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A CLASS OF SPANNED, SIMPLE AND BIUNIFORM VECTOR BUNDLES ON THE SMOOTH QUADRIC SURFACE

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Abstract: We study a class of spanned, simple and biuniform vector bundles on a smooth quadric surface Q (the duals of general surjections $\mathcal{O}_Q^{(r+1)} \to \mathcal{O}_Q(a,b)$).

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1. Introduction

Let $Q := \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth quadric surface. In this note we study the following family of spanned vector bundles on Q. Fix positive integers a, b, r such that $2 \le r \le ab + a + b$. Let $S(a, b, r)$ be the set of all surjective maps $\mathcal{O}_Q^{\oplus(r+1)} \to \mathcal{O}_Q(a, b)$. Since $r + 1 \ge 3 > \dim(Q)$, a dimensional count gives that the set $S(a, b, r)$ is a non-empty open subset of the vector space $H^0(\mathcal{O}_Q(a, b)^{\oplus(r+1)})$ and hence it is an integral variety of dimension $(a+1)(b+1)(r+1)$. For each $\phi \in S(a, b, r)$ the sheaf $\ker(\phi)$ is a vector bundle on Q with determinant isomorphic to $\mathcal{O}_Q(-a, -b)$. The sheaf $\ker(\phi)$ is a subsheaf

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of $\mathcal{O}_Q^{\oplus(r+1)}$ and $\mathcal{O}_Q^{\oplus(r+1)}/\ker(\phi) \cong \mathcal{O}_Q(a,b)$. Since the dual of a short exact sequence of vector bundles is exact, the vector bundle $\ker(\phi)^\vee$ is spanned. Let $G(a,b,r)$ be the set of all vector bundles $\{\ker(\phi)^\vee\}_{\phi\in S(a,b,r)}$. Any $E \in G(a,b,r)$ has rank r, $c_1(E) = (a,b)$, and it is spanned by its global sections. Since E is spanned, then $h^2(E) = h^0(E^\vee(-2,-2)) = 0$. It fits in an exact sequence

$$0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}_Q^{\oplus(r+1)} \to E \to 0.$$

Since $a > 0$ and $b > 0$, we have $h^i(\mathcal{O}_Q(-a,-b)) = 0$, $i = 0,1$. Hence $h^0(E) = r+1$ and $h^1(E) = h^2(\mathcal{O}_Q(-a,-b)) = (a-1)(b-1)$. We may do the same definition even if $r \geq (a+1)(b+1)$, but if $r \geq (a+1)(b+1)$, then $E \in G(a,b,r)$ $\Leftrightarrow$ $E \cong F \oplus \mathcal{O}_Q^{\oplus(r-ab-a-b)}$ for some $F \in G(a,b,ab+a+b)$. Assume again $2 \leq r \leq ab+a+b$. Let $S'(a,b,r)$ be the set of all $f \in S(a,b,r)$ associated to $r+1$ linearly independent sections of $\mathcal{O}_Q(a,b)$. Let $G'(a,b,r)$ the the set of all vector bundles $\{\ker(\phi)^\vee\}_{\phi\in S'(a,b,r)}$. It is easy to check that $E \in G'(a,b,r)$ if and only if $E \in G(a,b,r)$ and $\mathcal{O}_Q$ is not a factor of E.

Let E be a rank r vector bundle on Q. We say that E is *biuniform* if there are integers $a_1 \geq \cdots \geq a_r$ and $b_1 \geq \cdots \geq b_r$ such that $E|D$ has splitting type $(a_1,\cdots,a_r)$ for all $D \in |\mathcal{O}_Q(0,1)|$ and splitting type $(b_1,\ldots,b_r)$ for all $T \in |\mathcal{O}_Q(1,0)|$ ([1]). We say that a splitting type $c_1 \geq \cdots \geq c_r$ is *balanced* if $c_1 \leq c_r + 1$. In this note we prove the following result.

Theorem 1. *Assume* $2 \leq r \leq ab+a+b$.

