A CLASS OF SPANNED, SIMPLE AND BIUNIFORM VECTOR BUNDLES ON THE SMOOTH QUADRIC SURFACE

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Abstract: We study a class of spanned, simple and biuniform vector bundles on a smooth quadric surface $Q$ (the duals of general surjections $\mathcal{O}_Q^{(r+1)} \to \mathcal{O}_Q(a,b)$).

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1. Introduction

Let $Q := \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth quadric surface. In this note we study the following family of spanned vector bundles on $Q$. Fix positive integers $a, b, r$ such that $2 \leq r \leq ab + a + b$. Let $S(a, b, r)$ be the set of all surjective maps $\mathcal{O}_Q^{\oplus(r+1)} \to \mathcal{O}_Q(a,b)$. Since $r + 1 \geq 3 > \dim(Q)$, a dimensional count gives that the set $S(a, b, r)$ is a non-empty open subset of the vector space $H^0(\mathcal{O}_Q(a,b)^{\oplus(r+1)})$ and hence it is an integral variety of dimension $(a + 1)(b + 1)(r + 1)$. For each $\phi \in S(a, b, r)$ the sheaf $\ker(\phi)$ is a vector bundle on $Q$ with determinant isomorphic to $\mathcal{O}_Q(-a, -b)$. The sheaf $\ker(\phi)$ is a subsheaf...
of $\mathcal{O}^{(r+1)}_Q$ and $\mathcal{O}^{(r+1)}_Q/\ker(\phi) \cong \mathcal{O}_Q(a,b)$. Since the dual of a short exact sequence of vector bundles is exact, the vector bundle $\ker(\phi)\nu$ is spanned. Let $G(a,b,r)$ be the set of all vector bundles $\{\ker(\phi)\nu\}_{\phi \in S(a,b,r)}$. Any $E \in G(a,b,r)$ has rank $r$, $c_1(E) = (a,b)$, and it is spanned by its global sections. Since $E$ is spanned, then $h^2(E) = h^0(E\nu(-2,-2)) = 0$. It fits in an exact sequence

$$0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}^{(r+1)}_Q \to E \to 0.$$ 

Since $a > 0$ and $b > 0$, we have $h^i(\mathcal{O}_Q(-a,-b)) = 0$, $i = 0,1$. Hence $h^0(E) = r + 1$ and $h^1(E) = h^2(\mathcal{O}_Q(-a,-b)) = (a - 1)(b - 1)$. We may do the same definition even if $r \geq (a + 1)(b + 1)$, but if $r \geq (a + 1)(b + 1)$, then $E \in G(a,b,r)$ if and only if $E \in G'(a,b,r)$ if and only if $E \in G(a,b,r)$ and $\mathcal{O}_Q$ is not a factor of $E$.

Let $E$ be a rank $r$ vector bundle on $Q$. We say that $E$ is biuniform if there are integers $a_1 \geq \cdots \geq a_r$ and $b_1 \geq \cdots \geq b_r$ such that $E|D$ has splitting type $(a_1, \ldots, a_r)$ for all $D \in |\mathcal{O}_Q(0,1)|$ and splitting type $(b_1, \ldots, b_r)$ for all $T \in |\mathcal{O}_Q(1,0)|(1)]$. We say that a splitting type $c_1 \geq \cdots \geq c_r$ is balanced if $c_1 \leq c_r + 1$. In this note we prove the following result.

**Theorem 1.** Assume $2 \leq r \leq ab + a + b$.

1. Every $E \in G'(a,b,r)$ is simple.

2. Assume that neither $a$ nor $b$ is divisible by $r$. Then a general $E \in G(a,b,r)$ is biuniform with balanced splitting types.

Part 1 easily follows from the fact that the spanned bundle $E$ has no trivial factor and that $\text{rank}(E) = h^0(E) - 1$ (see step (a) of the proof of Theorem 1). Part 2 is related to the following result concerning simple bundles. For all $c_1 \in \text{Pic}(Q)$, $c_2 \in \mathbb{Z}$ and $r \geq 2$ let $S(c_1,c_2;r)$ denote the moduli space of all rank $r$ simple vector bundles $F$ on $Q$ with $c_1(F) = c_1$ and $c_2(F) = c_2$. It exists ([3], [4]), although it may be empty for some $c_1, c_2, r$. It is everywhere smooth (Remark 2). We need to adapt to the bundles in $G'(a,b,r)$ a proof of the following well-known result (true because the anticanonical line bundle $\omega_Q^\nu \cong \mathcal{O}_Q(2,2)$ is ample and spanned).

