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A NOTE ON THE HYPERCOMPLEX RIEMANN-CAUCHY LIKE RELATIONS FOR QUATERNIONS AND LAPLACE EQUATIONS

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Abstract: In this Note it is worked out a new set of Laplace-Like equations for quaternions through Riemann-Cauchy hypercomplex relations otained earlier [1]. As in the theory of functions of a complex variable, it is expected that this new set of Laplace-Like equations might be applied to a large number of Physical problems, providing new insights in the Classical Fields Theory.

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1. Cauchy-Riemann Equations (Functions of One Complex Variable)

In order to fix ideas will be considered theorem that relates the partial derivatives for the case of a function $f(z)$ of a complex variable $f(z) = u(x, y)+iv(x, y)$ [2], which here will be called Riemann-Cauchy conditions. These relations say that the first order partial derivatives of functions $u(x, y)$ and $v(x, y)$ satisfy relations according to the following theorem:

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Theorem 1. *Is $f(z) = u(x, y) + iv(x, y)$ a function defined and continues in a neighborhood of the point $z = x + yi$ and differentiable at z. Then the partial derivatives of the first order of $u(x, y)$ and $v(x, y)$ exist and satisfy the relations:*

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \tag{1}$$

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}. \tag{2}$$

Thus, if $f(z)$ is analytic in a domain Γ, its partial derivatives exist and satisfy the set of relations (1) and (2) over all point in *Gamma*. Moreover, with the above functions class C^2 using Schwartz's Theorem for partial derivatives immediately follows the following equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{4}$$

the above equations are called Laplace's equations.

2. Cauchy-Riemann Conditions for Quaternionic Functions

Now we may consider a set of conditions presented as the Riemann-Cauchy like relations for quaternionic functions. It follows the Theorem [1]:

Theorem 2. *For any pair pontis a and b and any path joining them simply conect subdomain of the four-dimmensional space, the integral $\int_a^b f dq$ is independent form the given path if and only if there is a function $F = F_1 + F_2 i + F_3 j + F_4 k$ such that $\int_a^b f dq = F(a) - F(b)$, and satisfying the following relations:*

$$\frac{\partial F_1}{\partial x_1} = \frac{\partial F_2}{\partial x_2} = \frac{\partial F_3}{\partial x_3} = \frac{\partial F_4}{\partial x_4}, \tag{5}$$

$$\frac{\partial F_2}{\partial x_1} = -\frac{\partial F_1}{\partial x_2} = -\frac{\partial F_3}{\partial x_4} = \frac{\partial F_4}{\partial x_3}, \tag{6}$$

$$\frac{\partial F_3}{\partial x_1} = -\frac{\partial F_1}{\partial x_3} = -\frac{\partial F_2}{\partial x_4} = \frac{\partial F_4}{\partial x_2}, \tag{7}$$

$$\frac{\partial F_4}{\partial x_1} = \frac{\partial F_1}{\partial x_4} = -\frac{\partial F_2}{\partial x_3} = -\frac{\partial F_3}{\partial x_2}. \tag{8}$$

Proof. The proof of this theorem can be analyzed in greater detail in [1]. □

3. The Laplace's Equations

In this section we show that a new set of hypercomplex Laplace equations may be generated in four dimensions, through the use of Riemann-Cauchy like relations [1] Therefore, the functions that make up the quaternionic function, depend on x_1,x_2,x_3 and x_4 and are supposed to be of class C^2 and thus the Schwartz's theorem is valid.