1. *Every* $E \in G'(a,b,r)$ *is simple.*
2. *Assume that neither* a *nor* b *is divisible by* r. *Then a general* $E \in G(a,b,r)$ *is biuniform with balanced splitting types.*

Part 1 easily follows from the fact that the spanned bundle E has no trivial factor and that $\mathrm{rank}(E) = h^0(E) - 1$ (see step (a) of the proof of Theorem 1). Part 2 is related to the following result concerning simple bundles. For all $c_1 \in \mathrm{Pic}(Q)$, $c_2 \in \mathbb{Z}$ and $r \geq 2$ let $\mathcal{S}(c_1,c_2;r)$ denote the moduli space of all rank r simple vector bundles F on Q with $c_1(F) = c_1$ and $c_2(F) = c_2$. It exists ([3], [4]), although it may be empty for some c_1, c_2, r. It is everywhere smooth (Remark 2). We need to adapt to the bundles in $G'(a,b,r)$ a proof of the following well-known result (true because the anticanonical line bundle $\omega_Q^\vee \cong \mathcal{O}_Q(2,2)$ is ample and spanned).

Proposition 1. *Fix* c_1, c_2, r *such that* $r \geq 2$ *and* $\mathcal{S}(c_1,c_2,r) \neq \emptyset$. *Take* $c_1 = (a,b)$ *and assume that neither* a *nor* b *is divisible by* r. *Let* $\mathcal{S}$ *be any connected component of* $\mathcal{S}(c_1,c_2,r)$. *Then a general* $E \in \mathcal{S}$ *is biuniform.*

When b is large we also show how to construct many non-biuniform vector bundles $E \in G'(a,b,r)$ with prescribed splitting type at several elements of $|\mathcal{O}_Q(0,1)|$ (Proposition 2).

Remark 1. There is a unique bundle $E \in G'(a,b,ab+a+b)$ (up to isomorphisms), because any $E \in G'(a,b,ab+a+b)$ is induced by the complete linear system $|\mathcal{O}_Q(a,b)|$. Hence $f^*(E) \cong E$ for all $f \in \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$ (the connected component of the identity of the algebraic group $\mathrm{Aut}(Q)$). If $a = b$, then $\sigma^*(E) \cong E$ for the order two automorphism of Q which exchanges the two rulings of Q. Take $r \geq (a+1)(b+1)$ and a general $A \in G(a,b,r)$. Since $A \cong E \oplus \mathcal{O}_Q^{\oplus(r-ab-a-b)}$ for some $E \in G'(a,b,ab+a+b)$, we get that A is biuniform (but not simple).

2. The Proof

We fix integers a, b, r such that $a > 0$, $b > 0$ and $2 \leq r \leq ab+a+b$. Let A be a rank r vector bundle on $\mathbb{P}^1$. The splitting type of A is balanced if and only if $h^1(A \otimes A^\vee) = 0$.

Lemma 1. *Let E be a simple vector bundle on Q. Then $h^2(End(E)(c,d)) = 0$ for all integers $c \geq -1$ and $d \geq -1$.*

Proof. Since E is simple, every element of $H^0(End(E))$ is induced by the multiplication by a scalar. Hence $h^0(End(E)(-2-c,-2-d)) = 0$ for all $c \geq -1$ and $d \geq -1$. □

Remark 2. Let E be a simple vector bundle. Set $c_i := c_i(E)$ and $r := \mathrm{rank}(E)$. Since $h^2(End(E)) = 0$ (Lemma 1), $\mathcal{S}(c_1,c_2,r)$ is smooth at E and of dimension $h^1(End(E))$.

Proof of Proposition 1. Since $\mathcal{S}$ is smooth (Remark 2) and connected, it is irreducible. Hence it is sufficient to prove the existence of non-empty open subsets U_1 and U_2 of $\mathcal{S}$ such that each $E_i \in U_i$ is uniform with respect to the ruling of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to the i-th projection $Q \to \mathbb{P}^1$. Just to fix the notation we prove the existence of this open subset of $\mathcal{S}$ with respect to the projection $\pi : Q \to \mathbb{P}^1$ such that $\mathcal{O}_Q(0,1) \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Write $c_1 = \mathcal{O}_Q(u,v)$. By assumption $u/r \notin \mathbb{Z}$. For all $u_1 \geq \cdots \geq u_r$ such that $u_1 + \cdots + u_r$ and each $D \in |\mathcal{O}_Q(0,1)|$ let $\mathcal{S}(D;u_1,\dots,u_r)$ be the set of all $E \in \mathcal{S}$ such that $E|D$ has splitting type $u_1,\dots,u_r$. Since $\mathcal{S}$ is an algebraic variety, the semicontinuity theorem for cohomology gives that $\mathcal{S}(D;u_1,\dots,u_r) \neq \emptyset$ only for finitely many splitting types. Since $\dim(|\mathcal{O}_Q(0,1)|) = 1$, to conclude the proof it is sufficient to prove that for each unbalanced splitting type $u_1, \dots, u_r$ and for each $D \in |\mathcal{O}_Q(0,1)|$ the set $\mathcal{S}(D; u_1, \dots, u_r)$ has codimension at least two in $\mathcal{S}$. Set $F := \oplus_{i=1}^{r} \mathcal{O}_D(u_i)$ and call $\mathcal{U}$ a versal deformation space of F (smooth of dimension $h^1(D, End(F))$ with a prescribed point $o \in \mathcal{U}$ corresponding to F. Since $u/r \notin \mathbb{Z}$ and $u_1, \dots, u_r$ is unbalanced, we have $h^1(D, End(F)) \geq 2$. Fix any $E \in \mathcal{S}(D; u_1, \dots, u_r)$. Since $h^2(E(0,-1)) = 0$ (Lemma 1), the restriction map $H^1(End(E)) \to H^1(D, End(F))$ is surjective. Hence $\mathcal{S}(D; u_1, \dots, u_r)$ has codimension $h^1(D, End(F))$ in $\mathcal{S}$. Since $h^1(D, End(F)) \geq 2$, we are done. □

