International Journal of Pure and Applied Mathematics
Volume 91 No. 1 2014, 103-111
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i1.11

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NEGATION OF THE CONJECTURE FOR ODD ZETA VALUES

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Abstract: It is known the Euler formula: For even zeta values, $\zeta(2n) = \alpha_n \pi^{2n}$, where α_n is a rational number. It seems natural to conjecture that we can have $\zeta(n) = \alpha_n \pi^n$ for every n, but this paper gives the negation of this conjecture for odd zeta values.

AMS Subject Classification: 11A25, 11M06, 11Y60, 14G10
Key Words: Riemann zeta function, odd zeta values, multiple sine functions

1. Introduction

For even positive integers, special values of the Riemann zeta function can be given by

$$\zeta(2n) = (-1)^n B_{2n} \frac{(2\pi)^{2n}}{2\,(2n)!}\,,$$

where B_{2n} is the Bernoulli's number.

But, no equivalent formula of $\zeta(m)$ are known for odd positive integers, see [1]. From the special values of even zetas to be $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, Euler firstly conjectured that $\zeta(3) = \pi^3/m$ for some integers falling between

Received: December 7, 2013

6 and 90, but it became obvious to be hardly a promising result from the numerical calculation [2]. Numerical calculations has shown that if $\zeta(3)$ has the form $(p/q)\pi^3$, then the denominator q has a very large number of digits and it is strongly expected that equivalent formulas for odd zetas do not exist [3].

But it has been conjectured by some researchers [4] that odd zeta values can be represented in the form of $\zeta(2n+1) = (p_n/q_n)\pi^{2n+1}$ for $n = 1, 2, 3, \cdots$, where p_n and q_n are positive integers as shown in Fig.1.

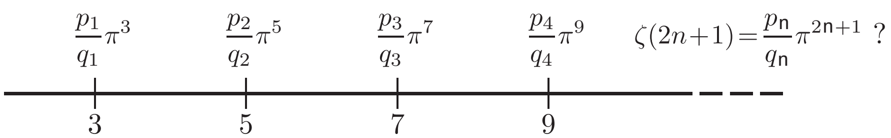

Figure 1: Conjecture for odd zeta values

Contrary to their conjecture, this paper has shown the negation of "$\zeta(2n+1) = (p_n/q_n)\pi^{2n+1}$ for $n = 1, 2, 3, \cdots$".

2. Negation of the Conjecture for Odd Zeta Values

Definitions.

(1) Let ***(AI)*** be the statement that:

$$\zeta(2n+1) = (p_n/q_n)\pi^{2n+1}$$

for $n \geq 1$, where p_n and q_n are positive integers.

(2) Let ***(AII)*** be the statement that: Only solution in integers of the equation, $mI_1 + nI_2 + lI_3 = 0$ is $m = n = l = 0$, where I_m is a real value given by the integral, $I_m = \frac{1}{\pi^{m+1}} \int_0^{\pi/2} x^m \log(\sin x) dx$.

At first, we prove following Lemmas.

Lemma 1. *I_1, I_2 and I_3 can be given by:*

(i) $I_1 = -\frac{1}{8}\log 2 + \frac{7}{16}\frac{\zeta(3)}{\pi^2}$,

(ii) $I_2 = -\frac{1}{24}\log 2 + \frac{3}{16}\frac{\zeta(3)}{\pi^2}$,

(iii) $I_3 = -\frac{1}{64}\log 2 + \frac{9}{64}\frac{\zeta(3)}{\pi^2} - \frac{93}{128}\frac{\zeta(5)}{\pi^4}$.

Proof. By the infinite sum shown as $\log(\sin x) = -\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n} - \log 2$ $(0 < x < \pi/2)$, we have

$$\pi^{m+1} I_m = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi/2} x^m \cos(2nx)dx - \frac{1}{m+1}\left(\frac{\pi}{2}\right)^{m+1}\log 2.$$

From the integration results given by:

$$m = 1; \quad \int_0^{\pi/2} x\cos(2nx)dx = \frac{-1+(-1)^n}{4n^2},$$

$$m = 2; \quad \int_0^{\pi/2} x^2\cos(2nx)dx = (-1)^n\frac{\pi}{4n^2},$$

$$m = 3; \quad \int_0^{\pi/2} x^3\cos(2nx)dx = \frac{3}{8n^4} - (-1)^n\frac{3}{8n^4} + (-1)^n\frac{3\pi^2}{16n^2},$$

we have

$$I_1 = \frac{1}{4\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\right) - \frac{1}{8}\log 2,$$

$$I_2 = -\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} - \frac{1}{24}\log 2,$$

$$I_3 = -\frac{3}{8\pi^4}\left(\sum_{n=1}^{\infty}\frac{1}{n^5} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\right) - \frac{3}{16\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} - \frac{1}{64}\log 2.$$

