\{\mathcal{B}\}} \\ &= \hat{P}_0\left[\sum_{k=0}^{m-1}\frac{\rho^k}{k!} + \frac{\rho^m}{m!}\sum_{k=1}^{c}\frac{(\rho/m)^{k-1}}{\prod_{j=1}^{k}(1+j\xi/m)}\right]. \end{aligned}$$

Let W_k be the waiting time of an accepted customer who has found k others in the waiting room upon arrival. The joint probability of his successful service and the mean waiting time is given by

$$E[W_k,\mathrm{Sr}] = P_k\{\mathrm{Sr}\}\sum_{j=0}^{k} m_j \qquad 0 \le k \le c-1.$$

Noting that the variance for the sum of independent random variables are the sum of variance for each random variable, we get the second moment

$$E[W_k^2,\mathrm{Sr}] = P_k\{\mathrm{Sr}\}\left[\sum_{j=0}^{k}(m_j)^2 + \left(\sum_{j=0}^{k} m_j\right)^2\right] \qquad 0 \le k \le c-1.$$

Therefore, the joint probabilities of successful service and the mean and the second moment of the waiting time for an accepted customer are given by

$$\begin{aligned} E[W,\mathrm{Sr}] &= \sum_{k=0}^{c-1}\hat{P}_{m+k}E[W_k,\mathrm{Sr}] \\ &= \frac{\hat{P}_0}{\lambda}\frac{\rho^m}{m!}\sum_{k=1}^{c}\frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}\sum_{j=1}^{k}\frac{1}{1+j\xi/m} \end{aligned}$$

and

$$E[W^2,\mathrm{Sr}] = \sum_{k=0}^{c-1}\hat{P}_{m+k}E[W_k^2,\mathrm{Sr}]$$

$$= \frac{\hat{P}_0}{m\lambda\mu}\frac{\rho^m}{m!}\sum_{k=1}^{c}\frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}$$
$$\times \left\{\left[\sum_{j=1}^{k}\frac{1}{1+j\xi/m}\right]^2+\sum_{j=1}^{k}\frac{1}{(1+j\xi/m)^2}\right\}.$$

The mean and second moment of the waiting time for a served customer are given by

$$E[W\,|\,\mathrm{Sr}] = \frac{E[W,\mathrm{Sr}]}{P\{\mathrm{Sr}\,|\,\mathcal{NB}\}}, \quad E[W^2\,|\,\mathrm{Sr}] = \frac{E[W^2,\mathrm{Sr}]}{P\{\mathrm{Sr}\,|\,\mathcal{NB}\}}.$$

The joint probability of successful service and the Laplace transform of the pdf of the waiting time for a customer who finds k others in the waiting room upon arrival is given by

$$W_k^*(s,\mathrm{Sr}) = P_k\{\mathrm{Sr}\}\prod_{j=0}^{k} w_j^*(s) = \frac{m\mu}{m\mu+(k+1)\theta}\prod_{j=0}^{k} w_j^*(s),$$

which leads to

$$W_k^*(0,\mathrm{Sr}) = \frac{m\mu}{m\mu+(k+1)\theta} = P_k\{\mathrm{Sr}\} \qquad 0\le k\le c-1.$$

Another expression is given by

$$W_k^*(s,\mathrm{Sr}) = m\mu\int_0^\infty e^{-(s+\theta)t}g_k^k(t)dt$$
$$= \frac{(m/\xi)^k}{k!}\left[\prod_{j=0}^{k}\left(1+\frac{j\xi}{m}\right)\right]\int_0^\infty e^{-(s+\theta+m\mu)t}(1-e^{-\theta t})^k dt.$$

Then the joint probability of successful service and the Laplace transform of the pdf $f_W(t,\mathrm{Sr})$ of the waiting time for an accepted customer is given by

$$W^*(s,\mathrm{Sr}) = \sum_{k=0}^{m-1}\hat{P}_k + \sum_{k=0}^{c-1}\hat{P}_{m+k}W_k^*(s,\mathrm{Sr}).$$

