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DETERMINANT OF ADJACENCY MATRIX OF SQUARE CYCLE GRAPH

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Abstract: Square Cycle, C_n^2 is a graph that has n vertices and two vertices u and v are adjacent if and only if distance between u and v not greater than 2. In this paper, we show that the determinant of adjacency matrix of square cycle C_n^2 are as follows

$$det(A(C_n^2)) = \begin{cases} 0, & n \equiv 0, 2, 4 \bmod 6, \\ 16, & n \equiv 3 \bmod 6, \\ 4, & n \equiv 1, 5 \bmod 6. \end{cases}$$

AMS Subject Classification: 05C50
Key Words: determinant, square cycle graph, adjacency matrix

1. Introduction

Let G be a simple graph with n vertices. We denote $det(A(G))$ is the determinant of adjacency matrix of G and $E(G;k)$ is k^{th} eigenvalues of the adjacency matrix which $det(A(G))$ and $E(G;k)$ are independent of the choice of vertices

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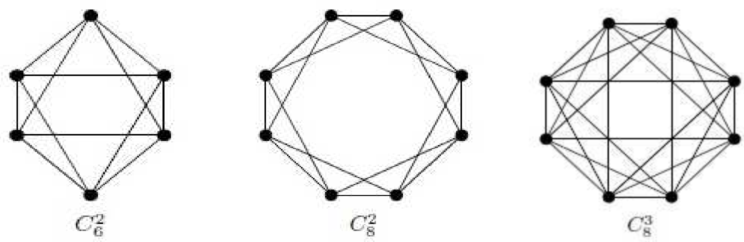

Figure 1: d-th power of cycle graph

in adjacency matrix and are an invariant of G.

In [2] and [4], they determined the determinant of adjacency matrix of some graphs, such as K_n, C_n, P_n and W_n. B. Gyurov and J. Cloud [7] has determined determinant of Pin-wheel graph. Moreover, there are studies of graph which satisfy some properties of determinant for example, M. Doob [5] construct circulant graph with $det(A(G)) = -deg(G)$, S. Hu [9] and A. Abdollahi [1] have found that the determinant of graphs with exactly one cycle and exactly two cycles, respectively.

Cycle power, C_n^d is a graph that has n vertices and distance each pair of vertex is less or equal d. For example,

If $d = 2$, $n \geq 6$, it is called *square cycle graph*.

Furthermore, there are studies of cycle power such as: C.N. Campos and C.P. de Mello [3], M. Krivelevich and A. Nachmias [10] studied about the colouring in cycle power, Y. Hoa, C. Woo and P. Chen [8] investigate the sandpile group in cycle power, D. Li and M. Liu [11] consider cycle power and their complements which satisfy Hadwiger's conjecture.

From figure 1 graph C_6^2 and graph C_8^2, we write to adjacency matrix

$$A(C_6^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A(C_8^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We see adjacency matrix of C_6^2 and C_8^2 is a circulant matrix because a main diagonal of matrix is equal to zero and entries in first row satisfy $a_{1j} = a_{1,(n-j+2)}$

for $j = 2, ..., n$ and $a_{ij} = a_{i+1,j+1}$, then a square cycle graph is a circulant graph. It is interesting to study determinant of adjacency matrix of square cycle graph.

Proposition 1. (see [2]) *Suppose that $[0, a_2, ..., a_n]$ is the first row of the adjacency matrix of a circulant graph G. Then the eigenvalues of graph G is denoted $E(G;k)$,*

$$E(G;k) = \sum_{j=1}^{n} a_j z^{j-1}$$

where $z = e^{\frac{2k\pi i}{n}}, k = 1, 2, ..., n$.

Square cycle graph is a circulant graph then eigenvalues of square cycle graph is

$$E(C_n^2;k) = z + z^2 + z^{n-2} + z^{n-1}. \tag{1}$$

Determinant of a square matrix can be find by eigenvalue its as below

Theorem 2. (see [6]) *Let $\lambda_1, ..., \lambda_n$ be a eigenvalues of a square matrix A. Then*

$$det(A) = \lambda_1 \lambda_2 ... \lambda_n.$$

Next, we present lemma that will be used in the proof of determinant of adjacency matrix of square cycle graph.