Thus, from the infinite sum given by

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = -\frac{3}{4}\zeta(3)$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5} = -\frac{15}{16}\zeta(5),$$

we can obtain formulas of (i), (ii) and (iii). □

Lemma 2. *There is no integers, a and b, which satisfy* $a\log 2 + b\pi = 0$.

Proof. From the Gelfond-Schneider's theorem, it can be derived that, if α_1, α_2, β_1, β_2 are non-zero algebraic number, and $\log\alpha_1$ and $\log\alpha_2$ are linearly independent over rational numbers, then $\beta_1\log\alpha_1 + \beta_2\log\alpha_2 \neq 0$ [5]. We can write $\pi = -i\log(-1)$, then $a\log 2 + b\pi = 0$ can be rewritten as $a\log 2 - ib\log(-1) = 0$. As $\log(2)$ and $\log(-1)$ are linearly independent over rational numbers, we have $a\log 2 - ib\log(-1) \neq 0$ from the Gelfond-Schneider's theorem. □

Theorem 1. *There is no integers p_i and q_i $(i = 1, 2)$, which satisfy* $\zeta(3) = (p_1/q_1)\pi^3$ *and* $\zeta(5) = (p_2/q_2)\pi^5$.

Proof. The proof of the theorem is consisted of two steps shown as follows:
Step 1. Let **(*BI*)** be the statement that:

There are integers p_1, q_1, p_2 and q_2 to satisfy $\zeta(3) = (p_1/q_1)\pi^3$ and $\zeta(5) = (p_2/q_2)\pi^5$.

First, we assume that: **(*AII*)** and **(*BI*)** are true.

From Lemma 1, we can obtain the matrix equation given by:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -1/8 & 7/16 & 0 \\ -1/24 & 3/16 & 0 \\ -1/64 & 9/64 & -93/128 \end{pmatrix} \begin{pmatrix} \log 2 \\ \zeta(3)/\pi^2 \\ \zeta(5)/\pi^4 \end{pmatrix}$$

By the inversion of the matrix, we have:

$$\begin{pmatrix} \log 2 \\ \zeta(3)/\pi^2 \\ \zeta(5)/\pi^4 \end{pmatrix} = \begin{pmatrix} -36 & 84 & 0 \\ -8 & 24 & 0 \\ -24/31 & 88/31 & -128/93 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}. \tag{1}$$

From **(*BI*)**, there exists positive integers, r and s satisfying:

$$\frac{\zeta(5)}{\pi^2\zeta(3)} = \frac{s}{r}. \tag{2}$$

By inserting equation (1) into equation (2), we have:

$$16rI_3 + (279s - 33r)I_2 + (9r - 93s)I_1 = 0.$$

From **(*AII*)**, we obtain the simultaneous equations given by:

$$\left.\begin{aligned} 16r &= 0 \\ 279s - 33r &= 0 \\ 9r - 93s &= 0 \end{aligned}\right\} \tag{3}$$

But there is no positive integers satisfying equation (3) and hence **(AII)** $\wedge$ **(BI)** is false.

Step 2. Next we suppose that: **(AII)** and the negation of **(BI)** are true. The negation of **(AII)** means that there are integers satisfying $mI_1 + nI_2 + lI_3 = 0$. By inserting formulas, (i), (ii) and (iii) into $mI_1 + nI_2 + lI_3 = 0$, we have:

$$\log 2 = \frac{3\{2p_1q_2(28m + 12n + 9l) - 93p_2q_1l\}}{2q_1q_2\{8(3m + n) + 3l\}}\pi\,. \tag{4}$$

If $2p_1q_2(28m + 12n + 9l) - 93p_2q_1l \neq 0$ and $8(3m + n) + 3l \neq 0$, it can be shown that there exists a rational number a/b satisfying $\pi/\log 2 = a/b$, which contradicts to Lemma.2. We can see that there is not such a case when either $2p_1q_2(28m + 12n + 9l) - 93p_2q_1l$ or $8(3m + n) + 3l$ becomes zero.

