QUASIPOLAR PROPERTY OF TRIVIAL MORITA CONTEXT

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Abstract: Let $T := \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a trivial Morita context. This article concerns the quasipolar properties of trivial Morita contexts over local rings. Necessary and sufficient conditions for a single matrix of $T$ (a trivial Morita context over local rings) to be quasipolar are obtained. And then we get a sufficient condition for $T$ to be a quasipolar ring.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. The Jacobson radical, the center and the group of units of a ring $R$ are denoted by $J(R)$, $C(R)$, and $U(R)$, respectively. In [5], Koliha and Patricio introduced quasipolar elements in a ring to study generalized Drazin invertible elements in rings and EP elements in involutory rings. Following [5], for any
element \( a \in R \), the commutant and double commutant of \( a \) in \( R \) are defined by \( \text{comm}(a) = \{ x \in R : xa = ax \} \) and \( \text{comm}^2(a) = \{ x \in R : xy = yx, x \in \text{comm}(a) \} \), respectively. Write \( R^{qnil} = \{ a \in R : 1 + xa \in U(R), x \in \text{comm}(a) \} \).

An element \( a \) is said to be quasinilpotent if \( a \in R^{qnil} \). Koliha and Patricio called an element \( a \in R \) quasipolar if there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \in U(R) \) and \( ap \in R^{qnil} \), and such an idempotent \( p \) is called a spectral idempotent of \( a \). By [5], a quasipolar element of a ring has a unique spectral idempotent. The notion of a quasipolar ring was introduced by Ying and Chen [11]. A ring is called quasipolar if each of its elements is quasipolar. Some properties of quasipolar rings were further studied in [2, 3, 4, 10].

Let \( R \) and \( S \) be rings, \( R M S \) and \( S N R \) bimodules. For \( a \in R, b \in S \), \( l_a \) (resp. \( r_a \)) and \( r_b \) (resp. \( l_b \)) will denote the abelian group homomorphisms of \( M \) (resp. \( N \)) given by left (resp. right) or right (resp. left) multiplication by \( a \) or \( b \). Given \( T : = \begin{pmatrix} R & M \\ N & S \end{pmatrix} \) as the trivial Morita context, which forms a ring with addition defined componentwise and with multiplication defined by

\[
\begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1m_2 + m_1b_2 \\ n_1a_2 + b_1n_2 & b_1b_2 \end{pmatrix}.
\]

In [9], it is proved that a trivial Morita context \( \begin{pmatrix} R & M \\ N & S \end{pmatrix} \) is strongly \( \pi \)-regular if and only if both \( R \) and \( S \) are strongly \( \pi \)-regular. In this note, we are motivated to investigate the quasipolar property of trivial Morita context \( S \). We showed some characterizations of the quasipolar properties about some elements in the trivial Morita context. In Theorem 3, we proved that if \( R \) and \( S \) are both local rings, \( u \in U(R) \) and \( j \in J(S) \) then \( l_u - r_j \) of \( M \) and \( r_u - l_j \) of \( N \) are both injective equivalent to that \( A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix} \) is quasipolar in \( T \). In Theorem 10, we proved that if \( R, S \) are both local rings, \( u \in U(R), j \in J(S) \), then \( r_u - l_j \) of \( N \) is injective and \( l_u - r_j \) of \( M \) is isomorphic if and only if for any \( m \in M \), \( A := \begin{pmatrix} u & m \\ 0 & j \end{pmatrix} \) is quasipolar in \( T \) with the spectral idempotent \( \begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix} \) for some \( m_1 \in M \). In Theorem 14, it is given that if \( R \) and \( S \) are both local rings, \( u \in U(R), j \in J(S) \), then \( l_u - r_j \) of \( M \) and \( r_u - l_j \) of \( N \) are both isomorphic if and only if for any \( m \in M, n \in N \), \( A = \begin{pmatrix} u & m \\ n & j \end{pmatrix} \) is quasipolar in \( T \) with the spectral idempotent \( \begin{pmatrix} 0 & m_1 \\ n_1 & 1 \end{pmatrix} \) for some \( m_1 \in M, n_1 \in N \).
2. Main Results

The first lemma decides all of the idempotents in the trivial Morita context.