Thus we have

$$W^*(0,\mathrm{Sr}) = \sum_{k=0}^{m-1}\hat{P}_k + \sum_{k=0}^{c-1}\hat{P}_{m+k}W_k^*(0,\mathrm{Sr})$$

$$= \sum_{k=0}^{m-1} \hat{P}_k + \sum_{k=0}^{c-1} \hat{P}_{m+k} P_k\{\mathrm{Sr}\} = P\{\mathrm{Sr}\,|\,\mathcal{NB}\},$$

which is decomposed into

$$\begin{aligned} P\{W = 0\} &= P\{W = 0, \mathrm{Sr}\} = \sum_{k=0}^{m-1} \hat{P}_k = \hat{P}_0 \sum_{k=0}^{m-1} \frac{\rho^k}{k!}, \\ P\{W > 0, \mathrm{Sr}\} &= \sum_{k=0}^{c-1} \hat{P}_{m+k} P_k\{\mathrm{Sr}\} = \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^{k-1}}{\prod_{j=1}^{k}(1 + j\xi/m)}. \end{aligned}$$

The Laplace transform of $f_W(t, \mathrm{Sr})$ is given by

$$\begin{aligned} &\int_0^\infty e^{-st} f_W(t, \mathrm{Sr})dt = \sum_{k=0}^{c-1} \hat{P}_{m+k} W_k^*(s, \mathrm{Sr}) \\ &= \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=1}^{k+1}(1 + j\xi/m)} \prod_{j=0}^{k} w_j^*(s) \\ &= \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^{k-1}}{\prod_{j=1}^{k}[1 + (s + j\theta)/(m\mu)]}. \end{aligned}$$

We can find the explicit expression for $f_W(t, \mathrm{Sr})$ as follows.

$$\begin{aligned} &f_W(t, \mathrm{Sr}) = m\mu e^{-\theta t} \sum_{k=0}^{c-1} \hat{P}_{m+k} g_k^k(t) \\ &= m\mu e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \hat{P}_{m+k} \frac{(m/\xi)^k}{k!} \left[\prod_{j=0}^{k} (1 + j\xi/m) \right] (1 - e^{-\theta t})^k \\ &= \hat{P}_0 \frac{\rho^m}{m!} m\mu e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} (1 - e^{-\theta t})^k \qquad t > 0. \end{aligned}$$

We are now in a position to obtain the joint probability of successful service and the distribution function of the waiting time for an accepted customer:

$$\begin{aligned} P\{W > t, \mathrm{Sr}\} &= \int_t^\infty f_W(x, \mathrm{Sr})dx \qquad t \geq 0, \\ P\{W \leq t, \mathrm{Sr}\} &= P\{W = 0\} + \int_0^t f_W(x, \mathrm{Sr})dx \qquad t \geq 0. \end{aligned}$$

Since

$$\int_t^\infty f_W(x,\mathrm{Sr})dx = \hat{P}_0 \frac{\rho^m}{m!} m\mu \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \int_t^\infty e^{-(m\mu+\theta)x}(1-e^{-\theta x})^k dx$$

and

$$\int_t^\infty e^{-(m\mu+\theta)x}(1-e^{-\theta x})^k dx = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{e^{-[m\mu+(j+1)\theta]t}}{m\mu+(j+1)\theta},$$

we get

$$P\{W>t,\mathrm{Sr}\} = \hat{P}_0 \frac{\rho^m}{m!} e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1+(j+1)\xi/m}.$$

Then we have

$$\begin{aligned} &P\{W=0\} + P\{0<W\le t,\mathrm{Sr}\} \\ &= P\{W\le t,\mathrm{Sr}\} = P\{\mathrm{Sr}\,|\,\mathcal{NB}\} - P\{W>t,\mathrm{Sr}\} \qquad t>0. \end{aligned}$$

The distribution $P\{W>t,\mathrm{Sr}\}$ yields the mean $E[W,\mathrm{Sr}]$ and the second moment $E[W^2,\mathrm{Sr}]$ that agree with the previous results.

