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ALEXANDROV L-TOPOLOGIES AND L-JOIN MEET APPROXIMATION OPERATORS

Yong Chan Kim

Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

Abstract: In this paper, we investigate the properties of L-fuzzy relations and L-join meet approximation operators induced by Alexandrov L-topologies in complete residuated lattices. We investigate relations among their operations, L-fuzzy relations and Alexandrov L-topologies.

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1. Introduction

Pawlak [7,8] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Radzikowska [9] developed fuzzy rough sets induced by various L-fuzzy relations in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Lai [5,6] introduced Alexandrov L-topologies induced by fuzzy rough sets. Algebraic structures of fuzzy rough sets are developed in many directions [3,4,10,11]. Kim [3,4] introduced L-join meet and L-meet join approximation operators as a generalization of fuzzy rough set in complete residuated lattices.

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In this paper, we investigate the properties of L-fuzzy relations and L-join meet approximation operators induced by Alexandrov L-topologies in complete residuated lattices. We investigate relations among their operations, L-fuzzy relations and Alexandrov L-topologies.

2. Preliminaries

Definition 1. [1,2] An algebra $(L, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, ^* \bot, \top)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \rightarrow A)(x) = \alpha \rightarrow A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(y) = \bot$, otherwise.

Lemma 2. [1,2] For each $x, y, z, x_i, y_i \in L$, we have the following properties:

(1) If $y \leq z$, $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \rightarrow y_i)$.

(3) $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma}(x_i \rightarrow y)$.

(4) $\bigwedge_{i \in \Gamma} y_i^* = (\bigvee_{i \in \Gamma} y_i)^*$ and $\bigvee_{i \in \Gamma} y_i^* = (\bigwedge_{i \in \Gamma} y_i)^*$.

(5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(6) $x \odot y = (x \rightarrow y^*)^*$.

(7) $x \odot (x \rightarrow y) \leq y$.

(8) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

Definition 3. [1,5] Let X be a set. A function $R : X \times X \rightarrow L$ is called:

(R1) *reflexive* if $R(x, x) = \top$ for all $x \in X$,

(R2) *symmetric* if $R(x, x) = \top$ for all $x \in X$,

(R3) *transitive* if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

(R4) *Euclidean* if $R(x, z) \odot R(y, z) \leq R(x, y)$, for all $x, y, z \in X$.

If R satisfies (R1) and (R3), R is called an *L-fuzzy preorder.*

If R satisfies (R1), (R2) and (R3), R is called an *L-fuzzy equivalence relation.*

Definition 4. [3,4] (1) A map $\mathcal{H} : L^X \to L^X$ is called an *L-upper approximation operator* iff it satisfies the following conditions

(H1) $A \leq \mathcal{H}(A)$,

(H2) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,

(H3) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.

(2) A map $\mathcal{K} : L^X \to L^X$ is called an *L-join meet approximation operator* iff it satisfies the following conditions

(K1) $\mathcal{K}(A) \leq A^*$,

(K2) $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A)$,

(K3) $\mathcal{K}(\bigvee_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{K}(A_i)$.

Definition 5. [3,4] A subset $\tau \subset L^X$ is called an *Alexandrov L-topology* if it satisfies:

(T1) $\bot_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\bot_X(x) = \bot$ for $x \in X$.

(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(T4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Theorem 6. [3,4] (1) *τ is an Alexandrov L-topology on X iff $\tau_* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov L-topology on X.*

(2) *If $\mathcal{K}$ is an L-join meet approximation operator, then $\tau_{\mathcal{K}} = \{A \in L^X \mid \mathcal{K}(A) = A^*\}$ is an Alexandrov L-topology on X.*

(3) *If $\mathcal{K}$ is an L-join meet approximation operator with $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$, then $\tau_{\mathcal{K}} = \{A \in L^X \mid \mathcal{K}(A) = A^*\} = \{\mathcal{K}^*(A) \mid A \in L^X\}$ is an Alexandrov L-topology on X.*

(4) *If $\mathcal{K}$ is an L-join meet approximation operator with $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$, then $\tau_{\mathcal{K}} = \{\mathcal{K}(A) \mid A \in L^X\} = (\tau_{\mathcal{K}})_*$ is an Alexandrov L-topology on X.*

Theorem 7. [3] (1) *A map $\mathcal{K} : L^X \to L^X$ is an L-join meet approximation operator iff there exists a reflexive L-fuzzy relation $R \in L^{X\times X}$ such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X} (A(x) \to R^*(x, y)).$$