2. Main Results

Lemma 3. *Let q be a positive number. Then*

$$\prod_{k=1}^{2q}(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})\prod_{k=2q+2}^{4q+1}(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})\prod_{k=4q+3}^{6q+2}(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) = 2^{-12q}. \tag{2}$$

Proof. The left hand side of (2) is

$$\prod_{k=1}^{2q}\frac{\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{2\sin\frac{3k\pi}{6q+3}2\sin\frac{k\pi}{6q+3}}\prod_{k=2q+2}^{4q+1}\frac{\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{2\sin\frac{3k\pi}{6q+3}2\sin\frac{k\pi}{6q+3}}\prod_{k=4q+3}^{6q+2}\frac{\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{2\sin\frac{3k\pi}{6q+3}2\sin\frac{k\pi}{6q+3}}$$

$$
\begin{aligned}
&= \frac{1}{2^{12q}} \left(\frac{\prod_{k=q+1}^{2q} \sin\frac{6k\pi}{6q+3} \sin\frac{2k\pi}{6q+3}}{\prod_{k=1}^{q} \sin\frac{(6k-3)\pi}{6q+3} \sin\frac{(2k-1)\pi}{6q+3}} \right) \\
&\quad \left(\frac{\prod_{k=2q+2}^{4q+1} \sin\frac{6k\pi}{6q+3} \sin\frac{2k\pi}{6q+3}}{\prod_{k=2q+2}^{4q+1} \sin\frac{3k\pi}{6q+3} \sin\frac{k\pi}{6q+3}} \right) \\
&\quad \left(\frac{\prod_{k=4q+3}^{6q+2} \sin\frac{6k\pi}{6q+3} \sin\frac{2k\pi}{6q+3}}{\prod_{k=4q+3}^{6q+2} \sin\frac{3k\pi}{6q+3} \sin\frac{k\pi}{6q+3}} \right) \\
&= \frac{1}{2^{12q}} \left(\frac{\prod_{k=1}^{q} \sin\frac{(6k-3)\pi}{6q+3} \sin\frac{(2k-1)\pi}{6q+3}}{\prod_{k=1}^{q} \sin\frac{(6k-3)\pi}{6q+3} \sin\frac{(2k-1)\pi}{6q+3}} \right) \\
&\quad \left(\frac{\prod_{k=2q+2}^{3q+1} \sin\frac{(6k-(6q+3))\pi}{6q+3} \sin\frac{(2k-(2q+1))\pi}{6q+3}}{\prod_{k=2q+2}^{3q+1} \sin\frac{(6k-(6q+3))\pi}{6q+3} \sin\frac{(2k-(2q+1))\pi}{6q+3}} \right) \\
&\quad \left(\frac{\prod_{k=4q+3}^{5q+2} \sin\frac{(6k-(12q+9))\pi}{6q+3} \sin\frac{(2k-(4q+3))\pi}{6q+3}}{\prod_{k=4q+3}^{5q+2} \sin\frac{(6k-(12q+9))\pi}{6q+3} \sin\frac{(2k-(4q+3))\pi}{6q+3}} \right) \\
&= 2^{-12q}. \qquad \square
\end{aligned}
$$