The conditional distribution of the waiting time for a served customer is given by

$$P\{W=0\,|\,\mathrm{Sr}\} = \frac{P\{W=0\}}{P\{\mathrm{Sr}\,|\,\mathcal{NB}\}} = \frac{\displaystyle\sum_{k=0}^{m-1} \frac{\rho^k}{k!}}{\displaystyle\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^{k-1}}{\prod_{j=1}^{k}(1+j\xi/m)}}$$

and

$$\begin{aligned} P\{W>t\,|\,\mathrm{Sr}\} &= \frac{P\{W>t,\mathrm{Sr}\}}{P\{\mathrm{Sr}\,|\,\mathcal{NB}\}} \\ &= \frac{\displaystyle\frac{\rho^m}{m!} e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1+(j+1)\xi/m}}{\displaystyle\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^{k-1}}{\prod_{j=1}^{k}(1+j\xi/m)}}. \end{aligned}$$

We also have

$$\begin{aligned} & P\{W = 0 \,|\, \text{Sr}\} + P\{0 < W \leq t \,|\, \text{Sr}\} \\ & = P\{W \leq t \,|\, \text{Sr}\} = 1 - P\{W > t \,|\, \text{Sr}\} \qquad t > 0. \end{aligned}$$

In Figure 19, we plot $P\{W > t \,|\, \text{Sr}\}$ with $\rho = 5, \mu = 1$ for different values of $\xi = 0.5$, 1.0, and 2.0.

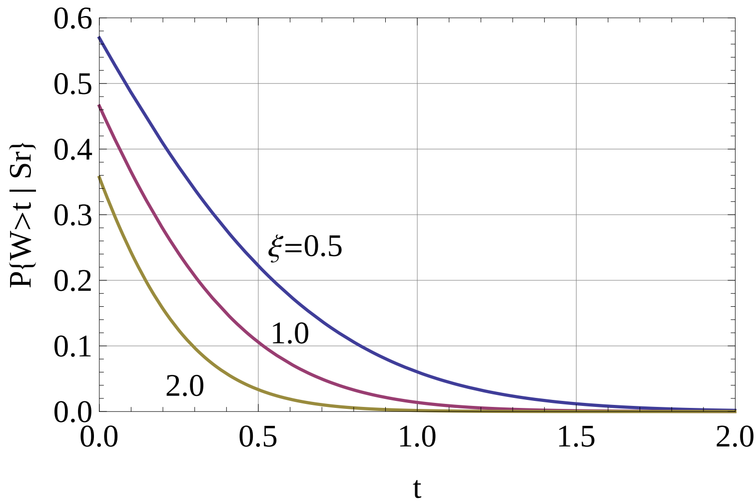

Figure 19: Complementary distribution function for the conditional waiting time $P\{W > t \,|\, \text{Sr}\}$ ($\rho = 5, \mu = 1$).

5.3. Time to Abandon

We can consider the time to abandon that a customer who abandons stays in the waiting room. Referring again to Figure 18, the probability that an accepted customer who finds k other customers in the waiting room upon arrival abandons is given by

$$\begin{aligned} P_0\{\text{Ab}\} &= \gamma_0 = \frac{\theta}{m\mu + \theta} = 1 - P_0\{\text{Sr}\}, \\ P_k\{\text{Ab}\} &= \gamma_k + (1 - \gamma_k)\gamma_{k-1} + (1 - \gamma_k)(1 - \gamma_{k-1})\gamma_{k-2} + \cdots \\ &+ (1 - \gamma_k)(1 - \gamma_{k-1}) \cdots (1 - \gamma_{j+1})\gamma_j + \cdots \\ &+ (1 - \gamma_k)(1 - \gamma_{k-1}) \cdots (1 - \gamma_1)\gamma_0 \end{aligned}$$

$$= \quad \gamma_k + \sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k} (1-\gamma_l) \qquad 1 \le k \le c-1.$$