Lemma 2. *Fix $D \in |\mathcal{O}_Q(0,1)|$ and integers $a_1 \geq \cdots \geq a_r \geq 0$ such that $a_1 + \cdots + a_r = a$. Set $F := \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$. Let A be the set of all surjections $\psi : \mathcal{O}_{\mathbb{P}^1}^{\oplus(r+1)} \to \mathcal{O}_{\mathbb{P}^1}(a)$ such that $ker(\psi) \cong F^\vee$. Then $A \neq \emptyset$. Fix $\psi \in A$. Let B (resp. B') be the set of all $E \in G(a,b,r)$ (resp. $E \in G'(a,b,r)$) of the form $ker(u)^\vee$ with $u : \mathcal{O}_Q^{\oplus(r+1)} \to \mathcal{O}_Q(a,b)$ a surjection and $u|D = \psi$. Then B is an irreducible variety of dimension $(r+1)(a+1)b$ and B' is a non-empty Zariski open subset of B.*

Proof. Every spanned vector bundle G on an integral variety T is spanned by at most $\dim(T) + \mathrm{rank}(G)$ sections (wait a few lines for the proof of this claim). Hence $A \neq \emptyset$. Fix $\psi \in A$. Let B'' be the set of all maps $f : \mathcal{O}_Q^{r+1} \to \mathcal{O}_Q(a,b)$ such that $a|D = \psi$. Since $b > 0$, $B'' \neq \emptyset$ and B'' is an irreducible variety of dimension $(r+1) \cdot h^0(\mathcal{O}_Q(a, b-1)) = (r+1)(a+1)b$. Notice that B is the set of all surjective elements of B''. Every $f \in B''$ is surjective at the points of P. Fix $P \in Q \setminus D$. Set $B_P := \{f \in B'' : f(P) \equiv 0\}$. Since $\mathcal{O}_Q(a, b-1)$ is spanned, B_P has codimension $r+1$ in B''. Since $\dim(Q) = 2 < r+1$, we get $B \neq \emptyset$. If $a_r > 0$, then $B' = B$. Now assume $a_r = 0$ and that $B' = \emptyset$. We get that all vector bundles associated to some element of B have $\mathcal{O}_Q$ as a factor. Taking out the factors $\mathcal{O}_D(a_r) = \mathcal{O}_D$ from F, we win counting the dimension for the data $a, b, r' := r-1$ and $F' := \oplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$, if we can do the case $r = 2$. Assume the existence of $E \in (G(a,b,2) \setminus G'(a,b,2))$ and write $E \cong G \oplus \mathcal{O}_Q$. We get $G \cong \mathcal{O}_Q(a,b)$ and hence $c_2(E) = 0$, a contradiction. □

