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A NOTE ON THE COMPUTATION OF INCIDENCE MATRICES OF SIMPLICIAL COMPLEXES

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Abstract: Incidence Matrices have been defined only for graphs. In this article we present a method to compute the incidence matrices for simplicial complexes.

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1. Introduction

H. Poincaré specifically emphsized the application of incidence matrices in his article [3]. Thereafter, Incidence Matrices are being used as a tool from linear algebra to derive important properties of long and complicated graphs. In this article, we generalise the notion of (vertex-edge) incidence matrix to $(\sigma^n - \sigma^{n-1})$ incidence matrix where σ^{n-1} and σ^n are $n-1$ and n dimensional simplexes of an orientated simplicial complex. We define these incidence matrices by using a concept from algebraic topology called incidence number [1]. In this article we give an alternative formula using incidence numbers. The entries of these incidence matrices are the incidence numbers of oriented simplexes which differ

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by 1 in their dimensions. We show that the alternative formula is equivalent to the definition of the (vertex-edge) incidence matrix of a graph as defined in (Chapter 2, [2]) . The importance of this formula is that it can be generalised to higher diemensions of directed graph i.e., oriented simplicial complex.

2. Incidence Number

Definition 1. [1] The **incidence number** associated with two simplexes which differ by 1 in their dimension (σ^{p+1}, σ^p) of an oriented simplicial complex K denoted by $[\sigma^{p+1}, \sigma^p]$ is defiend as follows:

(a) If σ^p is not a face of σ^{p+1}, we put $[\sigma^{p+1}, \sigma^p] = 0$.

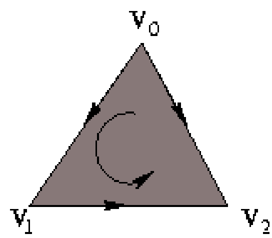
(b) If σ^p is a face of σ^{p+1}, This means all the vertices of σ^p are also the vertices of σ^{p+1} except one vertex as σ^{p+1} has $p+2$ vertices and σ^p has $p+1$ vertices. Let us label the vertices of σ^p so that $< v_0, v_1, \cdots, v_p >$ gets positive orientation. Let v be the additional vertex of σ^{p+1}. Then putting v as the first vertex $< v, v_0, \cdots, v_p >$, we get either σ^{p+1} is positively oriented or negatively oriented. We define

$$[\sigma^{p+1}, \sigma^p] = \begin{cases} 1 & \text{if} < v, v_0, \cdots, v_p >= +\sigma^{p+1} \\ -1 & \text{if} < v, v_0, \cdots, v_p >= -\sigma^{p+1}. \end{cases}$$

The above definition says that the incidence number $[\sigma^{p+1}, \sigma^p]$ of two simplexes where one is the face of other and their dimensions differ by 1 is either $+1$ or -1 depending on the orientation σ^{p+1} gets by putting the additional vertex at first order. If σ^{p+1} gets positive orientation then the incidence number is $+1$ or else -1.

Example 2.1. Let K the closure of σ^2, where $\sigma^2 =< v_0, v_1, v_2 >$, be oriented by the ordering $v_0 < v_1 < v_2$.Then the incidence numbers of simplices can be calculated as follows.

$$\begin{array}{cc} [\sigma^2, < v_0, v_1 >] = +1, & [< v_0, v_1 >, < v_1 >] = +1, \\ [\sigma^2, < v_1, v_2 >] = +1, & [< v_0, v_1 >, < v_0 >] = -1, \\ [\sigma^2, < v_0, v_2 >] = -1, & [< v_0, v_1 >, < v_2 >] = 0. \end{array}$$



3. Incidence Matrix of a Graph

A directed graph can be viewed as an oriented 1-dimensional simplicial complex with edges as 1-simplexes and the vertices as 0-simplexes, In this section we prove that an alternative formula can be given to the definition of incidence matrix of a graph which can be generalised to the higher dimensional oriented simplicial complexes.

