

MULTILINEAR STIELTJES INTEGRALS OF FUNCTIONS OF SEVERAL VARIABLES

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Abstract: In this paper we define and consider multilinear Stieltjes integrals for vector-valued functions defined on a closed bounded interval in \mathbb{R}^n . The convergence of the Stieltjes sums is considered in the Riemann, Moore-Pollard and Henstock-Kurzweil sense.

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1. Introduction

Multilinear integrals are Stieltjes-type integrals in which the number of integrands or the number of integrators (or both) can be greater than one, e.g. $\int_B \mathcal{A}(f_1, dg_1, f_2)$ or $\int_B \mathcal{A}(f_1, dg_1, f_2, dg_2)$. Multilinear integrals for vector-valued functions of one real variable have already been defined and studied by A Halilović, see for ex. [2]-[4]. In this article we expand the definition of the multilinear Stieltjes integral to n -dimensional space. The integrands

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$f_k : B \rightarrow X_k$, $k = 1, \dots, p$ and the integrators $g_l : B \rightarrow Y_l$, $l = 1, \dots, q$ are functions defined on a closed bounded interval $B \subset \mathbb{R}^n$ with values in linear normed spaces X_k , $k = 1, \dots, p$, Y_l , $l = 1, \dots, q$. To simplify the notation we consider multilinear integrals in the case when the integrands f_k are placed in the beginning and integrators g_k at the end of the expression $\mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$, but all the results are valid for any order of integrands and integrators. First we define Stieltjes quasi volume and then, using a bounded multilinear operator $\mathcal{A} : X_1 \times \dots \times X_p \times Y_1 \times \dots \times Y_q \rightarrow Z$, where Z is a Banach space, we define the multilinear integral

$$\int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$$

as the limit of Stieltjes sums. Stieltjes sums and the limiting process can be defined in different ways and this leads to different types of integrals. In this article we consider Riemann, Moore-Pollard and Henstock-Kurzweil multilinear integrals and give some basic properties of these integrals. Two existence theorems are given:

In the first, referred to here as Theorem 4.1, we prove the existence of the Riemann-Stieltjes multilinear integral assuming that:

1. The integrands f_k are continuous functions.
2. Stieltjes quasi volumes are of bounded semivariations.
3. The expression $\mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C), \dots, \sigma_{g_q}(C)]$ is interval additive.

In the second, referred to here as Theorem 5.1, we prove that the existence of the Moore-Pollard multilinear integral together with the interval additivity of $\mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C), \dots, \sigma_{g_q}(C)]$ implies the existence of the Henstock-Kurzweil multilinear integral.

2. Definitions and Terminology

We begin with some definitions and terminology that are used throughout this article.

Definition 2.1. By an *n-dimensional block* (or *n-dimensional closed interval*), or just *block*, we mean a set of points

$$B = \{(x_1, \dots, x_n) : a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n\},$$

where $a_1 \leq b_1, \dots, a_n \leq b_n$. We shall also occasionally write $B = [a, b]$ where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$.

The volume of B is

$$v(B) = \prod_{i=1}^n (b_i - a_i)$$

Any point $\gamma = (c_1, \dots, c_n)$, where $c_k = a_n$ or $c_k = b_n$, is called a *vertex* of B . Note that B has 2^n vertices.

Any set

$$\Lambda_k = \{(x_1, \dots, x_n) : a_1 \leq x_1 \leq b_1, \dots, x_k = c_k, \dots, a_n \leq x_n \leq b_n\},$$

where $c_k = a_k$ or $c_k = b_k$ will be called a *face* of the block B .

The interior of B we denote by $int(B)$.

The number $|B| = \max\{b_1 - a_1, \dots, b_n - a_n\}$ will be called the *size* of the block B .

A block B' is a *subblock* of a block B if $B' \subseteq B$.

Definition 2.2. We say that $\Pi = \{B_1, \dots, B_r\}$, consisting of a finite family of subblocks of B , forms a *partition* of B if $B = \cup B_k, k = 1, \dots, r$ and $i \neq j \Rightarrow int(B_i) \cap int(B_j) = \emptyset$.

The number

$$|\Pi| = \max\{|B_k|\}$$

is called the *norm* (or *mesh*) of the partition Π .

Definition 2.3. If a partition Π' is obtained by further subdividing the blocks of the partition Π we say that Π' is a *refinement* of Π .

Definition 2.4. (*Stieltjes quasi volume.*) Let $B = [a, b]$ with $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Let $g : B \rightarrow X$ be a vector-valued function defined on B with values in a Banach space X . Let $\gamma = (c_1, \dots, c_n)$ where $c_k = a_n$ or $c_k = b_n$ and $\nu(\gamma)$ denotes the number of terms in γ for which $c_k = a_n$. We define, in a common way (see for example [7] or [6]) a *Stieltjes quasi volume* $\sigma_g(B)$ of the block B associated to the function g by

$$\sigma_g(B) = \sum_{\gamma} (-1)^{\nu(\gamma)} g(\gamma)$$

where the summation is over all vertices $\gamma = (c_1, \dots, c_n)$ where $c_k = a_n$ or $c_k = b_n$, of the block B .

For example, when $n = 3$,

$$\begin{aligned} \sigma_g(B) &= g(b_1, b_2, b_3) + g(a_1, a_2, b_3) + g(a_1, b_2, a_3) + g(b_1, a_2, a_3) \\ &\quad - g(a_1, b_2, b_3) - g(b_1, a_2, b_3) - g(b_1, b_2, a_3) - g(a_1, a_2, a_3) \end{aligned}$$

Note that Stieltjes quasi volume is a generalization of the ordinary volume $v(B)$ of a block B . For example, we can compare $\sigma_g(B)$ in the above example with the volume of B :

$$v(B) = \prod_{i=1}^3 (b_i - a_i)$$

$$= b_1 b_2 b_3 + a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 b_2 a_3 - a_1 a_2 a_3.$$

Definition 2.5. A vector-valued function of blocks $F(C)$ is *interval-additive* if for any partition of C consisting of the blocks C_k , $k = 1, \dots, r$ we have $F(C) = \sum_{k=1}^r F(C_k)$.