However we have

$$\prod_{l=j+1}^{k} (1-\gamma_l) = \prod_{l=j+1}^{k} \frac{m\mu + l\theta}{m\mu + (l+1)\theta} = \frac{m\mu + (j+1)\theta}{m\mu + (k+1)\theta} = \frac{\gamma_k}{\gamma_j}.$$

Thus we get

$$\gamma_k + \sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k} (1-\gamma_l) = \gamma_k + k\gamma_k = (k+1)\gamma_k,$$

which leads to

$$P_k\{\mathrm{Ab}\} = \frac{(k+1)\theta}{m\mu + (k+1)\theta} = 1 - P_k\{\mathrm{Sr}\} \qquad 0 \le k \le c-1.$$

It follows that

$$\begin{aligned} P\{\mathrm{Ab}\,|\,\mathcal{NB}\} &= \sum_{k=0}^{c-1} P_k\{\mathrm{Ab}\}\hat{P}_{m+k} \\ &= \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(k+1)\theta}{m\mu + (k+1)\theta} \cdot \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)} \\ &= \frac{\theta}{\lambda} \cdot \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}, \end{aligned}$$

which agrees with the previous result.

Let W_k be the time to abandonment of a customer who finds k others in the waiting room upon arrival. The joint probability of abandonment and the mean time to abandon is given by

$$E[W_k, \mathrm{Ab}] = \begin{cases} \gamma_0 m_0 & k = 0, \\ \gamma_k m_k + \sum_{j=0}^{k-1} \gamma_j \left[\prod_{l=j+1}^{k} (1-\gamma_l) \right] \left[\sum_{l=j}^{k} m_l \right] & 1 \le k \le c-1, \end{cases}$$

which is simplified as

$$
\begin{aligned}
E[W_k, \mathrm{Ab}] &= \gamma_k m_k + \gamma_k \sum_{j=0}^{k-1} \sum_{l=j}^{k} m_l = \gamma_k \sum_{j=0}^{k} \sum_{l=j}^{k} m_l \\
&= \gamma_k \sum_{l=0}^{k} (l+1) m_l \\
&= \frac{\theta}{(m\mu)^2} \cdot \frac{1}{1+(k+1)\xi/m} \sum_{j=1}^{k+1} \frac{j}{1+j\xi/m}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
E[W, \mathrm{Ab}] &= \sum_{k=0}^{c-1} \hat{P}_{m+k} E[W_k, \mathrm{Ab}] \\
&= \frac{\theta \hat{P}_0}{\lambda(m\mu)} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)} \sum_{j=1}^{k} \frac{j}{1+j\xi/m}.
\end{aligned}
$$

We also get the joint probability of abandonment and the second moment of the time to abandon

$$
E[W_k^2, \mathrm{Ab}]
= \begin{cases}
2\gamma_0 (m_0)^2 & k = 0, \\
2\gamma_k (m_k)^2 + \sum_{j=0}^{k-1} \gamma_j \left[\prod_{l=j+1}^{k} (1-\gamma_l) \right] \left[\sum_{l=j}^{k} (m_l)^2 + \left(\sum_{l=j}^{k} m_l \right)^2 \right] & 1 \le k \le c-1,
\end{cases}
$$

which is simplified as

$$
\begin{aligned}
E[W_k^2, \mathrm{Ab}] &= 2\gamma_k (m_k)^2 + \gamma_k \sum_{j=0}^{k-1} \left[\sum_{l=j}^{k} (m_l)^2 + \left(\sum_{l=j}^{k} m_l \right)^2 \right] \\
&= \gamma_k \sum_{j=0}^{k} \left[\sum_{l=j}^{k} (m_l)^2 + \left(\sum_{l=j}^{k} m_l \right)^2 \right] \\
&= \frac{\theta}{(m\mu)^3} \cdot \frac{1}{1+(k+1)\xi/m}
\end{aligned}
$$

$$\times \sum_{j=0}^{k} \left[\sum_{l=j+1}^{k+1} \frac{1}{(1+l\xi/m)^2} + \left(\sum_{l=j+1}^{k+1} \frac{1}{1+l\xi/m} \right)^2 \right].$$