Theorem 3. *Let $f(q)$ is an quaternionic function. If $f(q)$ is of class C^2 and satisfies the Cauchy-Riemann conditions, then*

$$\begin{aligned} \Delta f_1 = 0, \\ \Delta f_2 = 0, \\ \Delta f_3 = 0, \\ \Delta f_4 = 0. \end{aligned} \tag{9}$$

Demonstration: The first step to obtain the Laplace equation is the derivation of equations (5), (6), (7) and (8) over x_1, x_2, x_3 and x_4. That will be done as follows: Firstly, in deriving the conditions of equation (5), we have:

$$\begin{aligned} \frac{\partial^2 F_1}{\partial x_1^2} = \frac{\partial^2 F_2}{\partial x_1 \partial x_2} = \frac{\partial^2 F_3}{\partial x_1 \partial x_3} = \frac{\partial^2 F_4}{\partial x_1 \partial x_4}, \\ \frac{\partial^2 F_1}{\partial x_1 \partial x_2} = \frac{\partial^2 F_2}{\partial x_2^2} = \frac{\partial^2 F_3}{\partial x_2 \partial x_3} = \frac{\partial^2 F_4}{\partial x_2 \partial x_4}, \\ \frac{\partial^2 F_1}{\partial x_3 \partial x_1} = \frac{\partial^2 F_2}{\partial x_3 \partial x_2} = \frac{\partial^2 F_3}{\partial x_3^2} = \frac{\partial^2 F_4}{\partial x_4 \partial x_3}, \\ \frac{\partial^2 F_1}{\partial x_1 \partial x_4} = \frac{\partial^2 F_2}{\partial x_4 \partial x_2} = \frac{\partial^2 F_3}{\partial x_4 \partial x_3} = \frac{\partial^2 F_4}{\partial x_4^2}. \end{aligned} \tag{10}$$

In deriving the conditions of equation (6), we have:

$$\begin{aligned} \frac{\partial^2 F_2}{\partial x_1^2} = -\frac{\partial^2 F_1}{\partial x_1 \partial x_2} = -\frac{\partial^2 F_3}{\partial x_1 \partial x_4} = \frac{\partial^2 F_4}{\partial x_1 \partial x_3}, \\ \frac{\partial^2 F_2}{\partial x_1 \partial x_2} = -\frac{\partial^1 F_1}{\partial x_2^2} = -\frac{\partial^2 F_3}{\partial x_2 \partial x_4} = \frac{\partial^2 F_4}{\partial x_3 \partial x_2}, \\ \frac{\partial^2 F_2}{\partial x_3 \partial x_1} = -\frac{\partial^2 F_1}{\partial x_3 \partial x_2} = -\frac{\partial^2 F_3}{\partial x_3 \partial x_4} = \frac{\partial^2 F_4}{\partial x_3^2}, \\ \frac{\partial^2 F_2}{\partial x_4 \partial x_1} = -\frac{\partial^2 F_1}{\partial x_4 \partial x_2} = -\frac{\partial^2 F_3}{\partial x_4^2} = \frac{\partial^2 F_4}{\partial x_4 \partial x_3}. \end{aligned} \tag{11}$$

In deriving the conditions of equation (7), we obtain that:

$$\begin{array}{l}
\dfrac{\partial^2 F_3}{\partial x_1^2} = -\dfrac{\partial^2 F_1}{\partial x_1 \partial x_3} = -\dfrac{\partial^2 F_2}{\partial x_1 \partial x_4} = \dfrac{\partial^2 F_4}{\partial x_1 \partial x_2}, \\
\dfrac{\partial^2 F_3}{\partial x_1 \partial x_2} = -\dfrac{\partial^1 F_1}{\partial x_2 \partial x_3} = -\dfrac{\partial^2 F_2}{\partial x_2 \partial x_4} = \dfrac{\partial^2 F_4}{\partial x_2^2}, \\
\dfrac{\partial^2 F_3}{\partial x_3 \partial x_1} = -\dfrac{\partial^2 F_1}{\partial x_3^2} = -\dfrac{\partial^2 F_2}{\partial x_4 \partial x_3} = \dfrac{\partial^2 F_4}{\partial x_3 \partial x_2}, \\
\dfrac{\partial^2 F_3}{\partial x_1 \partial x_4} = -\dfrac{\partial^2 F_1}{\partial x_4 \partial x_3} = -\dfrac{\partial^2 F_2}{\partial x_4^2} = \dfrac{\partial^2 F_4}{\partial x_4 \partial x_2}.
\end{array} \tag{12}$$