Remark 2.1. Using the operator

$$\Delta_{k a_k}^{b_k} g(x_1, \dots, x_n) = g(x_1, \dots, b_k, \dots, x_n) - g(x_1, \dots, a_k, \dots, x_n),$$

we can express Stieltjes quasi volume $\sigma_g(B)$ of the block B associated to the function g as

$$\sigma_g(B) = \Delta_{n a_n}^{b_n} \cdots \Delta_{2 a_2}^{b_2} \Delta_{1 a_1}^{b_1} g(x_1, \dots, x_n).$$

For example, in two dimensions

$$\begin{aligned} \sigma_g(B) &= \Delta_{2 a_2}^{b_2} \Delta_{1 a_1}^{b_1} g(x_1, x_2) = \Delta_{2 a_2}^{b_2} [g(b_1, x_2) - g(a_1, x_2)] \\ &= g(b_1, b_2) - g(b_1, a_2) - g(a_1, b_2) + g(a_1, a_2). \end{aligned}$$

Then it is easy to prove that $\sigma_g(B)$ is an interval-additive function.

Remark 2.2. The following properties follow from the definition of the Stieltjes quasi volume:

1. If B is a degenerated interval, that is $a_k = b_k$ for some k , then $\sigma_g(B) = 0$.
2. If $h = f + g$ then $\sigma_h(B) = \sigma_f(B) + \sigma_g(B)$.
3. If $h = kg$, where k is a real or complex number, then $\sigma_h(B) = k\sigma_g(B)$.

3. Riemann-Stieltjes Multilinear Integral in \mathbb{R}^n

Stieltjes sums

Let $X_k, k = 1, \dots, p, Y_l, l = 1, \dots, q$ and Z be linear normed spaces over the same field \mathbb{R} or \mathbb{C} . Let Z be a Banach space. Let

$$\mathcal{A} : X_1 \times \dots \times X_p \times Y_1 \dots \times Y_q \rightarrow Z$$

be a bounded multilinear operator and let

$$f_k : B \rightarrow X_k, k = 1, \dots, p,$$

$$g_l : B \rightarrow Y_l, l = 1, \dots, q$$

be vector-valued functions defined on a block B .

We define two types of Stieltjes sums, I and II, which differ in number of associated points in each subblock.

Type I. Let $\Pi = \{B_1, \dots, B_r\}$ be a partition of B . We arbitrarily choose p points s_1^i, \dots, s_p^i in each block B_i which we substitute into f_1, \dots, f_p , and form the Stieltjes sum of the first type

$$S_I(\Pi) = \sum_{i=1}^r \mathcal{A}[f_1(s_1^i), \dots, f_p(s_p^i), \sigma_{g_1}(B_i), \dots, \sigma_{g_q}(B_i)].$$

Type II. We choose one point s_i in each block B_i and form the Stieltjes sum of the second type

$$S_{II}(\Pi) = \sum_{i=1}^r \mathcal{A}[f_1(s_i), \dots, f_p(s_i), \sigma_{g_1}(B_i), \dots, \sigma_{g_q}(B_i)].$$

In the above sums $\sigma_{g_k}(B_i)$ denotes quasi volume of the block B_i associated to the function g_k .

Definition 3.1. We say that the type I multilinear Riemann-Stieltjes integral of f_1, \dots, f_p with respect to \mathcal{A} and g_1, \dots, g_q , exists on B if there exists a vector $J_I \in Z$ with the following property:

For every $\epsilon > 0$, there is a constant $\delta > 0$, such that for any partition $\Pi = \{B_1, \dots, B_r\}$ of the block B , and any choice of points s_1^i, \dots, s_p^i in $B_i, i = 1, \dots, r$ we have

$$|\Pi| < \delta \Rightarrow |J_I - S_I(\Pi)| < \epsilon$$

Then we write

$$J_I = (RS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q).$$

We shall also occasionally write

$$J_I = (RS_I) \int_B \mathcal{A}(F, dG)$$

and here F and dG stand for f_1, \dots, f_p and dg_1, \dots, dg_q respectively but this has the exact same meaning as the previous expression for J_I . Thus

$$(RS_I) \int_B \mathcal{A}(F, dG)$$

is not a bilinear integral; it is merely a less cumbersome notation for the multilinear integral.

Definition 3.2. By changing type I Stieltjes sums to type II sums in the above definition we get the definition of type II multilinear Riemann-Stieltjes integral

$$J_{II} = (RS_{II}) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q).$$

Remark 3.1. It is obvious that the existence of the type I integral implies the existence of the type II integral.

Remark 3.2. If we do not mention the type of a Stieltjes integral then, in this article, we mean that the integral we consider is of type I.

Using the above definitions and the fact that Z is a Banach space it is easy to prove the following lemmas.

Lemma 3.1. *If the integral $(RS_I) \int_B \mathcal{A}(F, dG)$ exists then it is uniquely determined.*

Lemma 3.2. *(Cauchy condition I) The integral*

$$(RS_I) \int_B \mathcal{A}(F, dG)$$

exists if and only if for every $\epsilon > 0$ there exists a constant $\delta > 0$, such that for any two partitions Π_1 and Π_2 satisfying $|\Pi_1| < \delta$ and $|\Pi_2| < \delta$ we have

$$|S_I(\Pi_1) - S_I(\Pi_2)| < \epsilon.$$

Lemma 3.3. (Cauchy condition II) *The integral*

$$(RS_I) \int_B \mathcal{A}(F, dG)$$

exists if and only if for every $\epsilon > 0$ there exists a constant $\delta > 0$, such that for any partition Π_1 satisfying $|\Pi_1| < \delta$ and its (arbitrary) refinement Π_2 we have

$$|S_I(\Pi_1) - S_I(\Pi_2)| < \epsilon.$$

4. Henstock-Kurzweil-Stieltjes Multilinear Integrals

The difference between Riemann-type integration and Henstock-Kurzweil (HK) integration is that δ in the definition of HK-integrals is not necessarily a constant but a positive function defined on B , that is

$$\delta : B \rightarrow \mathbb{R}_+ \quad (\text{where } \mathbb{R}_+ = (0, \infty)).$$

The function δ will be called a *gauge* on B . The basic concept in the theory of the HK integral is that of a *tagged δ -fine partition*; see [1], [5] or [6]. We begin by defining two types of tagged partitions which differ in number of points in each subinterval: type I with p associated points and type II with only one point associated to each subinterval.