(2) *A map* $\mathcal{K} : L^X \to L^X$ *is an L-join meet approximation operator with* $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ *iff there exists an L-fuzzy preorder* $R \in L^{X\times X}$ *such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X} (A(x) \to R^*(x, y)).$$

(3) *A map* $\mathcal{K} : L^X \to L^X$ *is an L-join meet approximation operator with* $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ *iff there exists a reflexive and Euclidean L-fuzzy relation* $R \in L^{X\times X}$ *such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X} (A(x) \to R^*(x, y)).$$

Theorem 8. [3] *Let* $R \in L^{X\times X}$ *be a relation. Define operators as follows*

$$\begin{array}{ll} \mathcal{K}_{R^*}(A)(y) & = \bigwedge_{x\in X}(A(x) \to R^*(x, y)), \\ \mathcal{K}_{R^{-1*}}(A)(y) & = \bigwedge_{x\in X}(A(x) \to R^{-1*}(x, y)). \end{array}$$

Then the following properties hold.

(1) *If R is reflexive, then* $\tau_{\mathcal{K}_{R^*}} = \tau_{(\mathcal{K}_{R^{-1*}})_*}$.

(2) *If R is an L-fuzzy preorder, then*

$$\begin{array}{ll} \tau_{\mathcal{K}_{R^*}} & = \{\bigvee_{x\in X}(A(x) \odot R(x, -)) \mid A \in L^X\} \\ \tau_{\mathcal{K}_{R^{-1*}}} & = \{\bigvee_{x\in X}(A(x) \odot R(-, x)) \mid A \in L^X\}. \end{array}$$

(3) *If R is reflexive and and Euclidean, then R is symmetric, R is an L-fuzzy preorder and* $\tau_{\mathcal{K}_{R^*}} = \tau_{\mathcal{K}_{R^{-1*}}} = \tau_{(\mathcal{K}_{R^{-1*}})_*}$ *such that*

$$\begin{array}{ll} \tau_{\mathcal{K}_{R^*}} & = \{\bigwedge_{x\in X}(A(x) \to R^*(x, -)) \mid A \in L^X\} \\ \tau_{\mathcal{K}_{R^{-1*}}} & = \{\bigwedge_{x\in X}(A(x) \to R^*(-, x)) \mid A \in L^X\}. \end{array}$$

3. Alexandrov *L*-Topologies and *L*-Join Meet Approximation Operators

Theorem 9. *Let* τ *be an Alexandrov L-topology on X. Then the following properties hold.*

(1) *There exists an L-fuzzy preorder* R_τ *such that* $\tau = \tau_{\mathcal{K}_{R^*_\tau}}$ *where*

$$\mathcal{K}_{R^*_\tau}(A)(y) = \bigwedge_{x\in X} (A(x) \to R^*_\tau(x, y)).$$

(2) *Define* $\mathcal{H}_\tau(A) = \bigwedge\{A_i \mid A \le A_i,\ A_i \in \tau\}$. *Then* $\mathcal{H}_\tau$ *is an* L*-upper approximation operator on* X *such that* $\mathcal{H}_\tau = \mathcal{K}^*_{R^*_\tau}$ *and* $\tau_{\mathcal{H}_\tau} = \tau = \tau_{\mathcal{K}_{R^*_\tau}}$ *with*

$$\mathcal{H}_\tau(A)(y) = \bigvee_{x\in X} (A(x) \odot R_\tau(x,y))$$

(3) *Define* $k_\tau(A) = \bigvee\{A_i \mid A_i \le A^*,\ A_i \in \tau\}$. *Then* k_τ *is an* L*-join meet approximation operator on* X *such that* $k_\tau(k^*_\tau(A)) = k_\tau(A)$ *for all* $A \in L^X$. *Moreover,* $\tau_{k_\tau} = \tau_*$ *and* $k_\tau(A) = (\mathcal{H}_{\tau_*}(A))^*$ *such that*

$$k_\tau(A)(y) = \bigwedge_{x\in X} (A(x) \to R^{-1*}_\tau(x,y)).$$

(4) *If* R_τ *is Euclidean, then* $k_\tau = \mathcal{K}_{R^*_\tau}$ *and* $\tau_* = \tau_{k_\tau} = \tau_{\mathcal{K}_{R^*_\tau}} = \tau$.