Let $S(a;r)$ be the set of all surjective maps $\mathcal{O}_{\mathbb{P}^1}^{\oplus(r+1)} \to \mathcal{O}_{\mathbb{P}^1}(a)$. Since $r+1 \geq 3 > \dim(\mathbb{P}^1)$, the set $S(a;r)$ is a non-empty open subset of the vector space $H^0(\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus(r+1)})$ and hence it is an integral variety of dimension $(a+1)(r+1)$. For each $\phi \in S(a;r)$ the sheaf $\ker(\phi)$ is a vector bundle on $\mathbb{P}^1$ with determinant isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a)$. The sheaf $\ker(\phi)$ is a subsheaf of $\mathcal{O}_{\mathbb{P}^1}^{\oplus(r+1)}$ and $\mathcal{O}_{\mathbb{P}^1}^{\oplus(r+1)}/\ker(\phi) \cong \mathcal{O}_{\mathbb{P}^1}(a)$. Since the dual of a short exact sequence of vector bundles is exact, the vector bundle $\ker(\phi)^\vee$ is spanned. Let $G(a;r)$ be the flat family $\{\ker(\phi)^\vee\}_{\phi\in S(a;r)}$ of vector bundles on $\mathbb{P}^1$. Any $E \in G(a;r)$ has rank r and it is spanned. Hence the splitting type $a_1 \ge \cdots \ge a_r$ of any $E \in G(a;r)$ satisfies the conditions $a_r \ge 0$ and $a_1 + \cdots + a_r = a$.

Lemma 3. *Fix integers $a_1 \ge \cdots \ge a_r$ such that $a_1 + \cdots + a_r = a$ and set $F := \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$. The set of all $E \in G(a;r)$ isomorphic to F is non-empty and it has dimension $(r+1)(a+1) - h^1(End(F))$.*

Proof. We have $h^0(F) = a + r$. Hence the Grassmannian of all $(r+1)$-dimensional linear subspaces of $H^0(F)$ has dimension $(r+1)(a-1)$. For any rank r vector bundle A on $\mathbb{P}^1$ we have $\chi(End(A)) = r^2$, because $End(A)$ has rank r^2 and degree 0. Hence $h^0(End(F)) = h^1(End(F)) + r^2$. We have $\dim(Aut(F)) = h^0(End(F))$. Two injective maps with locally free cokernels $u_i : F \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(r+1)}$, $i = 1,2$, have the same image if and only if $u_2 = u_1 \circ w$ for some automorphism w of F. □

We leave to the reader the proof of the following elementary lemma.

Lemma 4. *Let T be an integral projective variety. Let E be a rank r spanned torsion free sheaf on T. E is a trivial vector bundle if and only if it is spanned by an r-dimensional linear subspace of $H^0(E)$.*

Proof of Theorem 1. Fix $E \in G'(a,b,r)$ (it exists, because $r \le ab + a + b$). Since E is spanned and without trivial factors, we have $h^0(E^\vee) = 0$, i.e. there is no non-zero map $E \to \mathcal{O}_Q$.

(a) Assume that E is not simple. Hence there is a non-zero map $f : E \to E$ such that $G := Im(f)$ has rank $m < r$. Since $G \subset E$, G is torsion free. Let $V \subseteq H^0(G)$ be the image of the map $u : H^0(E) \to H^0(G)$ induced by the surjection $E \to G$. Since $G = f(E)$ is a quotient of the spanned sheaf E, the sheaf G is spanned by V. Set $W := \ker(u) \subseteq H^0(E)$. Since every map $E \to \mathcal{O}_Q$ is trivial, G is not a trivial vector vector bundle. Hence $\dim(V) \ge m+1$ (Lemma 4). Hence $\dim(W) \le r - m$. Let A be the saturation of G in E, i.e. the only rank m subsheaf of E containing G and such that E/A has no torsion. The sheaf E/A has rank $r - m$. Since E/A is a quotient of E/G, it is spanned by a quotient of W. Since $\dim(W) \le r - m$, the torsion free sheaf E/A is a trivial vector bundle (Lemma 4). Hence E has $\mathcal{O}_Q^{r-m}$ as a factor, a contradiction.