Definition 4.1. Let B be an n -dimensional block. Let $\Pi = \{B_1, \dots, B_r\}$ be a partition of B , and let

$$D_I = \{(\mathfrak{S}_i, B_i), i = 1, \dots, r\}$$

denote the collection of blocks of Π together with the associated points $\mathfrak{S}_i = \{s_j^i\}$, where $s_j^i \in B_i$, $j = 1, \dots, p$. We say that D_I is a *tagged partition* of type I of the block B and refer to the points s_j^i as the *tags* of D_I .

Remark 4.1. A type I tagged partition has the following properties:

1. A tagged partition D_I has p tags associated to the block B_i .
2. Some (or all) tags s_j^i , $j = 1, \dots, p$ can be equal.
3. Some (or all) tags s_j^i , $j = 1, \dots, p$ can belong to the faces of the block B_i .

Definition 4.2. A tagged partition which has (only) one point s_i associated to each block B_i will be denoted D_{II} . Thus

$$D_{II} = \{(s_i, B_i), i = 1, \dots, r\}.$$

D_{II} will be called a *tagged partition of type II*.

We now define δ -fine partition.

Definition 4.3. Let δ be a gauge, that is $\delta : B \rightarrow \mathbb{R}_+$. We say that a tagged partition D_I of the block B is δ -fine if for every i

$$|B_i| < \min(\delta(s_j^i), j = 1, \dots, p).$$

In other words, for each block B_i we have

$$s_j^i \in B_i \text{ and } |B_i| < \delta(s_j^i), \text{ for all } j = 1, \dots, p.$$

Changing s_j^i to s_i in the above definition we get the δ -fine partition of type II. Thus,

$$D_{II} = \{(s_i, B_i), i = 1, \dots, r\}$$

is δ -fine if

$$|B_i| < \delta(s_i), i = 1, \dots, r.$$

When the tagged partition D (type I or II) is δ -fine we say that D is *subordinate* to δ and we then write $D \ll \delta$.

Note that only tagged partitions can be δ -fine. Therefore we can (and will from now on) omit the word "tagged" in the expression " δ -fine tagged partition" and simply write " δ -fine partition".

Remark 4.2. Let $D_{II} = \{(s_i, B_i), i = 1, \dots, r\}$ be a δ -fine partition of type II. Then simply taking $s_1^i = s_2^i = \dots = s_p^i = s_i$ we get a δ -fine partition of type I, $D_I = \{(\{s_j^i\}, B_i), i = 1, \dots, r\}$.

The following lemma shows the existence of δ -fine partitions for any given positive function (gauge) $\delta : B \rightarrow \mathbb{R}_+$.

Lemma 4.1. *Let B be an n -dimensional block. For any gauge δ on B there exists at least one δ -fine partition D_I of B .*

Proof. A simple proof of the existence of a type II δ -fine partition can be found in [5], page 42. In other words, there exists at least one δ -fine partition

$$D_{II} = \{(s_i, B_i), i = 1, \dots, r\}$$

such that

$$|B_i| < \delta(s_i), \quad i = 1, \dots, r.$$

Now, putting $s_j^i = s_i, j = 1, \dots, p$ we define a type I partition

$$D_I = \{(\{s_j^i\}, B_i), j = 1, \dots, p \quad i = 1, \dots, r\}$$

which satisfies

$$s_j^i \in B_i \quad \text{and} \quad |B_i| < \delta(s_j^i), \quad \text{for all } j = 1, \dots, p.$$

Thus, there exists at least one δ -fine partition of type I. □

HKS-integral, type I

The associated Stieltjes sum of the first type we define in the same way as in the definition of the RS-integral,

$$S_I(D_I) = \sum_{i=1}^r \mathcal{A}[f_1(s_1^i), \dots, f_p(s_p^i), \sigma_{g_1}(B_i), \dots, \sigma_{g_q}(B_i)],$$

and now Lemma 4.1 makes the following definition possible.

Definition 4.4. (Henstock-Kurzweil-Stieltjes multilinear integral) We say that the type I multilinear Stieltjes integral of f_1, \dots, f_p with respect to \mathcal{A} and g_1, \dots, g_q , exists in the *Henstock-Kurzweil* sense on B , if there exists a vector $J_I \in Z$ with the following property:

For every $\epsilon > 0$, there is a gauge δ on B such that for any δ -fine partition D_I we have

$$|J_I - S_I(D_I)| < \epsilon$$

We then write

$$J_I = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$$

and occasionally we write simply $J_I = (HKS_I) \int_B \mathcal{A}(F, dG)$.

Definition 4.5. By changing type I Stieltjes sums to type II sums in the above definition we get the definition of type II multilinear Henstock-Kurzweil-Stieltjes integral

$$J_{II} = (HKS_{II}) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q).$$

Lemma 4.2. *Let $D_I^{(1)}$ and $D_I^{(2)}$ be any two partitions of B and assume that $D_I^{(1)}$ is δ_1 -fine and that $D_I^{(2)}$ is δ_2 -fine. Then there exists a tagged partition $D_I^{(3)}$ of B with the following properties:*

1. $D_I^{(3)}$ is both δ_1 -fine and δ_2 -fine
2. The set of tags of $D_I^{(3)}$ contains all tags of $D_I^{(1)}$ and $D_I^{(2)}$.