**Proposition 1.** Fix $c_1, c_2, r$ such that $r \geq 2$ and $S(c_1,c_2;r) \neq \emptyset$. Take $c_1 = (a,b)$ and assume that neither $a$ nor $b$ is divisible by $r$. Let $S$ be any connected component of $S(c_1,c_2;r)$. Then a general $E \in S$ is biuniform.
When $b$ is large we also show how to construct many non-biuniform vector bundles $E \in G'(a, b, r)$ with prescribed splitting type at several elements of $|\mathcal{O}_Q(0, 1)|$ (Proposition 2).

**Remark 1.** There is a unique bundle $E \in G'(a, b, ab + a + b)$ (up to isomorphisms), because any $E \in G'(a, b, ab + a + b)$ is induced by the complete linear system $|\mathcal{O}_Q(a, b)|$. Hence $f^*(E) \cong E$ for all $f \in \text{Aut}(\mathbb{P}^1) \times \text{Aut}(\mathbb{P}^1)$ (the connected component of the identity of the algebraic group $\text{Aut}(Q)$). If $a = b$, then $\sigma^*(E) \cong E$ for the order two automorphism of $Q$ which exchanges the two rulings of $Q$. Take $r \geq (a + 1)(b + 1)$ and a general $A \in G(a, b, r)$. Since $A \cong E \oplus \mathcal{O}_Q^{(r-2a-2b)}$ for some $E \in G'(a, b, ab + a + b)$, we get that $A$ is biuniform (but not simple).

### 2. The Proof

We fix integers $a, b, r$ such that $a > 0$, $b > 0$ and $2 \leq r \leq ab + a + b$. Let $A$ be a rank $r$ vector bundle on $\mathbb{P}^1$. The splitting type of $A$ is balanced if and only if $h^1(A \otimes A^\vee) = 0$.

**Lemma 1.** Let $E$ be a simple vector bundle on $Q$. Then $h^2(\text{End}(E)(c, d)) = 0$ for all integers $c \geq -1$ and $d \geq -1$.

**Proof.** Since $E$ is simple, every element of $H^0(\text{End}(E))$ is induced by the multiplication by a scalar. Hence $h^0(\text{End}(E)(-2-c, -2-d)) = 0$ for all $c \geq -1$ and $d \geq -1$. \hfill $\Box$

**Remark 2.** Let $E$ be a simple vector bundle. Set $c_i := c_i(E)$ and $r := \text{rank}(E)$. Since $h^2(\text{End}(E)) = 0$ (Lemma 1), $S(c_1, c_2, r)$ is smooth at $E$ and of dimension $h^1(\text{End}(E))$.

**Proof of Proposition 1.** Since $S$ is smooth (Remark 2) and connected, it is irreducible. Hence it is sufficient to prove the existence of non-empty open subsets $U_1$ and $U_2$ of $S$ such that each $E_i \in U_i$ is uniform with respect to the ruling of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to the $i$-th projection $Q \to \mathbb{P}^1$. Just to fix the notation we prove the existence of this open subset of $S$ with respect to the projection $\pi : Q \to \mathbb{P}^1$ such that $\mathcal{O}_Q(0, 1) \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Write $c_1 = \mathcal{O}_Q(u, v)$. By assumption $u/r \notin \mathbb{Z}$. For all $u_1 \geq \cdots \geq u_r$ such that $u_1 + \cdots + u_r$ and each $D \in |\mathcal{O}_Q(0, 1)|$ let $S(D; u_1, \ldots, u_r)$ be the set of all $E \in S$ such that $E|D$ has splitting type $u_1, \ldots, u_r$. Since $S$ is an algebraic variety, the semicontinuity theorem for cohomology gives that $S(D; u_1, \ldots, u_r) \neq \emptyset$ only for
Assume the existence of \( E \). We get \( B \) spanned, \( \neq B \) a balanced vector bundle associated to some element of \( \mathcal{O}_{Q}(0,1) \) with a prescribed point \( o \in U \) corresponding to \( F \). Since \( u/r \not\in \mathbb{Z} \) and \( u_{1}, \ldots, u_{r} \) is unbalanced, we have \( h^{1}(D, \text{End}(F)) \geq 2 \). Fix any \( E \in \mathcal{S}(D; u_{1}, \ldots, u_{r}) \). Since \( h^{2}(E(0,-1)) = 0 \) (Lemma 1), the restriction map \( H^{1}(\text{End}(E)) \rightarrow H^{1}(D, \text{End}(F)) \) is surjective. Hence \( \mathcal{S}(D; u_{1}, \ldots, u_{r}) \) has codimension \( h^{1}(D, \text{End}(F)) \) in \( \mathcal{S} \). Since \( h^{1}(D, \text{End}(F)) \geq 2 \), we are done.