And finally in deriving the conditions of equation (8), we have:

$$\begin{array}{l}
\dfrac{\partial^2 F_4}{\partial x_1^2} = \dfrac{\partial^2 F_1}{\partial x_1 \partial x_4} = -\dfrac{\partial^2 F_2}{\partial x_1 \partial x_3} = -\dfrac{\partial^2 F_3}{\partial x_1 \partial x_2}, \\
\dfrac{\partial^2 F_4}{\partial x_1 \partial x_2} = \dfrac{\partial^2 F_1}{\partial x_2 \partial x_4} = -\dfrac{\partial^2 F_2}{\partial x_2 \partial x_3} = -\dfrac{\partial^2 F_3}{\partial x_2^2}, \\
\dfrac{\partial^2 F_4}{\partial x_3 \partial x_1} = \dfrac{\partial^2 F_1}{\partial x_3 \partial x_4} = -\dfrac{\partial^2 F_2}{\partial x_3^2} = -\dfrac{\partial^2 F_3}{\partial x_3 \partial x_2}, \\
\dfrac{\partial^2 F_4}{\partial x_1 \partial x_4} = \dfrac{\partial^2 F_1}{\partial x_4^2} = -\dfrac{\partial^2 F_2}{\partial x_4 \partial x_3} = -\dfrac{\partial^2 F_3}{\partial x_4 \partial x_2}.
\end{array} \tag{13}$$

In correlating those groups of partial derivatives (9), (10), (11) and (12), immediately follow the Laplace Equations:

$$\frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial x_3^2} + \frac{\partial^2 F_1}{\partial x_4^2} = 0, \tag{14}$$

$$\frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_3^2} + \frac{\partial^2 F_2}{\partial x_4^2} = 0, \tag{15}$$

$$\frac{\partial^2 F_3}{\partial x_1^2} + \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial x_3^2} + \frac{\partial^2 F_3}{\partial x_4^2} = 0, \tag{16}$$

and

$$\frac{\partial^2 F_4}{\partial x_1^2} + \frac{\partial^2 F_4}{\partial x_2^2} + \frac{\partial^2 F_4}{\partial x_3^2} + \frac{\partial^2 F_4}{\partial x_4^2} = 0. \tag{17}$$

Therefore, it is more simplified manner, the set of equations appears below:

$$\begin{array}{l}
\Delta f_1 = 0, \\
\Delta f_2 = 0, \\
\Delta f_3 = 0, \\
\Delta f_4 = 0,
\end{array} \tag{18}$$

where,

$$\begin{aligned} f_1 &= f_1(x_1, x_2, x_3, x_4),\\ f_2 &= f_2(x_1, x_2, x_3, x_4),\\ f_3 &= f_3(x_1, x_2, x_3, x_4),\\ f_4 &= f_4(x_1, x_2, x_3, x_4). \end{aligned} \tag{19}$$

4. Conclusion

In this note it is showed the feasibility of obtaining the equations of Laplace through the Cauchy-Riemann conditions for quaternions. This fact will allow the relationship between equations that can explain many physical phenomena. You can also use the above equations as a way of stating a theorem for harmonic functions for quaternions that satisfy the conditions of Cauchy. It is worth mentioning the importance of [1] in performing all relations used in this work.

References

[1] J.M. Machado, M.F. Borges, New ramrks on the differenciability of hypercomplex functions, *International Journal of Pure and Applied Mathematics*, **8**, No. 1 (2002), 85-101.

[2] Kunihiko Kodaira, *Complex Analysis*, Cambridge Studies in Advanced Mathematics, Cambridge University Press Cambridge (2007), 406pp.