**Lemma 1.** Let $R$, $S$ be both local rings. If $E^2 = E \in T$, then $E$ must be one element of the set

$$E = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & m \\ n & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ n & 1 \end{pmatrix}, m \in M, n \in N \right\}.$$

**Lemma 2.** [10, Theorem 1.4] Let $R$ be a ring and let $a \in R$. If there exists $p^2 = p \in \text{comm}(a)$ such that $a + p \in U(R)$ and $a^kp \in J(R)$ for some $k \geq 1$, then $p$ is unique if and only if $p \in \text{comm}^2(a)$.

The next theorem showed that a characterization of the quasipolar properties of a part of elements in the trivial Morita context.

**Theorem 3.** Let $R$, $S$ be both local rings. For any $u \in U(R), j \in J(S)$. Then the following are equivalent:

1. $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both injective.
2. $A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in $T$.

**Proof.** (1) $\Rightarrow$ (2) Let $E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, $EA = AE \in J(T), E + A \in U(T)$. If there exists another idempotent $E_1$ of $T$ and some positive integer $k$ such that $E_1A = AE_1, E_1A^k \in J(T)$ and $E_1 + A \in U(T)$. Following from the condition $E_1 + A \in U(T)$, we have $E_1 \in \left\{ \begin{pmatrix} 0 & m \\ n & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, m \in M, n \in N \right\}$. Since for some positive integer $k$, $E_1A^k \in J(T)$, $E_1 = \begin{pmatrix} 0 & m \\ n & 1 \end{pmatrix}$ where $m \in M, n \in N$. $E_1A = AE_1$ and the conditions of (1) imply that $m = n = 0$. Thus $E$ is the uniquely idempotent of $T$ which satisfies all of the conditions of $EA = AE, EA^k \in J(T)$ for some positive integer $k$ and $E + A \in U(T)$. By Lemma 2, $E$ is the spectral idempotent of $A$ in $T$. Hence, $A$ is quasipolar in $T$.

(2) $\Rightarrow$ (1) We first show that $E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the spectral idempotent of $A$. Suppose not, there must exist another idempotent $E_1 = \begin{pmatrix} r & m \\ n & s \end{pmatrix} \in T$ which is the spectral idempotent of $A$. Since $EA = AE, EE_1 = E_1E$. Thus
$m = n = 0$ and $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Write $B = \begin{pmatrix} u^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. It can be easily check that $AB = BA$, but $I_2 - AB \notin U(T)$. This is a contradiction. Therefore, $E$ must be the spectral idempotent of $A$. Next, we prove that $r_u - l_j$ of $N$ is injective. Given $n \in N$ satisfies that $nu - jn = 0$. Set $C = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \in T$. Clearly, $AC = CA$. Thus $EC = CE$, it follows then $n = 0$. Therefore, $r_u - l_j$ of $N$ is injective. Similarly, $l_u - r_j$ of $M$ is injective. The proof is completed. 

Similarly, we can get the following theorem.

**Theorem 4.** Let $R, S$ be both local rings. For any $u \in U(S)$, $j \in J(R)$. Then the following are equivalent:

1. $l_u - r_j$ of $N$ and $r_u - l_j$ of $M$ are both injective.
2. $A := \begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in $T$.

**Corollary 5.** Let $R, S$ be both local rings. If $T$ is a quasipolar ring then:

1. For any $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both injective.
2. For any $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ of $M$ and $l_u - r_j$ of $N$ are both injective.

If $M = N = R = S$, then the trivial Morita context will be $K_0(R)$. In [1], the authors call a local ring $R$ bleached if, for any $j \in J(R)$ and any $u \in U(R)$, the abelian group endomorphisms $l_u - r_j$ and $l_j - r_u$ of $R$ are surjective. In [4], a local ring $R$ is called co-bleached if, for any $j \in J(R)$ and any $u \in U(R)$, the abelian group endomorphisms $l_u - r_j$ and $l_j - r_u$ of $R$ are injective. We have several corollaries.