(b) In this step we assume that neither a nor b is divisible by r and prove that a general $E \in G'(a,b,r)$ is biuniform. We adapt the proofs in [2]. We fix one of the two rulings of Q, say $|\mathcal{O}_Q(0,1)|$, since the proof for the other ruling is similar. Fix $D \in |\mathcal{O}_Q(0,1)|$. Fix integers $a_1 \ge \cdots \ge a_r \ge 0$ such that $a_1 + \cdots + a_r = a$ and $a_r \le a_1 - 2$. Call $u : G'(a,b,r) \times |\mathcal{O}_Q(0,1)| \to |\mathcal{O}_Q(0,1)|$ the projection onto the second factor. Let $\Gamma(a_1,\dots,a_r)$ be the set of all $(E,D) \in G'(a,b,r) \times |\mathcal{O}_Q(0,1)|$ such that $E|D$ has splitting type $(a_1,\dots,a_r)$. We have $\Gamma(a_1,\dots,a_r) = \emptyset$ if $a_r < 0$. Hence we only have finitely may non-empty sets $\Gamma(a_1,\dots,a_r)$. Since $\dim(|\mathcal{O}_Q(0,1)|) = 1$, to prove that a general $E \in G'(a,b,r)$ is uniform with respect to the ruling associated to $|\mathcal{O}_Q(0,1)|$ it is sufficient to prove that each non-empty $\Gamma(a_1,\dots,a_r)$ has codimension ≥ 2 in $G'(a,b,r) \times |\mathcal{O}_Q(0,1)|$. Fix $D \in |\mathcal{O}_Q(0,1)|$. It is sufficient to prove that $\dim(u^{-1}(D)) \le \dim(G'(a,b,r)) - 2$. Set $F := \oplus_{i=1}^r \mathcal{O}_D(a_i)$. Lemmas 2 and 3 give that $u^{-1}(D)$ has dimension $\dim(G'(a,b,r)) - h^1(End(F))$. Since F is unbalanced and $a/r \notin \mathbb{Z}$, we have $h^1(End(F)) \ge 2$. □

Lemma 5. *Let E be a rank r vector bundle on Q. Fix $T \in |\mathcal{O}_Q(1,0)|$ and let $b_1 \ge \cdots \ge b_r$ be the splitting type of $E|T$. Assume the existence of $c \in \mathbb{Z}$ such that $E|D$ has splitting type $(c,\dots,c)$ for all $D \in |\mathcal{O}_Q(0,1)|$. Then $E \cong \oplus_{i=1}^r \mathcal{O}_Q(c,b_i)$.*

Proof. Taking $E(-c,0)$ instead of E we reduce to the case $c = 0$. Let $\pi : Q \to \mathbb{P}^1$ denote the projection such that $\mathcal{O}_Q(0,1) \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Since $h^0(E|D) = r$ and $h^1(E|D) = 0$ for every fiber D of π, a theorem of base change gives that $\pi_*(E)$ is a rank r vector bundle ([5], page 11). We have $\pi_*(E) \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(d_i)$ for some integers $d_1 \ge \cdots \ge d_r$. For each fiber D of π the natural map $H^0(D, \pi^*(\pi_*(E))|D) \to H^0(D, E|D)$ is an isomorphism ([5], page 11). Since $E|D$ is trivial, we get that the natural map $\pi^*(\pi_*(E)) \to E$ is an isomorphism (see [5], page 53, for a similar proof). Since $E|T$ has splitting type $b_1 \ge \cdots \ge b_r$, we get $d_i = b_i$ for all i. □

Remark 3. Lemma 5 shows that if either $a/r \in \mathbb{Z}$ or $b/r \in \mathbb{Z}$, then every biuniform vector bundle with balanced splitting type is isomorphic to a direct sum of r line bundles.

Proposition 2. *Fix positive integers a, b, r, c, such that $r \ge 2$, $b > c$, $a_1(i) \ge \cdots \ge a_r(i) \ge 0$, $1 \le i \le c$, and $a_1(i) + \cdots + a_r(i) = a$ for all $i \in \{1,\dots,c\}$. Fix $D_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le c$, with $D_i \ne D_h$ for all $i \ne h$,. Then there is $E \in G'(a,b,r)$ such that $E|D_i$ has splitting type $a_1(i) \ge \cdots \ge a_r(i)$ for all i.*

Proof. Set $T := D_1 \cup \cdots \cup D_c$. Let G be the vector on D such that $G|D_i = \mathcal{O}_{D_i}(a_h(i))$ for all i. Fix a surjection $\psi : \mathcal{O}_T^{\oplus(r+1)} \to G$ (it exists, because G is spanned and $\dim(T) + \mathrm{rank}(G) = r+1$). Since $b - c > 0$, the proof of Lemma 2 works with T instead of D. □

Assume $a \geq 2$ and $b \geq 2$ and fix integers $a_1 \geq \cdots \geq a_r \geq 0$ such that $a_1 + \cdots + a_r = a + b$. Fix a smooth $T \in |\mathcal{O}_Q(1,1)|$. The proof of Proposition 2 shows the existence of $E \in G'(a,b,r)$ such that $E|T$ has splitting type $a_1, \ldots, a_r$.

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