Then we get

$$\begin{aligned} E[W^2, \mathrm{Ab}] &= \sum_{k=0}^{c-1} \hat{P}_{m+k} E[W_k^2, \mathrm{Ab}] \\ &= \frac{\theta \hat{P}_0}{\lambda(m\mu)^2} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)} \\ &\quad \times \sum_{j=0}^{k-1} \left[\sum_{l=j+1}^{k} \frac{1}{(1+l\xi/m)^2} + \left(\sum_{l=j+1}^{k} \frac{1}{1+l\xi/m} \right)^2 \right]. \end{aligned}$$

The joint probability of abandonment and the Laplace transform of the pdf of the time to abandon for a customer who finds k others in the waiting room upon arrival is given by

$$W_k^*(s, \mathrm{Ab}) = \begin{cases} \gamma_0 w_0^*(s) & k = 0, \\ \gamma_k w_k^*(s) + \sum_{j=0}^{k-1} \gamma_j \left[\prod_{l=j+1}^{k} (1-\gamma_l) \right] \left[\prod_{l=j}^{k} w_l^*(s) \right] & 1 \le k \le c-1. \end{cases}$$

This is also simplified as

$$W_k^*(s, \mathrm{Ab}) = \gamma_k \sum_{j=0}^{k} \prod_{l=j}^{k} w_l^*(s) \qquad 0 \le k \le c-1.$$

However, since

$$\begin{aligned} W_k^*(s, \mathrm{Ab}) &= \gamma_k w_k^*(s) + (1-\gamma_k) w_k^*(s) \\ &\quad \times \left[\gamma_{k-1} w_{k-1}^*(s) + \sum_{j=0}^{k-2} \gamma_j \prod_{l=j+1}^{k-1} (1-\gamma_l) \prod_{l=j}^{k-1} w_l^*(s) \right] \end{aligned}$$

we have the recursive relation

$$W_k^*(s, \mathrm{Ab}) = w_k^*(s)[\gamma_k + (1-\gamma_k) W_{k-1}^*(s, \mathrm{Ab})] \qquad 1 \le k \le c-1,$$

which can be shown to be satisfied by

$$\begin{aligned} W_k^*(s, \text{Ab}) &= \frac{\theta}{s+\theta}\left[1 - \frac{m\mu}{m\mu + (k+1)\theta}\prod_{j=0}^{k} w_j^*(s)\right] \\ &= \frac{\theta}{s+\theta}\left[1 - W_k^*(s, \text{Sr})\right] \qquad\qquad 0 \le k \le c-1. \end{aligned}$$

This form indicates that

$$W_k^*(s, \text{Ab}) = \theta \int_0^\infty e^{-(s+\theta)t} dt \int_t^\infty g_k^k(x) m\mu dx \qquad 0 \le k \le c-1.$$

The joint probability of abandonment and the Laplace transform of the pdf $f_W(t, \text{Ab})$ of the waiting time for an accepted customer is given by

$$W^*(s, \text{Ab}) = \int_0^\infty e^{-st} f_W(t, \text{Ab}) dt$$
$$= \hat{P}_0 \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1 + j\xi/m)} \cdot \frac{\theta}{s+\theta}\left[1 - \frac{m\mu}{m\mu + (k+1)\theta}\prod_{j=0}^{k} w_j^*(s)\right].$$

Therefore, we get

$$\begin{aligned} f_W(t, \text{Ab}) &= \theta e^{-\theta t} m\mu \sum_{k=0}^{c-1} \hat{P}_{m+k} \int_t^\infty g_k^k(x) dx = \theta P\{W > t\} \\ &= \theta \hat{P}_0 \frac{\rho^m}{m!} e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1 + j\xi/m}. \end{aligned}$$

