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QUASIPOLAR PROPERTY OF TRIVAL MORITA CONTEXT

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Abstract: Let $T := \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a trivial Morita context. This article concerns the quasipolar properties of trivial Morita contexts over local rings. Necessary and sufficient conditions for a single matrix of T (a trivial Morita context over local rings) to be quasipolar are obtained. And then we get a sufficient condition for T to be a quasipolar ring.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. The Jacobson radical, the center and the group of units of a ring R are denoted by $J(R)$, $C(R)$, and $U(R)$, respectively. In [5], Koliha and Patricio introduced quasipolar elements in a ring to study generalized Drazin invertible elements in rings and EP elements in involutory rings. Following [5], for any

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element $a \in R$, the commutant and double commutant of a in R are defined by $comm(a) = \{x \in R : xa = ax\}$ and $comm^2(a) = \{x \in R : xy = yx, x \in comm(a)\}$, respectively. Write $R^{qnil} = \{a \in R : 1 + xa \in U(R), x \in comm(a)\}$. An element a is said to be quasinilpotent if $a \in R^{qnil}$. Koliha and Patricio called an element $a \in R$ quasipolar if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$, and such an idempotent p is called a spectral idempotent of a. By [5], a quasipolar element of a ring has a unique spectral idempotent. The notion of a quasipolar ring was introduced by Ying and Chen [11]. A ring is called quasipolar if each of its elements is quasipolar. Some properties of quasipolar rings were further studied in [2, 3, 4, 10].

Let R and S be rings, ${}_RM_S$ and ${}_SN_R$ bimodules. For $a \in R$, $b \in S$, l_a (resp.r_a) and r_b(resp.l_b) will denote the abelian group homomorphisms of M(resp.N) given by left(resp. right) or right(resp. left) multiplication by a or b. Given $T := \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ as the trivial Morita context, which forms a ring with addition defined componentwise and with multiplication defined by

$$\begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1m_2 + m_1b_2 \\ n_1a_2 + b_1n_2 & b_1b_2 \end{pmatrix}.$$

In [9], it is proved that a trivial Morita context $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is strongly π-regular if and only if both R and S are strongly π-regular. In this note, we are motivated to investigate the quasipolar property of trivial Morita contextS. We showed some characterizations of the quasipolar properties about some elements in the trivial Morita context. In Theorem 3, we proved that if R and S are both local rings, $u \in U(R)$ and $j \in J(S)$ then $l_u - r_j$ of M and $r_u - l_j$ of N are both injective equivalent to that $A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in T. In Theorem 10, we proved that if R, S are both local rings, $u \in U(R)$, $j \in J(S)$, then $r_u - l_j$ of N is injective and $l_u - r_j$ of M is isomorphic if and only if for any $m \in M$, $A := \begin{pmatrix} u & m \\ 0 & j \end{pmatrix}$ is quasipolar in T with the spectral idempotent $\begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$ for some $m_1 \in M$. In Theorem 14, it is given that if R and S are both local rings, $u \in U(R)$, $j \in J(S)$, then $l_u - r_j$ of M and $r_u - l_j$ of N are both isomorphic if and only if for any $m \in M$, $n \in N$, $A = \begin{pmatrix} u & m \\ n & j \end{pmatrix}$ is quasipolar in T with the spectral idempotent $\begin{pmatrix} 0 & m_1 \\ n_1 & 1 \end{pmatrix}$ for some $m_1 \in M$, $n_1 \in N$.

2. Main Results

The first lemma decides all of the idempotents in the trivial Morita context.

Lemma 1. *Let R, S be both local rings. If $E^2 = E \in T$, then E must be one element of the set*

$$\mathbb{E} = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & m \\ n & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ n & 1 \end{pmatrix}, m \in M, n \in N\}.$$

Lemma 2. [10, Theorem 1.4] *Let R be a ring and let $a \in R$. If there exists $p^2 = p \in comm(a)$ such that $a + p \in U(R)$ and $a^k p \in J(R)$ for some $k \geq 1$, then p is unique if and only if $p \in comm^2(a)$.*

The next theorem showed that a characterization of the quasipolar properties of a part of elements in the trivial Morita context.

Theorem 3. *Let R, S be both local rings. For any $u \in U(R)$, $j \in J(S)$. Then the following are equivalent:*

(1) *$l_u - r_j$ of M and $r_u - l_j$ of N are both injective.*

(2) *$A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in T.*

Proof. (1) $\Rightarrow$ (2) Let $E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, $EA = AE \in J(T)$, $E + A \in U(T)$. If there exists another idempotent E_1 of T and some positive integer k such that $E_1 A = AE_1, E_1 A^k \in J(T)$ and $E_1 + A \in U(T)$. Following from the condition $E_1 + A \in U(T)$, we have $E_1 \in \{\begin{pmatrix} 0 & m \\ n & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, m \in M, n \in N\}$. Since for some positive integer k, $E_1 A^k \in J(T)$, $E_1 = \begin{pmatrix} 0 & m \\ n & 1 \end{pmatrix}$ where $m \in M$, $n \in N$. $E_1 A = AE_1$ and the conditions of (1) imply that $m = n = 0$. Thus E is the uniquely idempotent of T which satisfies all of the conditions of $EA = AE, EA^k \in J(T)$ for some positive integer k and $E + A \in U(T)$. By Lemma 2, E is the spectral idempotent of A in T. Hence, A is quasipolar in T.