For the case when $2p_1q_2(28m+12n+9l)-93p_2q_1l = 0$ and $8(3m+n)+3l = 0$, we have the following three equations:

$$\begin{aligned} &\text{a)} \quad I_1m + I_2n + I_3l = 0 \\ &\text{b)} \quad 56p_1q_2m + 24p_1q_2n + (18p_1q_2 - 93q_2p_1)l = 0 \\ &\text{c)} \quad 24m + 8n + 3l = 0 \end{aligned} \tag{5}$$

By considering two planes in a space, we have two cases, (A) and (B) as shown in Fig.2 .

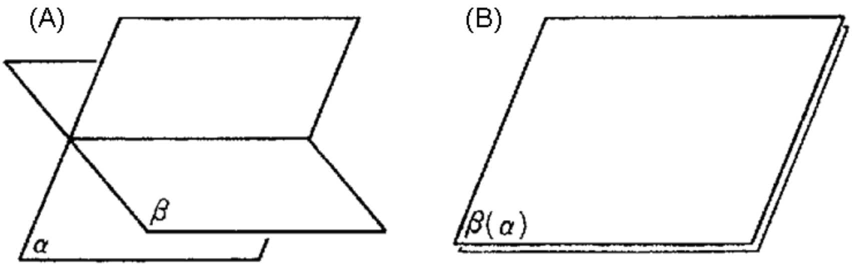

Figure 2: Two possible states of two planes in a space

The every plane containing the original point O for the case of (A) can be expressed as $\alpha(ax + by + cz) + \beta(a'x + b'y + c'z) = 0$, where α and β are arbitrary real numbers. From which, the case (B), when two planes coincide, satisfies $ka = a'$, $kb = b'$ and $kc = c'$, if we let $k = \alpha/\beta$.

Thus we obtain following three simultaneous equations for the case (B) from equation (5) as:

(1) From a) and c): $kI_1 = 24, \qquad kI_2 = 8, \qquad kI_3 = 3$

(2) From a) and b): $kI_1 = 56p_1q_2, \quad kI_2 = 24p_1q_2,$
$kI_3 = 18p_1q_2 - 93p_2q_1$

(3) From b) and c): $24k = 56p_1q_2, \quad 8k = 24p_1q_2,$
$3k = 18p_1q_2 - 93p_2q_1$

From numerical calculations, we obtain $I_1 = -0.033358\cdots$, $I_2 = -0.0060447\cdots$ and $I_3 = -0.0014374\cdots$, then it can be seen that there does not exist such a real number k satisfying simultaneous equations for each case, (1), (2) and (3), respectively.

Thus equation (5) has the only solution, $m = n = l = 0$, which contradicts to the negation of **(*BI*)**. Hence it is concluded that **(*AII*)** $\wedge\neg$**(*BI*)** is false.

From (step.1), we have already obtained that **(*AII*)** $\wedge$ **(*BI*)** is false, thus we can conclude **(*BI*)** is false and there is no integers to satisfy both of $\zeta(3) = (p_1/q_1)\pi^3$ and $\zeta(5) = (p_2/q_2)\pi^5$. □

Corollary. *" $\zeta(2n+1) = (p_n/q_n)\pi^{2n+1}$ for $n \geq 1$ h is false.*

Proof. It is clear from Theorem 1. □

3. Negation of the Conjecture for Higher Odd Zeta Values

M. Sato and J. Tate conjectured in their paper [6] that $\dfrac{\zeta(m+n)}{\zeta(m)\zeta(n)}$ is a rational number, from which odd zeta values can be represented in the form of $\zeta(2n+1) = (p_n/q_n)\gamma\pi^{2n+1}$ for $n = 1, 2, 3, \cdots$, where γ is a constant.

However their conjecture is not true from the following assumption for multiple sine functions.