Proof. (1) Define $R_\tau(x,y) = \bigwedge_{B\in\tau}(B(x) \to B(y))$. Then $R_\tau(x,x) = \top$ and $R_\tau(x,y) \odot R_\tau(y,z) \le R_\tau(x,z)$ because $(B(x) \to B(y)) \odot (B(y) \to B(z)) \le (B(x) \to B(z))$. Let $A \in \tau$.

$$\begin{aligned} A(x) \odot R_\tau(x,y) &= A(x) \odot \textstyle\bigwedge_{B\in\tau}(B(x) \to B(y)) \\ &\le A(x) \odot (A(x) \to A(y)) \le A(y). \end{aligned}$$

Thus $A^*(y) \le (A(x) \odot R_\tau(x,y))^* = A(x) \to R^*_\tau(x,y)$. Hence $\mathcal{K}_{R^*_\tau}(A) = A^*$, that is, $A \in \tau_{\mathcal{K}_{R^*_\tau}}$. Thus $\tau \subset \tau_{\mathcal{K}_{R^*_\tau}}$.

Let $A \in \tau_{\mathcal{K}_{R^*_\tau}}$ with $\mathcal{K}_{R^*_\tau}(A) = A^*$. Then $A^*(x) = \bigwedge_y(A(y) \to R^*_\tau(y,x))$. So, $A(x) = \bigvee_y(A(y) \odot R_\tau(y,x)) = \bigvee_y(A(y) \odot \bigwedge_{B\in\tau}(B(y) \to B(x))$. Since $\bigwedge_{B\in\tau}(B(y) \to B) \in \tau$ from (T4) and $\bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B(y) \to B)) \in \tau$ from (T3) and (T4), we have $A \in \tau$. Hence $\tau = \tau_{\mathcal{K}_{R^*_\tau}}$.

(2) Since R_τ is an L-fuzzy preorder, by Theorem 7(2), $\mathcal{K}_{R^*_\tau}(\mathcal{K}^*_{R^*_\tau}(A)) = \mathcal{K}_{R^*_\tau}(A)$. From the definition of $\mathcal{H}_\tau$, $\mathcal{H}_\tau(A) \le \mathcal{K}^*_{R^*_\tau}(A)$.

Since $\mathcal{H}_\tau(A) \in \tau = \tau_{\mathcal{K}_{R^*_\tau}}$, we have $\mathcal{K}_{R^*_\tau}(\mathcal{H}_\tau(A)) = \mathcal{H}^*_\tau(A)$. Since $\mathcal{K}^*_{R^*_\tau}$ is an increasing function by (K3), for $A \le \mathcal{H}_\tau(A)$, we have

$$\mathcal{K}^*_{R^*_\tau}(A) \le \mathcal{K}^*_{R^*_\tau}(\mathcal{H}_\tau(A)) = \mathcal{H}_\tau(A).$$

$$\begin{aligned} \mathcal{H}_\tau(A)(y) &= \mathcal{K}^*_{R^*_\tau}(A)(y) \\ &= (\textstyle\bigwedge_{x\in X}(A(x) \to R^*_\tau(x,y)))^* \\ &= \textstyle\bigvee_{x\in X}(A(x) \odot R_\tau(x,y)). \end{aligned}$$

Thus, $\mathcal{H}_\tau$ is an L-upper approximation operator.

(3) (K1) From the definition of k_τ, $k_\tau(A) \leq A^*$.

(K2) Since $B_i \leq \alpha \to \alpha \odot B_i$, we have

$$\begin{aligned} k_\tau(\alpha \odot A)(x) &= \bigvee\{B_i(x) \mid B_i \leq (\alpha \odot A)^* = \alpha \to A^*,\ B_i \in \tau\} \\ &\leq \alpha \to \bigvee\{\alpha \odot B_i(x) \mid \alpha \odot B_i \leq A^*,\ B_i \in \tau\} \\ &\leq \alpha \to k_\tau(A)(x). \end{aligned}$$

Suppose $k_\tau(\alpha \odot A) \not\geq \alpha \to k_\tau(A)$. Then there exists $x \in X$ such that

$$k_\tau(\alpha \odot A) \not\geq \alpha \to k_\tau(A).$$

From the definition of k_τ, there exists $A_i \in \tau$ such that $A_i \leq A^*$ with

$$k_\tau(\alpha \odot A) \not\geq \alpha \to A_i.$$

On the other hand, since $\alpha \to A_i \leq \alpha \to A^* = (\alpha \odot A)^*$, $k_\tau(\alpha \odot A) \geq \alpha \to A_i$. It is a contradiction. Thus, $k_\tau(\alpha \odot A) \geq \alpha \to k_\tau(A)$. Hence (K2) holds.