(2) $\Rightarrow$ (1) We first show that $E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the spectral idempotent of A. Suppose not, there must exist another idempotent $E_1 = \begin{pmatrix} r & m \\ n & s \end{pmatrix} \in T$ which is the spectral idempotent of A. Since $EA = AE$, $EE_1 = E_1 E$. Thus

$m = n = 0$ and $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Write $B = \begin{pmatrix} u^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. It can be easily check that $AB = BA$, but $I_2 - AB \notin U(T)$. This is a contradiction. Therefore, E must be the spectral idempotent of A. Next, we prove that $r_u - l_j$ of N is injective. Given $n \in N$ satisfies that $nu - jn = 0$. Set $C = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \in T$. Clearly, $AC = CA$. Thus $EC = CE$, it follows then $n = 0$. Therefore, $r_u - l_j$ of N is injective. Similarly, $l_u - r_j$ of M is injective. The proof is completed. □

Similarly, we can get the following theorem.

Theorem 4. *Let R, S be both local rings. For any $u \in U(S)$, $j \in J(R)$. Then the following are equivalent:*

(1) *$l_u - r_j$ of N and $r_u - l_j$ of M are both injective.*

(2) *$A := \begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in T.*

Corollary 5. *Let R, S be both local rings. If T is a quasipolar ring then:*

(1) *For any $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ of M and $r_u - l_j$ of N are both injective.*

(2) *For any $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ of M and $l_u - r_j$ of N are both injective.*

If $M = N = R = S$, then the trivial Morita context will be $K_0(R)$. In [1], the authors call a local ring R bleached if, for any $j \in J(R)$ and any $u \in U(R)$, the abelian group endomorphisms $l_u - r_j$ and $l_j - r_u$ of R are surjective. In [4], a local ring R is called co-bleached if, for any $j \in J(R)$ and any $u \in U(R)$, the abelian group endomorphisms $l_u - r_j$ and $l_j - r_u$ of R are injective. We have several corollaries.

Corollary 6. *Let R be a local ring. For any $u \in U(R)$, $j \in J(R)$, the following conditions are equivalent:*

(1) *$l_u - r_j$ and $r_u - l_j$ of R are both injective.*

(2) *$\begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in $K_0(R)$.*

(3) *$\begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in $K_0(R)$.*

Corollary 7. *Let R be a local ring. If $K_0(R)$ is quasipolar then R is co-bleached.*

If $N = 0$, we have the next corollary.

Corollary 8. *Let R, S be both local rings and ${}_RM_S$ a bimodule. $T_1 = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is the formal upper triangular matrix ring. Then the following conditions are satisfied:*

(1) For any $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ of M is injective if and only if $A := \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in T_1.

(2) For any $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ of M is injective if and only if $A := \begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in T_1.

If $M = R = S$, the next corollary can be given.

Corollary 9. *Let R be a local ring and $T_2 = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ the upper triangular matrix ring. Then for any $u \in U(R)$, $j \in J(R)$, the following conditions are satisfied:*

(1) $l_u - r_j$ of R is injective if and only if $\begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix}$ is quasipolar in T_2.

(2) $r_u - l_j$ of R is injective if and only if $\begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ is quasipolar in T_2.

The next theorem will give the characterization of quasipolar properties of another kind of elements in the trivial Morita context.

Theorem 10. *Let R, S be both local rings. For any $u \in U(R)$, $j \in J(S)$. Then the following are equivalent:*

(1) $r_u - l_j$ of N is injective and $l_u - r_j$ of M is an isomorphism.

(2) For any $m \in M$, $A := \begin{pmatrix} u & m \\ 0 & j \end{pmatrix}$ is quasipolar in T with the spectral idempotent $\begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$ for some $m_1 \in M$.

Proof. (1) $\Rightarrow$ (2) Since $l_u - r_j$ of M is an surjective, there exists $m_1 \in M$ such that $um_1 - m_1 j = -m$. Set $E = \begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$. Clearly, $E^2 = E$, $EA = AE \in J(T)$ and $E + A \in U(T)$. If there exists another idempotents $E_1 \in T$ such that $E_1 A = AE_1$, $E_1 + A \in U(T)$ and $E_1 A^k \in J(T)$ for some positive integer k. We can assume that $E_1 = \begin{pmatrix} 0 & m_0 \\ n_0 & 1 \end{pmatrix}$ where $m_0 \in M,\ n_0 \in N$. Following from $AE_1 = E_1A$, we can get $um_0 - m_0j = -m,\ jn_0 - n_0u = 0$. By the conditions of (1), $m_0 = m_1,\ n_0 = 0$. Thus E is the unique idempotent of T which satisfies all of the conditions of $EA = AE, EA^k \in J(T)$ for some positive integer k and $E + A \in U(T)$. By Lemma 2, E is the spectral idempotent of A in T. Hence, A is quasipolar in T.