**Corollary 6.** Let $R$ be a local ring. For any $u \in U(R)$, $j \in J(R)$, the following conditions are equivalent:

1. $l_u - r_j$ and $r_u - l_j$ of $R$ are both injective.
2. $\begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in $K_0(R)$.
3. $\begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in $K_0(R)$.

**Corollary 7.** Let $R$ be a local ring. If $K_0(R)$ is quasipolar then $R$ is co-bleached.
If $N = 0$, we have the next corollary.

**Corollary 8.** Let $R, S$ be both local rings and $RMS$ a bimodule. $T_1 = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is the formal upper triangular matrix ring. Then the following conditions are satisfied:

1. For any $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ of $M$ is injective if and only if $A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in $T_1$.

2. For any $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ of $M$ is injective if and only if $A := \begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in $T_1$.

If $M = R = S$, the next corollary can be given.

**Corollary 9.** Let $R$ be a local ring and $T_2 = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ the upper triangular matrix ring. Then for any $u \in U(R)$, $j \in J(R)$, the following conditions are satisfied:

1. $l_u - r_j$ of $R$ is injective if and only if $\begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in $T_2$.

2. $r_u - l_j$ of $R$ is injective if and only if $\begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in $T_2$.

The next theorem will give the characterization of quasipolar properties of another kind of elements in the trivial Morita context.

**Theorem 10.** Let $R, S$ be both local rings. For any $u \in U(R)$, $j \in J(S)$. Then the following are equivalent:

1. $r_u - l_j$ of $N$ is injective and $l_u - r_j$ of $M$ is an isomorphism.

2. For any $m \in M$, $A := \begin{pmatrix} u & m \\ 0 & j \end{pmatrix}$ is quasipolar in $T$ with the spectral idempotent $\begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$ for some $m_1 \in M$.

**Proof.** $(1) \Rightarrow (2)$ Since $l_u - r_j$ of $M$ is an surjective, there exists $m_1 \in M$ such that $um_1 - m_1j = -m$. Set $E = \begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$. Clearly, $E^2 = E$, $EA = AE \in J(T)$ and $E + A \in U(T)$. If there exists another idempotents $E_1 \in T$ such that $E_1A = AE_1$, $E_1 + A \in U(T)$ and $E_1A^k \in J(T)$ for some positive
integer \( k \). We can assume that \( E_1 = \begin{pmatrix} 0 & m_0 \\ n_0 & 1 \end{pmatrix} \) where \( m_0 \in M, \ n_0 \in N \).

Following from \( AE_1 = E_1A \), we can get \( um_0 - m_0j = -m, \ jn_0 - n_0u = 0 \). By the conditions of (1), \( m_0 = m_1, \ n_0 = 0 \). Thus \( E \) is the unique idempotent of \( T \) which satisfies all of the conditions of \( EA = AE, EA^k \in J(T) \) for some positive integer \( k \) and \( E + A \in U(T) \). By Lemma 2, \( E \) is the spectral idempotent of \( A \) in \( T \). Hence, \( A \) is quasipolar in \( T \).

\[(2) \Rightarrow (1)\]

By Theorem 3, we need only to proof that \( l_u - r_j \) of \( M \) is surjective. For any \( m \in M \), there exists \( m_1 \in M \) such that \( E = \begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix} \) is the spectral idempotent of \( A := \begin{pmatrix} u & -m \\ 0 & j \end{pmatrix} \) by hypothesis. So \( EA = AE \), it follows that \( m = um_1 - m_1j \). The proof is completed.

Similarly, the following theorem can be proved.