Then we obtain the joint probability of abandonment and the distribution function of the waiting time for a customer who abandons

$$P\{W > t, \text{Ab}\} = \int_t^\infty f_W(x, \text{Ab}) dx = \theta \int_t^\infty P\{W > x\} dx$$
$$= \frac{\theta \hat{P}_0}{m\mu} \frac{\rho^m}{m!} e^{-(m\mu+\theta)t} \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{(1 + j\xi/m)[1 + (j+1)\xi/m]}.$$

Note that we have

$$P\{\text{Ab} | \mathcal{NB}\} \;=\; P\{W > 0, \text{Ab}\} = \int_0^\infty f_W(t, \text{Ab}) dt$$

$$
\begin{aligned}
&= \theta \int_0^\infty P\{W > t\} dt = \theta E[W] \\
&= \frac{\theta \hat{P}_0}{\lambda} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1 + j\xi/m)}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&P\{0 < W \leq t, \mathrm{Ab}\} = \int_0^t f_W(x; \mathrm{Ab}) dx \\
&= P\{W \leq t, \mathrm{Ab}\} = P\{\mathrm{Ab} | \mathcal{NB}\} - P\{W > t, \mathrm{Ab}\} \qquad t > 0,
\end{aligned}
$$

where $P\{W = 0, \mathrm{Ab}\} = 0$ because no customers abandon immediately after acceptance.

The joint probability of abandonment and the mean time to abandon is given by

$$
\begin{aligned}
E[W, \mathrm{Ab}] &= \int_0^\infty P\{W > t, \mathrm{Ab}\} dt = \int_0^\infty t f_W(t, \mathrm{Ab}) dt \\
&= \theta \int_0^\infty t P\{W > t\} dt = \frac{\theta}{2} E[W^2] \\
&= \frac{\theta \hat{P}_0}{\lambda(m\mu)} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1 + j\xi/m)} \sum_{j=1}^{k} \frac{j}{1 + j\xi/m}.
\end{aligned}
$$

The joint probability of abandonment and the second moment of the time to abandonment is given by

$$
\begin{aligned}
&E[W^2, \mathrm{Ab}] = \int_0^\infty (2t)\{W > t, \mathrm{Ab}\} dt = \int_0^\infty t^2 f_W(t, \mathrm{Ab}) dt \\
&= \theta \int_0^\infty t^2 P\{W > t\} dt = \frac{\theta}{3} E[W^3] \\
&= \frac{\hat{P}_0}{\lambda(m\mu)} \frac{\rho^m}{m!} \sum_{k=1}^{c} \frac{(\rho/m)^k}{\prod_{j=1}^{k}(1 + j\xi/m)} \\
&\times \left[\sum_{j=1}^{k} \frac{2j}{1 + j\xi/m} - \left(\sum_{j=1}^{k} \frac{1}{1 + j\xi/m} \right)^2 - \sum_{j=1}^{k} \frac{1}{(1 + j\xi/m)^2} \right],
\end{aligned}
$$

which can be shown to agree with the previous result by using the identity

$$
x \sum_{j=0}^{k-1} \left[\sum_{l=j+1}^{k} \frac{1}{(1 + lx)^2} + \left(\sum_{l=j+1}^{k} \frac{1}{1 + lx} \right)^2 \right]
$$

$$= \sum_{j=1}^{k} \frac{2j}{1+jx} - \left(\sum_{j=1}^{k} \frac{1}{1+jx} \right)^2 - \sum_{j=1}^{k} \frac{1}{(1+jx)^2} \qquad k \geq 1.$$

The mean and second moment of the time to abandon for a customer who abandons are given by

$$E[W \,|\, \text{Ab}] = \frac{E[W, \text{Ab}]}{P\{\text{Ab}|\mathcal{NB}\}}, \quad E[W^2 \,|\, \text{Ab}] = \frac{E[W^2, \text{Ab}]}{P\{\text{Ab}|\mathcal{NB}\}}.$$