The incidence matrix of a directed graph is is defined as follows:

Definition 2. [2] The incidence matrix of a directed graph G is a $p \times q$ matrix (a_{ij}) where p and q are the number of vertices and edges respectively, such that $a_{ij} = -1$ if the edge e_j leaves vertex v_i, $(a_{ij}) = 1$ if the edge e_j enters the vertex v_i and 0 otherwise.

Let us consider the following example:

Example 3.1. Consider the directed graph G shown in figure 3.1, with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$. The incidence matrix of G is given by

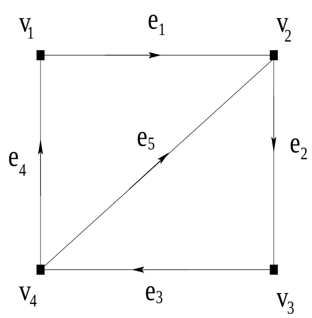


Figure 3.1: Directed Graph G

$$E = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}. \qquad (*)$$

Notations. As a directed graph as 1-dimensional oriented simplicial complex. We denote the edges as 1-simplexes i.e. $\{\sigma_1^1, \sigma_2^1 \cdots \sigma_n^1\}$ and all the vertex as 0-simplexes namely, $\{\sigma_0^0, \sigma_1^0 \cdots \sigma_m^0\}$. We now prove the following Theorem.

Theorem 3. *The incidence matrix of a directed graph G with m vertices and n edges is the following matrix*

$$I_G = (a_{ij})_{m\times n}$$

where a_{ij} is the incidence number $[\sigma_j^1, \sigma_i^0]$. Where σ_j^1 is the j$^{\text{th}}$ *directed edge and σ_i^0 is* i$^{\text{th}}$ *vertex.*

Proof. Let us lebel all the edges and vertices of the graph as given in notation abve. If any two vertex are non-adjacent i.e., they are not connected by any edge then their incidence number is zero. Let the 1-simplex σ_j^1 be any arbitrary edge joining the vertices σ_i^0 and σ_j^0. If the edge σ_j^1 leaves the vertex σ_i^0 and enters the vertex σ_j^0 then the edge $\sigma_j^1 =< \sigma_i^0, \sigma_j^0 >$ gets the orientation as $\sigma_i^0 < \sigma_j^0$.

Calculating the incidence numbers $[\sigma_j^1, \sigma_i^0]$ and $[\sigma_j^1, \sigma_j^0]$ we get $[\sigma_j^1, \sigma_i^0] = -1$ which is same as a_{ij} by Definition 3.1 and $[\sigma_j^1, \sigma_j^0] = 1 = a_{jj}$. Thus the formula given in the statement of the theorem is equivalent to the definition of the incidence matrix. □

Example 3.2. Consider the directed graph G as in Example 3.1. We calculate the incidence matrix of G by using incidence number. By above Theorem we get

$$E = \begin{pmatrix} [e_1, v_1] & [e_2, v_1] & [e_3, v_1] & [e_4, v_1] & [e_5, v_1] \\ [e_1, v_2] & [e_2, v_2] & [e_3, v_2] & [e_4, v_2] & [e_5, v_2] \\ [e_1, v_3] & [e_2, v_3] & [e_3, v_3] & [e_4, v_3] & [e_5, v_3] \\ [e_1, v_4] & [e_2, v_4] & [e_3, v_4] & [e_4, v_4] & [e_5, v_4] \end{pmatrix},$$

$$E = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

Thus both the defintions are equivalent.

4. Incidence Matrices of Oriented Simplicial Complex

In this section, we generalize the formula obtained for computing the incidence matrix of a graph in Theorem 3.1 to the oriented simplicial complex K.