Lemma 4. *Let q be a positive integer. Then*

$$\prod_{k=1}^{6q} (\cos\frac{3k\pi}{6q+1} \cos\frac{k\pi}{6q+1}) = 2^{-12q}.$$

Proof. It can be proved by

$$
\begin{aligned}
\prod_{k=1}^{6q} (\cos\frac{3k\pi}{6q+1} \cos\frac{k\pi}{6q+1}) &= \prod_{k=1}^{6q} \frac{\sin\frac{6k\pi}{6q+1} \sin\frac{2k\pi}{6q+1}}{2\sin\frac{3k\pi}{6q+1} 2\sin\frac{k\pi}{6q+1}} \\
&= \frac{1}{2^{12q}} \left(\frac{\prod_{k=3q+1}^{6q} \sin\frac{6k\pi}{6q+1} \sin\frac{2k\pi}{6q+1}}{\prod_{k=1}^{3q} \sin\frac{(6k-3)\pi}{6q+1} \sin\frac{(2k-1)\pi}{6q+1}} \right) \\
&= \frac{1}{2^{12q}} \left(\frac{\prod_{k=1}^{3q} \sin\frac{(6k-3)\pi}{6q+1} \sin\frac{(2k-1)\pi}{6q+1}}{\prod_{k=1}^{3q} \sin\frac{(6k-3)\pi}{6q+1} \sin\frac{(2k-1)\pi}{6q+1}} \right) \\
&= 2^{-12q}. \qquad \square
\end{aligned}
$$

Lemma 5. *Let q be a positive integer. Then*

$$\prod_{k=1}^{6q+4}(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}) = 2^{-2(6q+4)}.$$

Proof. It can be proved by

$$\begin{aligned}\prod_{k=1}^{6q+4}(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}) &= \prod_{k=1}^{6q+4}\frac{\sin\frac{6k\pi}{6q+5}\sin\frac{2k\pi}{6q+5}}{2\sin\frac{3k\pi}{6q+5}2\sin\frac{k\pi}{6q+5}}\\ &= \frac{1}{2^{2(6q+4)}}\left(\frac{\prod_{k=3q+3}^{6q+4}\sin\frac{6k\pi}{6q+5}\sin\frac{2k\pi}{6q+5}}{\prod_{k=1}^{3q+2}\sin\frac{(6k-3)\pi}{6q+5}\sin\frac{(2k-1)\pi}{6q+5}}\right)\\ &= \frac{1}{2^{2(6q+4)}}\left(\frac{\prod_{k=1}^{3q+2}\sin\frac{(6k-3)\pi}{6q+5}\sin\frac{(2k-1)\pi}{6q+5}}{\prod_{k=1}^{3q+2}\sin\frac{(6k-3)\pi}{6q+5}\sin\frac{(2k-1)\pi}{6q+5}}\right)\\ &= 2^{-2(6q+4)}.\end{aligned}$$

□

Next, we use Lemma 3, 4 and 5 to find determinant of adjacency matrix of square cycle graph.

Theorem 6. *Let C_n^2 be a square cycle graph with n vertices and n be a positive integer. Then*

$$det(A(C_n^2)) = \begin{cases} 0, & n \equiv 0, 2, 4 \bmod 6,\\ 16, & n \equiv 3 \bmod 6,\\ 4, & n \equiv 1, 5 \bmod 6.\end{cases}.$$

Proof. Let $E(C_n^2; k)$ be a k^{th} eigenvalue of adjacency matrix of square cycle graph C_n^2. From (1), We get

$$\begin{aligned}E(C_n^2; k) &= e^{\frac{2k\pi i}{n}} + e^{\frac{4k\pi i}{n}} + e^{\frac{2k(n-2)\pi i}{n}} + e^{\frac{2k(n-1)\pi i}{n}}\\ &= e^{\frac{2k\pi i}{n}} + e^{\frac{4k\pi i}{n}} + e^{\frac{2kn\pi i}{n}}\cdot e^{\frac{-4k\pi i}{n}} + e^{\frac{2kn\pi i}{n}}\cdot e^{\frac{-2k\pi i}{n}}.\end{aligned}$$

By Euler's formula, we obtain

$$\begin{aligned}E(C_n^2; k) = {} & (\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}) + (\cos\frac{4k\pi}{n} + i\sin\frac{4k\pi}{n}) +\\ & (\cos\frac{-4k\pi}{n} + i\sin\frac{-4k\pi}{n}) + (\cos\frac{-2k\pi}{n} + i\sin\frac{-2k\pi}{n})\\ = {} & 2\cos\frac{2k\pi}{n} + 2\cos\frac{4k\pi}{n}.\end{aligned}$$

We can rewrite

$$E(C_n^2;k) = 4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}). \quad (3)$$