**Lemma 2.** Fix \( D \in |\mathcal{O}_{Q}(0,1)| \) and integers \( a_{1} \geq \cdots \geq a_{r} \geq 0 \) such that \( a_{1} + \cdots + a_{r} = a \). Set \( F := \oplus_{i=1}^{r} \mathcal{O}_{D}(u_{i}) \) and call \( U \) a versal deformation space of \( F \) (smooth of dimension \( h^{1}(D, \text{End}(F)) \)). Since \( \text{dim}(\mathcal{O}_{Q}(0,1)) = 1 \), to conclude the proof it is sufficient to prove that for each unbalanced splitting type \( u_{1}, \ldots, u_{r} \) and for each \( D \in |\mathcal{O}_{Q}(0,1)| \) the set \( \mathcal{S}(D; u_{1}, \ldots, u_{r}) \) has codimension at least two in \( \mathcal{S} \). Set \( F := \oplus_{i=1}^{r} \mathcal{O}_{D}(u_{i}) \) and call \( U \) a versal deformation space of \( F \) (smooth of dimension \( h^{1}(D, \text{End}(F)) \)) with a prescribed point \( o \in U \) corresponding to \( F \). Since \( u/r \not\in \mathbb{Z} \) and \( u_{1}, \ldots, u_{r} \) is unbalanced, we have \( h^{1}(D, \text{End}(F)) \geq 2 \). Fix any \( E \in \mathcal{S}(D; u_{1}, \ldots, u_{r}) \). Since \( h^{2}(E(0,-1)) = 0 \) (Lemma 1), the restriction map \( H^{1}(\text{End}(E)) \rightarrow H^{1}(D, \text{End}(F)) \) is surjective. Hence \( \mathcal{S}(D; u_{1}, \ldots, u_{r}) \) has codimension \( h^{1}(D, \text{End}(F)) \) in \( \mathcal{S} \). Since \( h^{1}(D, \text{End}(F)) \geq 2 \), we are done.

**Proof.** Every spanned vector bundle \( G \) on an integral variety \( T \) is spanned by at most \( \text{dim}(T) + \text{rank}(G) \) sections (wait a few lines for the proof of this claim). Hence \( A \neq \emptyset \). Fix \( \psi \in A \). Let \( B' \) be the set of all surjective maps \( \psi : \mathcal{O}_{p_{1}}(r+1) \rightarrow \mathcal{O}_{p_{1}}(a) \) such that \( \text{ker}(\psi) \cong F^{\vee} \). Then \( A \neq \emptyset \). Fix \( \psi \in A \). Let \( B \) (resp. \( B' \)) be the set of all \( E \in G(a,b,r) \) (resp. \( E \in G'(a,b,r) \)) of the form \( \text{ker}(u)^{\vee} \) with \( u : \mathcal{O}_{Q}^{(r+1)} \rightarrow \mathcal{O}_{Q}(a,b) \) a surjection and \( u|D = \psi \). Then \( B \) is an irreducible variety of dimension \( (r+1)(a+1)b \) and \( B' \) is a non-empty Zariski open subset of \( B \).

Let \( S(a;r) \) be the set of all surjective maps \( \mathcal{O}_{p_{1}}^{(r+1)} \rightarrow \mathcal{O}_{p_{1}}(a) \). Since \( r+1 \geq 3 > \text{dim}(\mathbb{P}^{1}) \), the set \( S(a;r) \) is a non-empty open subset of the vector space \( H^{0}(\mathcal{O}_{p_{1}}(a)^{(r+1)}) \) and hence it is an integral variety of dimension \( (a+1)(r+1) \). For each \( \phi \in S(a;r) \) the sheaf \( \text{ker}(\phi) \) is a vector bundle on \( \mathbb{P}^{1} \) with determinant isomorphic to \( \mathcal{O}_{p_{1}}(-a) \). The sheaf \( \text{ker}(\phi) \) is a subsheaf of \( \mathcal{O}_{p_{1}}^{(r+1)} \) and \( \mathcal{O}_{p_{1}}^{(r+1)}/\text{ker}(\phi) \cong \mathcal{O}_{p_{1}}(a) \). Since the dual of a short exact sequence of
vector bundles is exact, the vector bundle \( \ker(\phi)^{\vee} \) is spanned. Let \( G(a; r) \) be the flat family \( \{ \ker(\phi)^{\vee} \}_{\phi \in S(a; r)} \) of vector bundles on \( \mathbb{P}^1 \). Any \( E \in G(a; r) \) has rank \( r \) and it is spanned. Hence the splitting type \( a_1 \geq \cdots \geq a_r \) of any \( E \in G(a; r) \) satisfies the conditions \( a_r \geq 0 \) and \( a_1 + \cdots + a_r = a \).