Definition 4. Let K be a oriented simplicial complex of dimension k. Let the total number of $k-1$ simplexes be m and the total number of k simplexes be n Then incidence matrix of order k, I_k is defined as follows:

$$I_k = M_{m\times n} = (a_{ij})_{m\times n}.$$

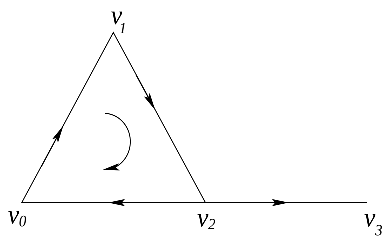
Here

$$a_{ij} = [\sigma_j^k, \sigma_i^{k-1}] \quad \text{for } i = 1, \cdots, m,\ j = 1, \cdots, n,$$

and $[\sigma_j^k, \sigma_i^{k-1}] =$ the incidence number of pair $(\sigma_j^k, \sigma_i^{k-1})$.

Consider the following example.

Example 4.1. Let K be the following simplicial complex We observe that the dimension of K is 2 as the largest simplex it has is a 2-simplex namely, $< v_0, v_1, v_2 >$. In the simplicial complex K we note that K has only one 2-simplex, four 1-simplex and four 0-simplex. We give an orientation to K as follows $v_0 < v_1 < v_2 < v_3$.



Simplicial Complex K

First we define the incidence Matrix I_2 as the matrix $M_{m\times n}$ where n is the number of 2-simplices i.e. $n = 1$ and m is the number of 1-simplices i.e., $m = 4$. Thus in this example the incidence matrix I_2 is the Matrix $M_{4\times 1}$.

We name all the simplices as follows:

$$\begin{aligned} \sigma^2 &= < v_0, v_1, v_2 >, \\ \sigma_1^1 &= < v_0, v_1 >, \end{aligned}$$

$$\begin{aligned}\sigma_2^1 &= < v_1, v_2 >,\\ \sigma_3^1 &= < v_2, v_3 >,\\ \sigma_4^1 &= < v_2, v_0 > .\end{aligned}$$

Now we define I_2 as follows:

$$I_2 = M_{4\times 1} = \begin{pmatrix}[\sigma^2, \sigma_1^1]\\ [\sigma^2, \sigma_2^1]\\ [\sigma^2, \sigma_3^1]\\ [\sigma^2, \sigma_4^1]\end{pmatrix} = \begin{pmatrix}[< v_0, v_1, v_2 >, < v_0, v_1 >]\\ [< v_0, v_1, v_2 >, < v_1, v_2 >]\\ [< v_0, v_1, v_2 >, < v_2, v_3 >]\\ [< v_0, v_1, v_2 >, < v_2, v_0 >]\end{pmatrix} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}.$$

Here $[\sigma^2, \sigma_i^1]$ denotes the incidence number of the pair (σ^2, σ_i^1).

Similarly, we will define the incidence matrix I_1. We find that in K there are four 1-simplex and four 0-simplex. Therefor by above argument the order of the incidence matrix I_1 will be 4×4. Proceeding same as above, we get

$$I_1 = M_{4\times 4} = \begin{pmatrix}[\sigma_1^1, \sigma_1^0] & [\sigma_2^1, \sigma_1^0] & [\sigma_3^1, \sigma_1^0] & [\sigma_4^1, \sigma_1^0]\\ [\sigma_1^1, \sigma_2^0] & [\sigma_2^1, \sigma_2^0] & [\sigma_3^1, \sigma_2^0] & [\sigma_4^1, \sigma_2^0]\\ [\sigma_1^1, \sigma_3^0] & [\sigma_2^1, \sigma_3^0] & [\sigma_3^1, \sigma_3^0] & [\sigma_4^1, \sigma_3^0]\\ [\sigma_1^1, \sigma_4^0] & [\sigma_2^1, \sigma_4^0] & [\sigma_3^1, \sigma_4^0] & [\sigma_4^1, \sigma_4^0]\end{pmatrix}$$

$$= \begin{pmatrix}-1 & 0 & 0 & 1\\ 1 & -1 & 0 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

Thus we have an alternative formula for computing the incidence matrix of directed graphs which is generalised to compute the incidence matrices of higher orders for an oriented simplicial complex.

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