(K3) Since $k_\tau(B) \leq k_\tau(A)$ for $A \leq B$,

$$k_\tau(\bigvee_{i\in\Gamma} A_i) \leq \bigwedge_{i\in\Gamma} k_\tau(A_i).$$

Suppose $k_\tau(\bigvee_{i\in\Gamma} A_i) \not\geq \bigwedge_{i\in\Gamma} k_\tau(A_i)$. From the definition of $k_\tau(A_i)$, for all $i \in \Gamma$, there exists $B_i \in \tau$ such that $B_i \leq A_i^*$ with

$$k_\tau(\bigvee_{i\in\Gamma} A_i) \not\geq \bigwedge_{i\in\Gamma} B_i.$$

On the other hand, since $\bigwedge_{i\in\Gamma} B_i \leq \bigwedge_{i\in\Gamma} A_i^* = (\bigvee_{i\in\Gamma} A_i)^*$,

$$k_\tau(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} B_i.$$

It is a contradiction. Thus, $k_\tau(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} k_\tau(A_i)$. Hence (K3) holds.

Since $k_\tau(A) \in \tau$, by the definition of k_τ, $k_\tau(A) \leq k_\tau(k_\tau^*(A)) \leq k_\tau(A)$. Then $k_\tau(k_\tau^*(A)) = k_\tau(A)$.

Let $A \in \tau_*$. Then $A^* \in \tau$. By the definition of k_τ, $k_\tau(A) = A^*$. So, $A \in \tau_{k_\tau}$. Thus $\tau_* \subset \tau_{k_\tau}$. Conversely, it similarly proved.

$$\begin{aligned} (\mathcal{H}_{\tau_*}(A))^* &= (\bigwedge\{A_i \mid A \leq A_i,\ A_i \in \tau_*\})^* \\ &= \bigvee\{A_i^* \mid A_i^* \leq A^*,\ A_i^* \in \tau\} = k_\tau(A). \end{aligned}$$

By (2), since $R_\tau^{-1}(x,y) = \bigwedge_{B\in\tau}(B(y) \to B(x)) = \bigwedge_{B\in\tau}(B^*(x) \to B^*(y)) = \bigwedge_{A\in\tau^*}(A(x) \to A(y))$, we have

$$\begin{aligned} k_\tau(A)(y) &= (\mathcal{H}_{\tau_*}(A)(y))^* = (\bigvee_{x\in X}(A(x) \odot R_{\tau_*}(x,y)))^* \\ &= \bigwedge_{x\in X}(A(x) \to R_\tau^{-1}(x,y)). \end{aligned}$$

(4) Since R_τ is Euclidean, then $\bigvee_{x\in X}(R_\tau(y,x) \odot R_\tau(z,x)) \le R_\tau(y,z)$. It follows

$$\begin{aligned} \mathcal{K}_{R_\tau^*}(\mathcal{K}_{R_\tau^*}(A))(x) &= \bigwedge_{y\in X}(\mathcal{K}_{R_\tau^*}(A)(y) \to R_\tau^*(y,x)) \\ &= \bigwedge_{y\in X}(\bigwedge_{z\in X}(A(z) \to R_\tau^*(z,y)) \to R_\tau^*(y,x)) \\ &= \bigwedge_{y\in X}(R_\tau(y,x) \to (\bigwedge_{z\in X}(A(z) \to R_\tau^*(z,y)))^*) \\ &= \bigwedge_{y\in X}(R_\tau(y,x) \to \bigvee_{z\in X}(A(z) \odot R_\tau(z,y))) \\ &\ge \bigvee_{z\in X}(A(z) \odot R_\tau(z,x)) = (\bigwedge_{z\in X}(A(z) \to R_\tau(z,x)))^*. \\ &= (\mathcal{K}_{R_\tau^*}(A))^*(x). \end{aligned}$$

Thus $\mathcal{K}_{R_\tau^*}(\mathcal{K}_{R_\tau^*}(A)) = \mathcal{K}^*_{R_\tau^*}(A)$ and $\mathcal{K}_{R_\tau^*}(A) \le A^*$. Since $\tau = \tau_{\mathcal{K}_{R_\tau^*}}$, by the definition of k_τ, $\mathcal{K}_{R_\tau^*}(A) \le k_\tau(A)$.