**Lemma 3.** Fix integers \( a_1 \geq \cdots \geq a_r \) such that \( a_1 + \cdots + a_r = a \) and set 
\[
F := \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i).
\]
The set of all \( E \in G(a; r) \) isomorphic to \( F \) is non-empty and it has dimension \((r+1)(a+1) - h^1(\text{End}(F))\).

**Proof.** We have \( h^0(F) = a + r \). Hence the Grassmannian of all \((r+1)\)-dimensional linear subspaces of \( H^0(F) \) has dimension \((r+1)(a-1)\). For any rank \( r \) vector bundle \( A \) on \( \mathbb{P}^1 \) we have \( \chi(\text{End}(A)) = r^2 \), because \( \text{End}(A) \) has rank \( r^2 \) and degree 0. Hence \( h^0(\text{End}(F)) = h^1(\text{End}(F)) + r^2 \). We have \( \dim(\text{Aut}(F)) = h^0(\text{End}(F)) \). Two injective maps with locally free cokernels \( u_i : F \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(r+1)}, i = 1, 2 \), have the same image if and only if \( u_2 = u_1 \circ w \) for some automorphism \( w \) of \( F \).

We leave to the reader the proof of the following elementary lemma.

**Lemma 4.** Let \( T \) be an integral projective variety. Let \( E \) be a rank \( r \) spanned torsion free sheaf on \( T \). \( E \) is a trivial vector bundle if and only if it is spanned by an \( r \)-dimensional linear subspace of \( H^0(E) \).

**Proof of Theorem 1.** Fix \( E \in G'(a, b, r) \) (it exists, because \( r \leq ab + a + b \)). Since \( E \) is spanned and without trivial factors, we have \( h^0(E^{\vee}) = 0 \), i.e. there is no non-zero map \( E \to \mathcal{O}_Q \).

(a) Assume that \( E \) is not simple. Hence there is a non-zero map \( f : E \to E \) such that \( G := \text{Im}(f) \) has rank \( m < r \). Since \( G \subset E \), \( G \) is torsion free. Let \( V \subset H^0(G) \) be the image of the map \( u : H^0(E) \to H^0(G) \) induced by the surjection \( E \to G \). Since \( G = f(E) \) is a quotient of the spanned sheaf \( E \), the sheaf \( G \) is spanned by \( V \). Set \( W := \ker(u) \subset H^0(E) \). Since every map \( E \to \mathcal{O}_Q \) is trivial, \( G \) is not a trivial vector bundle. Hence \( \dim(V) \geq m+1 \) (Lemma 4). Hence \( \dim(W) \leq r - m \). Let \( A \) be the saturation of \( G \) in \( E \), i.e. the only rank \( m \) subsheaf of \( E \) containing \( G \) and such that \( E/A \) has no torsion. The sheaf \( E/A \) has rank \( r - m \). Since \( E/A \) is a quotient of \( E/G \), it is spanned by a quotient of \( W \). Since \( \dim(W) \leq r - m \), the torsion free sheaf \( E/A \) is a trivial vector bundle (Lemma 4). Hence \( E \) has \( \mathcal{O}_Q^{r-m} \) as a factor, a contradiction.

(b) In this step we assume that neither \( a \) nor \( b \) is divisible by \( r \) and prove that a general \( E \in G'(a, b, r) \) is biuniform. We adapt the proofs in [2]. We fix one of the two rulings of \( Q \), say \( |\mathcal{O}_Q(0,1)| \), since the proof for the other ruling is similar. Fix \( D \in |\mathcal{O}_Q(0,1)| \). Fix integers \( a_1 \geq \cdots \geq a_r \geq 0 \) such
that \( a_1 + \cdots + a_r = a \) and \( a_r \leq a_1 - 2 \). Call \( u : G'(a, b, r) \times |\mathcal{O}_Q(0, 1)| \to |\mathcal{O}_Q(0, 1)| \) the projection onto the second factor. Let \( \Gamma(a_1, \ldots, a_r) \) be the set of all \((E, D) \in G'(a, b, r) \times |\mathcal{O}_Q(0, 1)|\) such that \( E|D \) has splitting type \((a_1, \ldots, a_r)\). We have \( \Gamma(a_1, \ldots, a_r) = \emptyset \) if \( a_r < 0 \). Hence we only have finitely many non-empty sets \( \Gamma(a_1, \ldots, a_r) \). Since \( \dim(|\mathcal{O}_Q(0, 1)|) = 1 \), to prove that a general \( E \in G'(a, b, r) \) is uniform with respect to the ruling associated to \( |\mathcal{O}_Q(0, 1)| \) it is sufficient to prove that each non-empty \( \Gamma(a_1, \ldots, a_r) \) has codimension \( \geq 2 \) in \( G'(a, b, r) \times |\mathcal{O}_Q(0, 1)| \). Fix \( D \in |\mathcal{O}_Q(0, 1)| \). It is sufficient to prove that \( \dim(u^{-1}(D)) \leq \dim(G'(a, b, r)) - 2 \). Set \( F := \oplus_{i=1}^r \mathcal{O}_D(d_i) \). Lemmas 2 and 3 give that \( u^{-1}(D) \) has dimension \( \dim(G'(a, b, r)) - h^1(\text{End}(F)) \). Since \( F \) is unbalanced and \( a/r \notin \mathbb{Z} \), we have \( h^1(\text{End}(F)) \geq 2 \). \( \square \)