Proof. Let $D_I^{(1)} = \{(\{s_j^i\}, B_i)\}$ and $D_I^{(2)} = \{(\{t_l^k\}, C_k)\}$ be any two partitions of B which are δ_1 -fine and δ_2 -fine respectively. Let $E = \{s_j^i\} \cup \{t_l^k\} = \{Y_1, Y_2, \dots, Y_N\}$ denote the finite set of all different tags in both $D_I^{(1)}$ and $D_I^{(2)}$. Let $d(P, Q)$ denote the (non-euclidian) distance

$$d(P, Q) = \max\{|p_k - q_k|, k = 1, \dots, n\}$$

between points $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$. Let

$$\delta_3 = \min\{d(Y_k, Y_l), k \neq l\}$$

and

$$d(P, E) = \min\{d(P, Y_k), k = 1, \dots, N\}.$$

We define

$$\delta = \begin{cases} \min\{\delta_1(P), \delta_2(P), \frac{1}{2}\delta_3\} & \text{if } P \in E, \\ \min\{\delta_1(P), \delta_2(P), \frac{1}{2}d(P, E), \frac{1}{2}\delta_3\} & \text{if } P \in B \setminus E. \end{cases}$$

Then we have:

1. The function δ is a gauge on B ; thus, according to Lemma 4.1, there exists a δ -fine partition of B .
2. Since $\delta \leq \delta_1$ and $\delta \leq \delta_2$, any partition that is δ -fine will also be δ_1 -fine and δ_2 -fine.
3. If Y_k belongs to a block B_j of a δ -fine partition then Y_k must be a tag. Thus, any δ -fine partition (type I or II) will contain all tags $\{Y_1, Y_2, \dots, Y_N\}$.

□

Lemma 4.3. *If the integral $(HKS_I) \int_B A(F, dG)$ exists then it is uniquely determined.*

Proof. Assume that both J_I and J'_I in Z fulfill the requirements of Definition 4.4 for an arbitrary $\epsilon > 0$. Denote by δ and δ' the corresponding gauge functions. Then $|J_I - S_I(D_I)| < \epsilon$ for any δ -fine partition D_I and $|J_I - S_I(D_I)| < \epsilon$ for any δ' -fine partition D'_I . Let $\delta'' = \min(\delta, \delta')$ and let D''_I be a δ'' -fine partition. Then D''_I is a partition of B which is both δ -fine and δ' -fine. Therefore $|J_I - S_I(D''_I)| < \epsilon$ and $|J'_I - S_I(D''_I)| < \epsilon$ giving

$$|J_I - J'_I| \leq |J_I - S_I(D''_I)| + |S_I(D''_I) - J'_I| < \epsilon + \epsilon = 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary we conclude that $J_I = J'_I$. □

Lemma 4.4. (Cauchy condition) *The integral*

$$(HKS_I) \int_B \mathcal{A}(F, dG)$$

exists if and only if for every $\epsilon > 0$ there exists a gauge δ on B , such that for any δ -fine partitions D'_I and D''_I we have

$$|S_I(D'_I) - S_I(D''_I)| < \epsilon.$$

Proof. (\Rightarrow) Assume that $(HKS_I) \int_B \mathcal{A}(F, dG)$ exists and has the value $J \in Z$. Let $\epsilon > 0$ be arbitrary and choose a gauge function δ such that whenever a partition D_I of B is δ -fine we have $|S_I(D_I) - J| < \epsilon/2$ regardless of the choice of D_I . For arbitrary partitions $D_I^{(1)}$ and $D_I^{(2)}$ that are δ -fine we then have the estimate

$$|S_I(D'_I) - S_I(D''_I)| \leq |S_I(D'_I) - J| + |J - S_I(D''_I)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

(\Leftarrow) Assume the converse and, for each $n = 1, 2, \dots$, choose δ_n such that for any δ_n -fine partitions D'_I and D''_I we have

$$|S_I(D'_I) - S_I(D''_I)| < \frac{1}{n}. \tag{4.1}$$

We may assume that $\delta_{n+1} \leq \delta_n$; otherwise we replace δ_{n+1} by

$$\eta_{n+1} = \min\{\delta_1, \dots, \delta_{n+1}\}.$$

For each n , choose a δ_n -fine partition $D_I^{(n)}$. Now, if $m > n$ then $\delta_m < \delta_n$ so that both $D_I^{(m)}$ and $D_I^{(n)}$ are δ_n -fine. Therefore, if $m > n$, using (4.1) we have

$$|S_I(D_I^{(n)}) - S_I(D_I^{(m)})| < \frac{1}{n}. \tag{4.2}$$

This shows that the sequence

$$\{S_I(D_I^{(k)}), k = 1, 2, \dots, \}$$

is a Cauchy sequence in the complete space Z and therefore convergent. Let

$$J = \lim_{k \rightarrow \infty} S_I(D_I^{(k)}).$$

Passing to the limit as $m \rightarrow \infty$ in (4.2) we get

$$|S_I(D_I^{(n)}) - J| \leq \frac{1}{n}. \quad (4.3)$$

Given arbitrary $\epsilon > 0$, let N be chosen so that $\frac{1}{N} < \frac{\epsilon}{2}$. Let D_I be an arbitrary δ_N -fine partition. Then, using (4.1) and (4.3), we have

$$\begin{aligned} |S_I(D_I) - J| &\leq |S_I(D_I) - S_I(D_I^{(N)})| + |S_I(D_I^{(N)}) - J| \\ &\leq \frac{1}{N} + \frac{1}{N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $(HKS_I) \int_B \mathcal{A}(F, dG)$ exists and the proof is complete. \square

Here are some properties of the HKS-integral.