From (3) We have

$$\begin{aligned} det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\ &= \prod_{k=1}^{n} 4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}). \quad (4) \end{aligned}$$

Consider n as follows

Case I. $n \equiv 0, 2, 4 \bmod 6$.
Since n is even and $1 \leq k \leq n$, consider (3) when $k = \frac{n}{2}$. Then

$$\begin{aligned} E(C_n^2;\frac{n}{2}) &= 4(\cos\frac{3\frac{n}{2}\pi}{n}\cos\frac{\frac{n}{2}\pi}{n}) \\ &= 0. \end{aligned}$$

From (4), we obtain

$$\begin{aligned} det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\ &= 0. \end{aligned}$$

Therefore, $det(A(C_n^2)) = 0$ when $n \equiv 0, 2, 4 \bmod 6$.

Case II. $n \equiv 3 \bmod 6$ Then $n = 6q+3, \exists q \in \mathbb{Z}^+$.
From (4), we obtain

$$\begin{aligned} det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\ &= \prod_{k=1}^{6q+3} 4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}) \\ &= 4(\cos\frac{3(2q+1)\pi}{6q+3}\cos\frac{(2q+1)\pi}{6q+3})4(\cos\frac{3(4q+2)\pi}{6q+3}\cos\frac{(4q+2)\pi}{6q+3}) \\ &\quad 4(\cos\frac{3(6q+3)\pi}{6q+3}\cos\frac{(6q+3)\pi}{6q+3})2^{12q}\prod_{k=1}^{2q} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \end{aligned}$$

$$\prod_{k=2q+2}^{4q+1} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \prod_{k=4q+3}^{6q+2} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})$$

$$= (-2)(-2)(4)2^{12q}\prod_{k=1}^{2q} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})$$

$$\prod_{k=2q+2}^{4q+1} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})$$

$$\prod_{k=4q+3}^{6q+2} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}).$$

Using Lemma 3, we have

$$\begin{aligned} det(A(C_n^2)) &= (-2)(-2)(4)(2^{12q})(2^{-12q}) \\ &= 16. \end{aligned}$$

Therefore $det(A(C_n^2)) = 16$ when $n \equiv 3 \bmod 6$.

Case III. $n \equiv 1 \bmod 6$ and $n \equiv 5 \bmod 6$. We consider 2 subcases.
Subcase 3.1, $n \equiv 1 \bmod 6$, by (4), we obtain

$$\begin{aligned} det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\ &= \prod_{k=1}^{6q+1} 2^2(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}) \\ &= 4(\cos\frac{3(6q+1)\pi}{6q+1}\cos\frac{(6q+1)\pi}{6q+1})2^{12q}(\prod_{k=1}^{6q} 4(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}). \end{aligned}$$

Using Lemma 4, we have

$$\begin{aligned} det(A(C_n^2)) &= 4(2^{12q})(2^{-12q}) \\ &= 4. \end{aligned}$$

Subcase 3.2, $n \equiv 5 \bmod 6$, by (4), we obtain

$$\begin{aligned} det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\ &= \prod_{k=1}^{6q+5} 2^2(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}) \end{aligned}$$

$$= 4(\cos\frac{3(6q+5)\pi}{6q+5}\cos\frac{(6q+5)\pi}{6q+5})2^{2(6q+4)}(\prod_{k=1}^{6q+4}4(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5})).$$

Using Lemma 5, we have

$$\begin{aligned} det(A(C_n^2)) &= 4(2^{2(6q+4)})(2^{-2(6q+4)}) \\ &= 4. \end{aligned}$$

From subcase 3.1 and 3.2, we obtain

$$det(A(C_n^2)) = 4 \text{ for } n \equiv 1,5 \bmod 6.$$

From case I, II and III,

$$det(A(C_n^2)) = \begin{cases} 0, & n \equiv 0,2,4 \bmod 6, \\ 16, & n \equiv 3 \bmod 6, \\ 4, & n \equiv 1,5 \bmod 6. \end{cases} \qquad \square$$

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