The conditional distribution of the time to abandon is given by

$$P\{W > t \,|\, \text{Ab}\} = \frac{P\{W > t, \text{Ab}\}}{P\{\text{Ab}|\mathcal{NB}\}}$$

$$= \frac{\dfrac{\lambda}{m\mu} e^{-(m\mu+\theta)t} \displaystyle\sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{(1+j\xi/m)[1+(j+1)\xi/m]}}{\displaystyle\sum_{k=1}^{c} \frac{k(\rho/m)^k}{\prod_{j=1}^{k}(1+j\xi/m)}}$$

and

$$P\{0 < W \leq t \,|\, \text{Ab}\} = 1 - P\{W > t \,|\, \text{Ab}\} \qquad t > 0.$$

In Figure 20, we plot $P\{W > t \,|\, \text{Ab}\}$ with $\rho = 5, \mu = 1$ for different values of $\xi = 0.5$, 1.0, and 2.0.

By now we have derived the four service measures for call centers mentioned at the beginning of this chapter.

5.4. Time-Dependent Probabilities of Service and Abandonment

By unconditioning on the successful service and abandonment gives the unconditional distribution of the waiting time for all accepted customers as follows:

$$\begin{aligned}
1 &= P\{\text{Sr} \mid \mathcal{NB}\} + P\{\text{Ab} \mid \mathcal{NB}\}, \\
E[W] &= E[W, \text{Sr}] + E[W, \text{Ab}], \\
E[W^2] &= E[W^2, \text{Sr}] + E[W^2, \text{Ab}], \\
f_W(t) &= f_W(t, \text{Sr}) + f_W(t, \text{Ab}), \\
W^*(s) &= W^*(s, \text{Sr}) + W^*(s, \text{Ab}), \\
P\{W > t\} &= P\{W > t, \text{Sr}\} + P\{W > t, \text{Ab}\} \qquad t \geq 0,
\end{aligned}$$

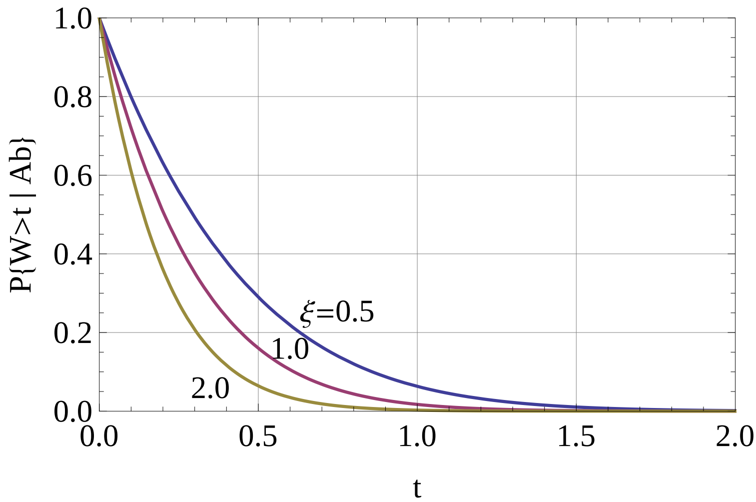

Figure 20: Complementary distribution function for the time to abandon $P\{W > t | \text{Ab}\}$ ($\rho = 5, \mu = 1$).

and

$$P\{0 < W \le t\} = P\{0 < W \le t, \text{Sr}\} + P\{0 < W \le t, \text{Ab}\} \quad t > 0.$$

We may also consider the Laplace transform of the pdf for the time interval from the instant at which there are k other customers in front of a customer in the waiting room to the instant at which either his position advances by one in the waiting room or he abandons:

$$\begin{aligned} & W_k^*(s) = W_k^*(s, \text{Sr}) + W_k^*(s, \text{Ab}) \\ & = \frac{\theta}{s+\theta} + \frac{s}{s+\theta} W_k^*(s, \text{Sr}) \\ & = \frac{1}{s+\theta}\left[\theta + \frac{m\mu s}{m\mu + (k+1)\theta} \prod_{j=0}^{k} w_j^*(s)\right] \qquad 0 \le k \le c-1. \end{aligned}$$