We consider the multiple sine functions given by [7]:

$$S_r(x) = e^{x^{r-1}/(r-1)} \prod_{n=1}^{\infty} \left(P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}}$$

where

$$P_r(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right)$$

For $r = 2, 3, 4, \cdots$, we have [8]:

$$\int_0^{\pi/2} x^{r-2} \log(\sin x) dx = -\frac{\pi^{r-1}}{r-1} \log S_r(1/2) , \tag{6}$$

From which

$$I_2 = -\frac{1}{3} \log S_4(1/2), \quad I_3 = -\frac{1}{4} \log S_5(1/2),$$
$$I_4 = -\frac{1}{5} \log S_6(1/2), \quad I_5 = -\frac{1}{6} \log S_7(1/2)$$

Thus we have:

$$S_4(1/2) = \exp(-3I_2), S_5(1/2) = \exp(-4I_3), S_6(1/2) = \exp(-5I_4)$$
and $S_7(1/2) = \exp(-6I_5)$

Then, for these multiple sine values, we assume that:

(C) ; $S_4(1/2)^k S_5(1/2)^l S_6(1/2)^m S_7(1/2)^n = 1 \quad (k, l, m, n \in Z)$,
then $k = l = m = n = 0$.

From the calculation results of I_4 and I_5 by the same process at the proof of Lemma.1, we have:

$$\begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} -1/24 & 3/16 & 0 & 0 \\ -1/64 & 9/64 & -93/128 & 0 \\ -1/160 & 3/32 & -45/64 & 0 \\ -4/1536 & 90/1536 & -1350/1536 & 5715/1536 \end{bmatrix} \cdot \begin{bmatrix} \log 2 \\ \zeta(3)/\pi^2 \\ \zeta(5)/\pi^4 \\ \zeta(7)/\pi^6 \end{bmatrix}$$

From which, we can obtain

$$\begin{bmatrix} \log 2 \\ \zeta(3)/\pi^2 \\ \zeta(5)/\pi^4 \\ \zeta(7)/\pi^6 \end{bmatrix} = \begin{bmatrix} -420 & 1800 & -1860 & 0 \\ -88 & 400 & -1240/3 & 0 \\ -8 & 112/3 & -40 & 0 \\ -304/381 & 480/127 & -1616/381 & 512/1905 \end{bmatrix} \cdot \begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}$$

Then we have:

$$\zeta(5)/\pi^4 = -8I_2 + \frac{112}{3} I_3 - 40 I_4$$
$$\zeta(7)/\pi^6 = -\frac{304}{381} I_2 + \frac{480}{127} I_3 - \frac{1616}{381} I_4 + \frac{512}{1905} I_5$$

If we suppose $\dfrac{\zeta(7)}{\pi^2\zeta(5)} = \dfrac{s}{r}$ (s, r are positive integers), then we have:

$$\left(-\frac{304}{381}r + 8s\right) I_2 + \left(\frac{480}{127}r - \frac{112}{3}s\right) I_3 + \left(-\frac{1616}{381}r + 40s\right) I_4 + \frac{512r}{1905} I_5 = 0$$

From assumption **(C)**, we can see that I_2, I_3, I_4 and I_5 are linearly independent over Q, hence we have:

$$-\frac{304}{381}r + 8s = 0\,, \quad \frac{480}{127}r - \frac{112}{3}s = 0\,, \quad -\frac{1616}{381}r + 40s = 0$$

$$\text{and} \quad \frac{512}{1905}r = 0\,.$$

But there are no numbers r, s satisfying the above equations and we can conclude that there are no rational numbers to satisfy $\zeta(5) = c\gamma\pi^5$ and $\zeta(7) = c'\gamma\pi^7$ ($c, c' \in Q$, and γ: constant).

Similarly the same result can be obtained for higher zeta values from the following assumption for the values of multiple sine functions:

"$S_n(1/2)^{m_0} \cdots S_{n+k}(1/2)^{m_k} = 1 \quad (n \geq 4, m_0, \cdots, m_k \in Z)$,
then $m_0 = \cdots = m_k = 0\,$."

Then it can been seen that $\dfrac{\zeta(2n+3)}{\pi^2\zeta(2n+1)}$ is irrational for $n \geq 1$, which means that there is no real number γ for odd zeta values which satisfies $\zeta(2n+1) = (p_n/q_n)\gamma\pi^{2n+1}$ for $n \geq 1$.

4. Conclusion

It has been proved that the conjecture for odd zeta values that $\zeta(2n+1) = (p_n/q_n)\pi^{2n+1}$ for $n \geq 1$ is not true. Furthermore, from the assumption for values of multiple sine functions at 1/2, it can be seen that there is no real number γ for odd zeta values which satisfy $\zeta(2n+1) = (p_n/q_n)\gamma\pi^{2n+1}$ for $n \geq 1$.

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