Lemma 4.5. a) Let $f_i = f_{i,a} + f_{i,b}$ for some $i, 1 \leq i \leq p$. If the integrals

$$J_a = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_{i,a}, \dots, f_p, dg_1, \dots, dg_q)$$

and

$$J_b = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_{i,b}, \dots, f_p, dg_1, \dots, dg_q)$$

exist then the integral

$$J = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_i, \dots, f_p, dg_1, \dots, dg_q)$$

also exists and $J = J_a + J_b$.

b) Let $g_j = g_{j,a} + g_{j,b}$ for some $j, 1 \leq j \leq q$. If the integrals

$$J_a = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_{j,a}, \dots, dg_q)$$

and

$$J_b = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_{j,b}, \dots, dg_q)$$

exist then the integral

$$J = (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_j, \dots, dg_q)$$

also exists and $J = J_a + J_b$.

c) Let k be a real (or complex) number. If the integral

$$(HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$$

exists then

$c_1)$

$$\begin{aligned} & (HKS_I) \int_B \mathcal{A}(f_1, \dots, kf_i, \dots, f_p, dg_1, \dots, dg_q) \\ &= k \cdot (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_i, \dots, f_p, dg_1, \dots, dg_q) \end{aligned}$$

and

$c_2)$

$$\begin{aligned} & (HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, d(kg_j), \dots, dg_q) \\ &= k \cdot (HKS_I) \int_B \mathcal{A}(f_1, v f_p, dg_1, \dots, dg_j, \dots, dg_q). \end{aligned}$$

d) Let

$$\mathcal{A}_1, \mathcal{A}_2 : X_1 \times \dots \times X_p \times Y_1 \times \dots \times Y_q \rightarrow Z$$

be bounded multilinear operators and let k_1, k_2 be real (or complex) numbers.

If the integrals

$$(HKS_I) \int_B \mathcal{A}_1(F, dG) \quad \text{and} \quad (HKS_I) \int_B \mathcal{A}_2(F, dG)$$

exist then

$$\begin{aligned} & (HKS_I) \int_B (k_1 \mathcal{A}_1 + k_2 \mathcal{A}_2)(F, dG) \\ &= k_1 \cdot (HKS_I) \int_B \mathcal{A}_1(F, dG) + k_2 \cdot (HKS_I) \int_B \mathcal{A}_2(F, dG). \end{aligned}$$

Proof. We prove only *b*). Choose $\epsilon > 0$ arbitrarily and let δ_a and δ_b fulfill $D \ll \delta_a \Rightarrow |S_I^{(a)}(D) - J_a| < \epsilon/2$ and $D \ll \delta_b \Rightarrow |S_I^{(b)}(D) - J_b| < \epsilon/2$ where $S_I^{(a)}(D)$ and $S_I^{(b)}(D)$ are arbitrary Stieltjes sums belonging to the HKS-integrals with $g_{i,a}$ and $g_{i,b}$ as integrators respectively. Set $\delta = \min(\delta_a, \delta_b)$ and let D be any δ -fine partition of B . By the multilinearity of \mathcal{A} and the linearity of the Stieltjes quasi volume, see Remark 2.1, we can write

$$S_I(D) = S_I^{(a)}(D) + S_I^{(b)}(D).$$

Now let $J = J_a + J_b$. We then have

$$\begin{aligned} |S_I(D) - J| &= |(S_I^{(a)}(D) + S_I^{(b)}(D)) - (J_a + J_b)| \leq \\ &|S_I^{(a)}(D) - J_a| + |S_I^{(b)}(D) - J_b| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This means that the integral

$$(HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_i, \dots, dg_q)$$

exists and has the value J . □

Remark 4.3. If we have only one integrator g_1 then the expression

$$\mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C)]$$

preserves the property of interval additivity in the following meaning: If $\{C_1, \dots, C_m\}$ is a partition of a block $C \subseteq B$ and $s_1, \dots, s_p \in C$ then, by the multilinearity of \mathcal{A} and the linearity of the Stieltjes quasi volume, we have:

$$\begin{aligned} \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C)] &= \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sum_{i=1}^m \sigma_{g_1}(C_i)] \\ &= \sum_{i=1}^m \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C_i)]. \end{aligned}$$

But, if we have more than one integrator then the expression

$$\mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C), \dots, \sigma_{g_q}(C)]$$

is not necessarily interval additive. Consider, for example, the case where we have two integrators (and one integrand)

$\mathcal{A}[f_1(s_1), \sigma_{g_1}(C), \sigma_{g_2}(C)]$. Let $\{C_1, C_2\}$ be a partition of a block $C \subseteq B$. We have

$$\begin{aligned} \mathcal{A}[f_1(s_1), \sigma_{g_1}(C), \sigma_{g_2}(C)] &= \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_1 \cup C_2), \sigma_{g_2}(C_1 \cup C_2)] \\ &= \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_1) + \sigma_{g_1}(C_2), \sigma_{g_2}(C_1) + \sigma_{g_2}(C_2)] \\ &= \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_1), \sigma_{g_2}(C_1)] + \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_1), \sigma_{g_2}(C_2)] \\ &\quad + \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_2), \sigma_{g_2}(C_1)] + \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_2), \sigma_{g_2}(C_2)] \end{aligned}$$

Thus, in the general case,

$$\mathcal{A}[f_1(s_1), \sigma_{g_1}(C), \sigma_{g_2}(C)]$$

differs from

$$\mathcal{A}[f_1(s_1), \sigma_{g_1}(C_1), \sigma_{g_2}(C_1)] + \mathcal{A}[f_1(s_1), \sigma_{g_1}(C_2), \sigma_{g_2}(C_2)].$$

Therefore, we must include the interval additivity of the expression $\mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C)]$ as an extra assumption, if we need this property to prove a lemma or a theorem (as in Lemma 4.6, and Theorems 4.1 and 5.1).