We then get

$$\begin{aligned} & \int_0^\infty e^{-st} P\{W > t \mid \hat{N} = m+k\} dt = \frac{1 - W_k^*(s)}{s} \\ & = \frac{1}{s+\theta}\left[1 - \frac{m\mu}{m\mu + (k+1)\theta} \prod_{j=0}^{0} w_j^*(s)\right] = \frac{1}{\theta} W_k^*(s, \text{Ab}). \end{aligned}$$

By unconditioning, we obtain

$$\int_0^\infty e^{-st} P\{W > t\} dt = \frac{1}{\theta} W^*(s, \mathrm{Ab})$$
$$= \frac{\hat{P}_0}{s+\theta} \frac{\rho^m}{m!} \sum_{k=0}^{c-1} \frac{(\rho/m)^k}{\prod_{j=0}^{k}(1+j\xi/m)} \left[1 - \frac{m\mu}{m\mu + (k+1)\theta} \prod_{j=0}^{k} w_j^*(s) \right],$$

which agrees with Eq. (2).

Finally, by conditioning on the waiting time, we get the probability of successful service after having waited more than t

$$P\{\mathrm{Sr} | W > t\} = \frac{P\{W > t, \mathrm{Sr}\}}{P\{W > t\}}$$
$$= \frac{\sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1+(j+1)\xi/m}}{\sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1+j\xi/m}} \qquad t \geq 0.$$

On the other hand, the probability of abandonment after having waited more than t is given by

$$P\{\mathrm{Ab} | W > t\} = \frac{P\{W > t, \mathrm{Ab}\}}{P\{W > t\}}$$
$$= \frac{\theta \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{(1+j\xi/m)[1+(j+1)\xi/m]}}{m\mu \sum_{k=0}^{c-1} \frac{(\lambda/\theta)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j e^{-j\theta t}}{1+j\xi/m}} \qquad t \geq 0$$

so that

$$P\{\mathrm{Sr} | W > t\} + P\{\mathrm{Ab} | W > t\} = 1.$$

In Figure 21, we plot $P\{\mathrm{Sr} \,|\, W > t\}$ with $\rho = 5, \mu = 1$ for different values of $\xi = 0.5$, 1.0, and 2.0. This probability increases slowly as t increases, which means that $P\{\mathrm{Ab} | W > t\}$ decreases slowly as t increases.

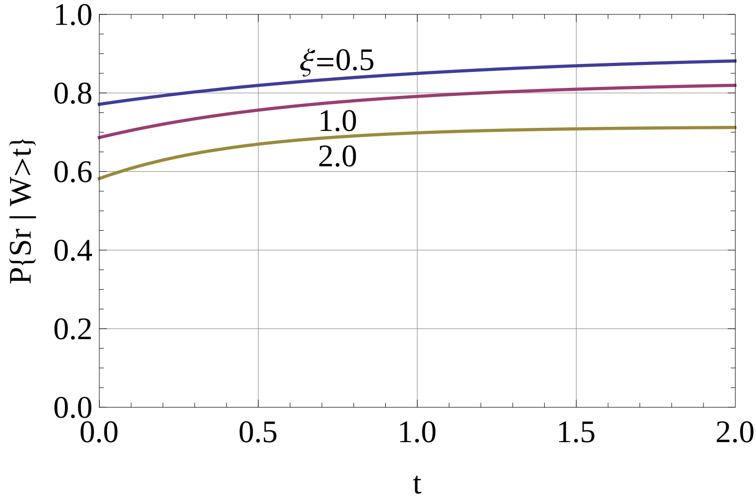

Figure 21: Probability of successful service after having waited more than t ($\rho = 5, \mu = 1$).

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