**Lemma 5.** Let \( E \) be a rank \( r \) vector bundle on \( Q \). Fix \( T \in |\mathcal{O}_Q(1, 0)| \) and let \( b_1 \geq \cdots \geq b_r \) be the splitting type of \( E|T \). Assume the existence of \( c \in \mathbb{Z} \) such that \( E|D \) has splitting type \((c, \ldots, c)\) for all \( D \in |\mathcal{O}_Q(0, 1)| \). Then \( E \cong \oplus_{i=1}^r \mathcal{O}_Q(c, b_i) \).

**Proof.** Taking \( E(-c, 0) \) instead of \( E \) we reduce to the case \( c = 0 \). Let \( \pi : Q \to \mathbb{P}^1 \) denote the projection such that \( \mathcal{O}_Q(0, 1) \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \). Since \( h^0(E|D) = r \) and \( h^1(E|D) = 0 \) for every fiber \( D \) of \( \pi \), a theorem of base change gives that \( \pi_*(E) \) is a rank \( r \) vector bundle ([5], page 11). We have \( \pi_*(E) \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(d_i) \) for some integers \( d_1 \geq \cdots \geq d_r \). For each fiber \( D \) of \( \pi \) the natural map \( H^0(D, \pi^*(\pi_*(E))|D) \to H^0(D, E|D) \) is an isomorphism ([5], page 11). Since \( E|D \) is trivial, we get that the natural map \( \pi^*(\pi_*(E)) \to E \) is an isomorphism (see [5], page 53, for a similar proof). Since \( E|T \) has splitting type \( b_1 \geq \cdots \geq b_r \), we get \( d_i = b_i \) for all \( i \). \( \square \)

**Remark 3.** Lemma 5 shows that if either \( a/r \in \mathbb{Z} \) or \( b/r \in \mathbb{Z} \), then every biuniform vector bundle with balanced splitting type is isomorphic to a direct sum of \( r \) line bundles.

**Proposition 2.** Fix positive integers \( a, b, r, c \), such that \( r \geq 2 \), \( b > c \), \( a_1(i) \geq \cdots \geq a_r(i) \geq 0 \), \( 1 \leq i \leq c \), and \( a_1(i)+\cdots+a_r(i) = a \) for all \( i \in \{1, \ldots, c\} \). Fix \( D_i \in |\mathcal{O}_Q(0, 1)| \), \( 1 \leq i \leq c \), with \( D_i \neq D_h \) for all \( i \neq h \). Then there is \( E \in G'(a, b, r) \) such that \( E|D_i \) has splitting type \( a_1(i) \geq \cdots \geq a_r(i) \) for all \( i \).

**Proof.** Set \( T := D_1 \cup \cdots \cup D_c \). Let \( G \) be the vector on \( D \) such that \( G|D_i = \mathcal{O}_{D_i}(a_h(i)) \) for all \( i \). Fix a surjection \( \psi : \mathcal{O}_T^{\oplus(r+1)} \to G \) (it exists, because \( G \) is spanned and \( \dim(T) + \text{rank}(G) = r + 1 \)). Since \( b - c > 0 \), the proof of Lemma 2 works with \( T \) instead of \( D \). \( \square \)
Assume $a \geq 2$ and $b \geq 2$ and fix integers $a_1 \geq \cdots \geq a_r \geq 0$ such that $a_1 + \cdots + a_r = a + b$. Fix a smooth $T \in |\mathcal{O}_Q(1,1)|$. The proof of Proposition 2 shows the existence of $E \in G'(a,b,r)$ such that $E|T$ has splitting type $a_1, \ldots, a_r$.

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References