Lemma 4.6. *Let $\Pi = \{B_1, \dots, B_r\}$, be a partition of B . Assume that the two following conditions hold:*

1. *(Interval additivity) If $\{C_1, \dots, C_m\}$ is a partition of a block $C \subseteq B$ and $s_1, \dots, s_p \in C$ then*

$$\begin{aligned} &\mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C), \dots, \sigma_{g_q}(C)] \\ &= \sum_{i=1}^m \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C_i), \dots, \sigma_{g_q}(C_i)]. \end{aligned}$$

2. *The integrals $(HKS_I) \int_{B_k} \mathcal{A}(F, dG), k = 1, \dots, r$ exist.*

Then the integral $(HKS_I) \int_B \mathcal{A}(F, dG)$ exists and

$$(HKS_I) \int_B \mathcal{A}(F, dG) = \sum_{k=1}^r (HKS_I) \int_{B_k} \mathcal{A}(F, dG).$$

Proof. For simplicity, we shall prove the case when $B = B_1 \cup B_2$, where B_1 and B_2 are two adjacent blocks with one face, M , in common. We denote $J_1 = (HKS_I) \int_{B_1} \mathcal{A}(F, dG)$ and $J_2 = (HKS_I) \int_{B_2} \mathcal{A}(F, dG)$. Our aim is to prove that $(HKS_I) \int_B \mathcal{A}(F, dG)$ exists and that

$$(HKS_I) \int_B \mathcal{A}(F, dG) = J_1 + J_2.$$

Given $\epsilon > 0$, let δ_1 and δ_2 be defined on B_1 and B_2 such that if the partition D_1 of B_1 is δ_1 -fine, we have $|S_I^{(1)}(D_1) - J_1| < \epsilon/2$ and if the partition D_2 is δ_2 -fine we have $|S_I^{(2)}(D_2) - J_2| < \epsilon/2$. Here $S_I^{(1)}(D_1)$ and $S_I^{(2)}(D_2)$ denote Stieltjes sums for the integrals over B_1 and B_2 respectively. Denote by $d(P, M)$ the distance from a point P to the set M and define the gauge

$$\delta = \begin{cases} \min\{\delta_1(P), \delta_2(P)\} & \text{if } P \in M; \\ \min\{\delta_1(P), \frac{1}{2}d(P, M)\} & \text{if } P \in B_1 \setminus M; \\ \min\{\delta_2(P), \frac{1}{2}d(P, M)\} & \text{if } P \in B_2 \setminus M. \end{cases}$$

Let D be a δ -fine partition of B consisting of subblocks C_k and associated tags. Consider the associated Stieltjes sum $S_I(D)$. First we divide all the subblocks into three groups. The first group consists of blocks denoted here by $C_{1,i}$ which are entirely contained in B_1 , the second group consists of blocks denoted by $C_{2,j}$ which are entirely contained in B_2 and the third group consists of all other subblocks denoted by $C_{3,m}$. Thus, for a subblock $C_{3,m}$, we have $C_{3,m} \cap \text{int}(B_1) \neq \emptyset$ and $C_{3,m} \cap \text{int}(B_2) \neq \emptyset$, where $\text{int}(B_k)$ denotes the interior of B_k . Due to the definition of δ , all associated tags (s_1^m, \dots, s_p^m) of $C_{3,m}$ belong to the common face M .

We now split each subblock $C_{3,m}$ in into $E_m = C_{3,m} \cap B_1$ and $F_m = C_{3,m} \cap B_2$ and associate the same tags (s_1^m, \dots, s_p^m) to both E_m and F_m . By the interval additivity of \mathcal{A} we have

$$\begin{aligned} &\mathcal{A}[f_1(s_1^m), \dots, f_p(s_p^m), \sigma_{g_1}(C_{3,m}), \dots, \sigma_{g_q}(C_{3,m})] \\ &= \mathcal{A}[f_1(s_1^m), \dots, f_p(s_p^m), \sigma_{g_1}(E_m), \dots, \sigma_{g_q}(E_m)] \\ &\quad + \mathcal{A}[f_1(s_1^m), \dots, f_p(s_p^m), \sigma_{g_1}(F_m), \dots, \sigma_{g_q}(F_m)]. \end{aligned}$$

Adding the blocks E_m to the blocks $C_{1,i}$ (together with associated tags) we get a δ_1 -fine partition D' of B_1 . Similarly, adding the blocks F_m to the blocks $C_{2,j}$ we get a δ_2 -fine partition D'' of B_2 . Furthermore,

$$S_I(D) = S_I(D') + S_I(D'')$$

so that we have

$$|S_I(D) - (J_1 + J_2)| \leq |S_I(D') - J_1| + |S_I(D'') - J_2| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus the integral $(HK S_I) \int_B \mathcal{A}(F, dG)$ exists and

$$(HK S_I) \int_B \mathcal{A}(F, dG) = J_1 + J_2.$$

□

Note that in the case when we have only one integrator, the interval additivity condition in Lemma 4.6 is automatically fulfilled (as we mentioned in Remark 4.3).

Definition 4.6. *Semivariation.* Let $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \cdots \times Y_q \rightarrow Z$ be a multilinear operator. We say that Stieltjes quasi volumes $\sigma_{g_1}, \dots, \sigma_{g_q}$ are of *bounded semivariation on B with respect to A* if there exists a constant M_s such that for any partition $\Pi = \{B_1, \dots, B_r\}$ of B and arbitrary vectors $x_j^i \in X_j, i = 1, \dots, r, j = 1, \dots, p$ the following holds

$$\left| \sum_{i=1}^r \mathcal{A}[x_1^i, \dots, x_p^i, \sigma_{g_1}(B_i), \dots, \sigma_{g_q}(B_i)] \right| \leq M_s \cdot G_1 \cdot G_2 \cdots G_p.$$

where

$$G_j = \max\{|x_j^i|, i = 1, \dots, n\}$$

Theorem 4.1. *Let $X_1, \dots, X_p, Y_1, \dots, Y_q$ be normed spaces and let Z be a Banach space over the same field. Let $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \cdots \times Y_q \rightarrow Z$ be a multilinear operator. Assume that the following hold:*

1. *The integrands $f_j : B \rightarrow X_j$ are continuous functions.*
2. *Stieltjes quasi volumes $\sigma_{g_1}, \dots, \sigma_{g_q}$ are of bounded semivariation.*
3. *(Interval additivity) If $\{C_1, \dots, C_m\}$ is a partition of a block $C \subseteq B$ and $s_1, \dots, s_p \in C$ then*

$$\begin{aligned} & \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C), \dots, \sigma_{g_q}(C)] \\ &= \sum_{i=1}^m \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C_i), \dots, \sigma_{g_q}(C_i)] \end{aligned}$$

Then the Riemann-Stieltjes multilinear integral

$$(RS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$$

exists.