(2) $\Rightarrow$ (1) By Theorem 3, we need only to proof that $l_u - r_j$ of M is surjective. For any $m \in M$, there exists $m_1 \in M$ such that $E = \begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$ is the spectral idempotent of $A := \begin{pmatrix} u & -m \\ 0 & j \end{pmatrix}$ by hypothesis. So $EA = AE$, it follows that $m = um_1 - m_1j$. The proof is completed. □

Similarly, the following theorem can be proved.

Theorem 11. *Let $R,\ S$ be both local rings. For any $u \in U(R),\ j \in J(S)$. Then the following are equivalent:*

(1) *$l_u - r_j$ of M is injective and $r_u - l_j$ of N is an isomorphism.*

(2) *For any $m \in M$, $A := \begin{pmatrix} u & 0 \\ n & j \end{pmatrix}$ is quasipolar in T with the spectral idempotent $\begin{pmatrix} 0 & 0 \\ n_1 & 1 \end{pmatrix}$ for some $n_1 \in N$.*

Corollary 12. *Let $R,\ S$ be both local rings and ${}_RM_S$ is a bimodule. Write $T_1 = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ which is the formal upper triangular matrix ring. Then for any $u \in U(R),\ j \in J(S)$, the following conditions are equivalent:*

(1) *$l_u - r_j$ of M is an isomorphism.*

(2) *For any $m \in M$, $A := \begin{pmatrix} u & m \\ 0 & j \end{pmatrix}$ is quasipolar in T_1 with the spectral idempotent $\begin{pmatrix} 0 & m_1 \\ 0 & 1 \end{pmatrix}$ for some $m_1 \in M$.*

Corollary 13. [3, Lemma 2.4] *Let R be a local ring. Write $T_2 = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ which is the upper triangular matrix ring. Then for any $u \in U(R),\ j \in J(R)$, the following conditions are equivalent:*

(1) *$l_u - r_j$ of R is an isomorphism.*

(2) *For any* $r \in R$, $A := \begin{pmatrix} u & r \\ 0 & j \end{pmatrix}$ *is quasipolar in* T_2 *with the spectral idempotent* $\begin{pmatrix} 0 & e_{12} \\ 0 & 1 \end{pmatrix}$ *for some* $e_{12} \in R$.

Theorem 14. *Let* R, S *be both local rings, and let* $u \in U(R)$, $j \in J(S)$. *The following are equivalent:*

(1) $l_u - r_j$ *of* M *and* $r_u - l_j$ *of* N *are both isomorphic.*

(2) *For any* $m \in M$, $n \in N$, $A = \begin{pmatrix} u & m \\ n & j \end{pmatrix}$ *is quasipolar in* T *with the spectral idempotent* $\begin{pmatrix} 0 & m_1 \\ n_1 & 1 \end{pmatrix}$ *for some* $m_1 \in M$, $n_1 \in N$.

Proof. (1) $\Rightarrow$ (2) By the hypothesis that $l_u - r_j$ of M and $r_u - l_j$ of N are both surjective, for any $m \in M$, $n \in N$ there exist $m_1 \in M$, $n_1 \in N$ such that $um_1 - m_1 j = -m$, $n_1 u - jn_1 = -n$. Set $E = \begin{pmatrix} 0 & m_1 \\ n_1 & 1 \end{pmatrix}$. It is not difficult to check that $E^2 = E$, $EA = AE \in J(T)$ and $E + A \in U(T)$. Following from $l_u - r_j$ of M and $r_u - l_j$ of N are both injective, we have E is the unique idempotent of T which satisfies all of the conditions of $EA = AE, EA^k \in J(T)$ for some positive integer k and $E + A \in U(T)$. Again by Lemma 2, E is the spectral idempotent of A in T.

(2) $\Rightarrow$ (1) By Theorem 10 and Theorem 11. □

Theorem 15. *Let* R, S *be both local rings. If the following conditions are satisfied:*

(1) *For any* $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ *of* M *and* $r_u - l_j$ *of* N *are both surjective;*

(2) *For any* $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ *of* M *and* $l_u - r_j$ *of* N *are both surjective.*

Then the following are equivalent:

(1) *For any* $u \in U(R)$, $j \in J(S)$, $l_u - r_j$ *of* M *and* $r_u - l_j$ *of* N *are both injective; And for any* $u \in U(S)$, $j \in J(R)$, $r_u - l_j$ *of* M *and* $l_u - r_j$ *of* N *are both injective.*

(2) T *is a quasipolar ring.*

Corollary 16. *Let* R, S *be both local rings. If the following conditions are satisfied:*

(1) *For any* $u \in U(R),\ j \in J(S)$, $l_u - r_j$ *of* M *and* $r_u - l_j$ *of* N *are both isomorphic;*

(2) *For any* $u \in U(S),\ j \in J(R)$, $r_u - l_j$ *of* M *and* $l_u - r_j$ *of* N *are both isomorphic.*

Then T *is a quasipolar ring.*

Corollary 17. *Let* R *be a bleached local ring. Then* R *is co-bleached if and only if* $K_0(R)$ *is a quasipolar ring.*

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