**Theorem 11.** Let \( R, S \) be both local rings. For any \( u \in U(R), \ j \in J(S) \). Then the following are equivalent:

1. \( l_u - r_j \) of \( M \) is injective and \( r_u - l_j \) of \( N \) is an isomorphism.
2. For any \( m \in M \), \( A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix} \) is quasipolar in \( T \) with the spectral idempotent \( \begin{pmatrix} 0 & 0 \\ n_1 & 1 \end{pmatrix} \) for some \( n_1 \in N \).

**Corollary 12.** Let \( R, S \) be both local rings and \( R \) is a bimodule. Write \( T_1 = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \) which is the formal upper triangular matrix ring. Then for any \( u \in U(R), \ j \in J(S) \), the following conditions are equivalent:

1. \( l_u - r_j \) of \( M \) is an isomorphism.
2. For any \( m \in M \), \( A := \begin{pmatrix} u & m \\ 0 & j \end{pmatrix} \) is quasipolar in \( T_1 \) with the spectral idempotent \( \begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix} \) for some \( m_1 \in M \).

**Corollary 13.** [3, Lemma 2.4] Let \( R \) be a local ring. Write \( T_2 = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \) which is the upper triangular matrix ring. Then for any \( u \in U(R), \ j \in J(R) \), the following conditions are equivalent:

1. \( l_u - r_j \) of \( R \) is an isomorphism.
(2) For any $r \in R$, $A := \begin{pmatrix} u & r \\ 0 & j \end{pmatrix}$ is quasipolar in $T_2$ with the spectral idempotent $\begin{pmatrix} 0 & e_{12} \\ 0 & 1 \end{pmatrix}$ for some $e_{12} \in R$.

**Theorem 14.** Let $R, S$ be both local rings, and let $u \in U(R), j \in J(S)$. The following are equivalent:

1. $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both isomorphic.
2. For any $m \in M, n \in N$, $A = \begin{pmatrix} u & m \\ n & j \end{pmatrix}$ is quasipolar in $T$ with the spectral idempotent $\begin{pmatrix} 0 & m_1 \\ n_1 & 1 \end{pmatrix}$ for some $m_1 \in M, n_1 \in N$.

**Proof.** (1) $\Rightarrow$ (2) By the hypothesis that $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both surjective, for any $m \in M, n \in N$ there exist $m_1 \in M, n_1 \in N$ such that $um_1 - m_1 j = -m, n_1 u - jn_1 = -n$. Set $E = \begin{pmatrix} 0 & m_1 \\ n_1 & 1 \end{pmatrix}$. It is not difficult to check that $E^2 = E, EA = AE \in J(T)$ and $E + A \in U(T)$. Following from $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both injective, we have $E$ is the unique idempotent of $T$ which satisfies all of the conditions of $EA = AE, EA^k \in J(T)$ for some positive integer $k$ and $E + A \in U(T)$. Again by Lemma 2, $E$ is the spectral idempotent of $A$ in $T$.

(2) $\Rightarrow$ (1) By Theorem 10 and Theorem 11.

**Theorem 15.** Let $R, S$ be both local rings. If the following conditions are satisfied:

1. For any $u \in U(R), j \in J(S)$, $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both surjective;
2. For any $u \in U(S), j \in J(R)$, $r_u - l_j$ of $M$ and $l_u - r_j$ of $N$ are both surjective.

Then the following are equivalent:

1. For any $u \in U(R), j \in J(S)$, $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both injective; And for any $u \in U(S), j \in J(R)$, $r_u - l_j$ of $M$ and $l_u - r_j$ of $N$ are both injective.
2. $T$ is a quasipolar ring.

**Corollary 16.** Let $R, S$ be both local rings. If the following conditions are satisfied:
(1) For any $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ of $M$ and $r_u - l_j$ of $N$ are both isomorphic;

(2) For any $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ of $M$ and $l_u - r_j$ of $N$ are both isomorphic.

Then $T$ is a quasipolar ring.

**Corollary 17.** Let $R$ be a bleached local ring. Then $R$ is co-bleached if and only if $K_0(R)$ is a quasipolar ring.

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