Proof. To simplify notation we assume that we only have two integrands f_1 and f_2 . The proof of the general case can be carried out in the same way.

Now we let $\epsilon > 0$ have an arbitrary value. Let $M_1 = \max\{|f_1(s)|, s \in B\}$ and $M_2 = \max\{|f_2(s)|, s \in B\}$. The functions f_1, f_2 are continuous

on B and hence they are uniformly continuous on B . We can therefore choose $\delta > 0$ such that for any partition $\Pi = \{B_k, k = 1, \dots, r\}$ of B with $|\Pi| < \delta$ we have $|f_j(s) - f_j(s')| < \epsilon$ for $j = 1, 2$ and all $s, s' \in B$, if s and s' belong to the same subblock in Π . Let $\Pi' = \{B_{k,j}\}$, where $B_k = \cup B_{k,j}$, be any refinement of Π . We then have

$$|S_I(\Pi) - S_I(\Pi')| = \left| \sum_k \mathcal{A}[f_1(s_1^k), f_2(s_2^k), \sigma_{g_1}(B_k), \dots, \sigma_{g_q}(B_k)] - \sum_{k,j} \mathcal{A}[f_1(s_1^{kj}), f_2(s_2^{kj}), \sigma_{g_1}(B_{k,j}), \dots, \sigma_{g_q}(B_{k,j})] \right|.$$

Using the interval additivity of the expression, this can be rewritten as

$$\begin{aligned} & \left| \sum_{k,j} \mathcal{A}[f_1(s_1^k), f_2(s_2^k), \sigma_{g_1}(B_{k,j}), \dots, \sigma_{g_q}(B_{k,j})] \right. \\ & \left. - \sum_{k,j} \mathcal{A}[f_1(s_1^{kj}), f_2(s_2^{kj}), \sigma_{g_1}(B_{k,j}), \dots, \sigma_{g_q}(B_{k,j})] \right| \\ &= \left| \sum_{k,j} \mathcal{A}(f_1(s_1^k), f_2(s_2^k), \sigma_{kj}) - \mathcal{A}(f_1(s_1^{kj}), f_2(s_2^{kj}), \sigma_{kj}) \right| \end{aligned} \tag{4.4}$$

where we simplified notation by writing σ_{kj} instead of $\sigma_{g_1}(B_{k,j}), \dots, \sigma_{g_q}(B_{k,j})$. If we here add and subtract $\mathcal{A}(f_1(s_1^k), f_2(s_2^{kj}), \sigma_{kj})$ we can use the triangle inequality to estimate 4.4 upwards with $\Sigma_1 + \Sigma_2$ where

$$\Sigma_1 = \left| \sum_{k,j} \mathcal{A}(f_1(s_1^k), f_2(s_2^k), \sigma_{kj}) - \mathcal{A}(f_1(s_1^k), f_2(s_2^{kj}), \sigma_{kj}) \right|$$

and

$$\Sigma_2 = \left| \sum_{k,j} \mathcal{A}(f_1(s_1^k), f_2(s_2^{kj}), \sigma_{kj}) - \mathcal{A}(f_1(s_1^{kj}), f_2(s_2^{kj}), \sigma_{kj}) \right|.$$

By the multilinearity of \mathcal{A} and the bounded semivariability of the Stieltjes quasi volume we then get

$$\Sigma_1 = \left| \sum_{k,j} \mathcal{A}(f_1(s_1^k), f_2(s_2^k) - f_2(s_2^{kj}), \sigma_{kj}) \right| \leq M_s \cdot M_1 \cdot \epsilon. \tag{4.5}$$

For Σ_2 in the same way to obtain the estimate

$$\Sigma_2 \leq M_s \cdot \epsilon \cdot M_2. \tag{4.6}$$

Combining 4.5 and 4.6 we get

$$|S_I(\Pi) - S_I(\Pi')| < M_s \cdot M_1 \cdot \epsilon + M_s \cdot \epsilon \cdot M_2 = \epsilon M_s (M_1 + M_2)$$

This proves the existence of $(RS_I) \int_B \mathcal{A}(f_1, f_2, dg_1, \dots, dg_q)$ by Lemma 3.3. As stated above, an analogous procedure can be employed when dealing with integrals having p integrands. □

5. Moore-Pollard-Stieltjes Multilinear Integrals in \mathbb{R}^n

The difference between Moore-Pollard-Stieltjes (MPS) and Riemann-Stieltjes integrals (RS) is the limiting process. The limit of the Stieltjes sums for the MPS-integral is controlled by refinements while the limit for the RS-integral is defined by the norm of partitions.

Definition 5.1. We say that the type I multilinear Moore-Pollard-Stieltjes integral of $f_i, i = 1, \dots, p$ with respect to \mathcal{A} and $g_j, j = 1, \dots, q$, exists on B if there exists a vector $J_I \in Z$ with the following property:

For every $\epsilon > 0$, there is a partition Π_ϵ , such that for any partition $\Pi = \{B_1, \dots, B_r\}$ of the block B that is a refinement of Π_ϵ , and any choice of points s_1^i, \dots, s_p^i in $B_i, i = 1, \dots, r$ we have

$$|J_I - S_I(\Pi)| < \epsilon.$$

Then we write

$$J_I = (MPS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q).$$

We can also more briefly write

$$J_I = (MPS_I) \int_B \mathcal{A}(F, dG)$$

analogous with earlier notation.

Definition 5.2. By changing type I Stieltjes sums to type II sums in the above definition we get the definition of type II multilinear Moore-Pollard-Stieltjes integral

$$J_{II} = (MPS_{II}) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q).$$

Remark 5.1. It is obvious that the existence of the type I (type II) RS-integral RS_I implies the existence of type I (type II) MPS-integral.

In the next theorem we prove that the existence of the MPS integral, supplied with the interval additivity condition, implies the existence of the HKS integral on the block B .

Theorem 5.1. Let $\Pi = \{B_1, \dots, B_r\}$, be a partition of B . Assume that the following conditions hold:

1. (Interval additivity) If $\{C_1, \dots, C_m\}$ is a partition of a block $C \subseteq B$ and $s_1, \dots, s_p \in C$ then

$$\begin{aligned} & \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C), \dots, \sigma_{g_q}(C)] \\ &= \sum_{i=1}^m \mathcal{A}[f_1(s_1), \dots, f_p(s_p), \sigma_{g_1}(C_i), \dots, \sigma_{g_q}(C_i)]. \end{aligned}$$

2. The integral $(MPS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$ exists and has the value J_I .

Then the integral $(HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$ exists and has the same value.

Proof. We shall prove this theorem first in the the case when B is a block in the space \mathbb{R}^2 . Fix $\epsilon > 0$ and denote $MPS_I \int_B \mathcal{A}(F, dG)$ by J_I . Then there exists a partition Π_ϵ of B such that for any partition Π that is a refinement of Π_ϵ we have

$$|S_I(\Pi) - J_I| < \epsilon. \tag{5.1}$$

We will now construct a gauge δ on B to show that the integral $HKS_I \int_B \mathcal{A}(F, dG)$ also exists with the value J_I . Let $\{B_1, \dots, B_r\}$ be the partition Π_ϵ specified above. We assume further that

$\{B_1, \dots, B_r\}$ are non-degenerated intervals. (We can do that since the Stieltjes volume of a degenerated interval is 0.) Let M be the set of all boundary points and let $V = \{v_1, \dots, v_B\}$ be the set of all different vertices of all the blocks B_1, \dots, B_r in Π_ϵ .

Let $d(P, Q) = \max\{|p_k - q_k|, k = 1, \dots, n\}$ denote the (non-euclidian) distance between points $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$. Define

$$\delta_1 = \min\{d(v_k, v_l), k \neq l\}.$$

Thus, δ_1 is the shortest distance between two (different) vertices and $\delta_1 > 0$. Now we define $\delta(P)$:

$$\delta(P) = \begin{cases} \delta_1/4, & \text{if } P \in V \\ d(P, V)/4 & \text{if } P \in M \setminus V \\ d(P, M)/4 & \text{if } P \in B \setminus M. \end{cases}$$

Here $d(P, M) = \min\{d(P, X), X \in M\}$ denotes the distance between the point P and the closed set M . Similarly, $d(P, V)$ is the distance between P and the set V .

Let $D_I = \{(\{s_j^i\}, C_i)\}$ be a δ -fine partition of B . Using blocks C_i we construct a new partition that is a refinement of Π_ϵ . First we divide all the subblocks into two groups. The first group consists of blocks, denoted here by C_l , which are entirely contained in some B_i . The second group consists of all other subblocks denoted here by C_m . Blocks C_l (from the first group) we keep unchanged. Blocks C_m from the second group (who contain interior points of two or more blocks) we divide in subblocks. From the definition of δ we have the following properties:

1. If a block C_m intersects the set of boundary points M then associated points must be in M .
2. Two different vertices can not belong to the same block C_m .
3. If a vertex v_k belongs to C_m then all associated points s_j^m are equal to v_k , i.e. $s_j^m = v_k, j = 1, \dots, p$.

Let C_m , with associated points s_j^m , be a block that contains interior points of two or more blocks B_i . We split C_m in subblocks $C_{m,k}$, where $C_{m,k} \subseteq B_{i_k}$. The properties 1, 2 and 3 show that we can associate the same points s_j^m to every $C_{m,k}$. For the related term in the Stieltjes sum, using the assumption of interval additivity, we then get

$$\begin{aligned} & \mathcal{A}[f_1(s_1^m), \dots, f_p(s_p^m), \sigma_{g_1}(C_m), \dots, \sigma_{g_q}(C_m)] \\ &= \sum_k \mathcal{A}[f_1(s_1^m), \dots, f_p(s_p^m), \sigma_{g_1}(C_{m,k}), \dots, \sigma_{g_q}(C_{m,k})]. \end{aligned}$$

Let D'_I be a partition consisting of the blocks $C_{m,k}$ and those blocks C_l lying entirely in some B_i . Then

$$S_I(D_I) = S_I(D'_I).$$

Since D'_I is a refinement of Π_ϵ , from (5.1) we have

$$|S_I(D'_I) - J_I| < \epsilon$$

and therefore

$$|S_I(D_I) - J_I| < \epsilon.$$

This means that $(HKS_I) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q)$ exists and has the value J_I , which completes the proof in \mathbb{R}^2 .

Defining δ in a similar way we can prove this theorem in the space \mathbb{R}^n . For example, in the space \mathbb{R}^3 we define

$$\delta(P) = \begin{cases} \delta_1/4, & \text{if } P \in V \\ d(P, V)/4 & \text{if } P \in M_1 \setminus V \\ d(P, M_1)/4 & \text{if } P \in M_2 \setminus V \\ d(P, M_2)/4 & \text{if } P \in B \setminus M_2 \end{cases}$$

where V denotes all different vertices, M_1 points on edges and M_2 boundary points for all blocks $\{B_1, \dots, B_r\}$ of the partition Π_ϵ . Then we repeat the same procedure as in the case of the space \mathbb{R}^2 . \square

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