Since $k_\tau(A) \le A^*$ iff $A \le k_\tau^*(A) = \mathcal{K}_{R_\tau^*}(k_\tau(A))$ and $\tau = \tau_{\mathcal{K}_{R_\tau^*}}$, by the definition of k_τ, then

$$\mathcal{K}_{R_\tau^*}(A) \ge \mathcal{K}_{R_\tau^*}(\mathcal{K}_{R_\tau^*}(k_\tau(A))) = \mathcal{K}^*_{R_\tau^*}(k_\tau(A)) = k_\tau(A).$$

□

Theorem 10. *Let τ be an Alexandrov L-topology on X. Then the following properties hold.*

(1) *There exists an L-fuzzy preorder R_τ^{-1} such that $\tau_* = \tau_{\mathcal{K}_{R_\tau^{-1*}}}$ where*

$$\mathcal{K}_{R^{-1*}}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^{-1*}(x,y)).$$

(2) *Define $\mathcal{H}_{\tau_*}(A) = \bigwedge\{A_i \mid A \le A_i,\ A_i \in \tau_*\}$. Then $\mathcal{H}_{\tau_*}$ is an L-upper approximation operator on X such that $\mathcal{H}_{\tau_*} = \mathcal{K}^*_{R_\tau^{-1*}}$ and $\tau_* = \tau_{\mathcal{H}_{\tau_*}} = \tau_{\mathcal{K}_{R_\tau^{-1*}}}$ with*

$$\mathcal{H}_{\tau_*}(A)(y) = \bigvee_{x\in X}(A(x) \odot R_\tau^{-1}(x,y))$$

(3) *Define* $k_{\tau_*}(A) = \bigvee\{A_i \mid A_i \le A^*,\ A_i \in \tau_*\}$. *Then* k_{τ_*} *is an* L*-join meet approximation operator on* X $k_{\tau_*}(k^*_{\tau_*}(A)) = k_{\tau_*}(A)$ *for all* $A \in L^X$. *Moreover,* $\tau_{k_{\tau_*}} = \tau$ *and* $k_{\tau_*}(A) = (\mathcal{H}_\tau(A))^*$ *such that*

$$k_{\tau_*}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^*_\tau(x,y)).$$

(4) *If* R^{-1}_τ *is Euclidean, then* $k_{\tau_*} = \mathcal{K}_{R^{-1*}_\tau}$ *and* $\tau = \tau_{k_{\tau_*}} = \tau_{\mathcal{K}_{R^{-1*}_\tau}} = \tau_*$.

Proof. (1) Define $R^{-1}_\tau(x,y) = \bigwedge_{B\in\tau}(B(y) \to B(x)) = \bigwedge_{B\in\tau}(B^*(x) \to B^*(y))$. Then $R^{-1}_\tau(x,x) = \top$ and $R^{-1}_\tau(x,y) \odot R^{-1}_\tau(y,z) \le R^{-1}_\tau(x,z)$. Let $A^* \in \tau_*$.

$$\begin{array}{rl} A^*(x) \odot R^{-1}_\tau(x,y) & = A^*(x) \odot \bigwedge_{B\in\tau}(B^*(x) \to B^*(y)) \\ & \le A^*(x) \odot (A^*(x) \to A^*(y)) \le A^*(y). \end{array}$$

Thus $A(y) \le (A^*(x) \odot R^{-1}_\tau(x,y))^* = A^*(x) \to R^{-1*}_\tau(x,y)$. Hence $\mathcal{K}_{R^{-1*}_\tau}(A^*) = A$, that is, $A^* \in \tau_{\mathcal{K}_{R^{-1*}_\tau}}$.

Let $A \in \tau_{\mathcal{K}_{R^{-1*}_\tau}}$ with $\mathcal{K}_{R^{-1*}_\tau}(A) = A^*$. Then $A^*(x) = \bigwedge_y(A(y) \to R^{-1*}_\tau(y,x))$. So, $A(x) = \bigvee_y(A(y)\odot R^{-1}_\tau(y,x)) = \bigvee_y(A(y)\odot\bigwedge_{B\in\tau}(B(x) \to B(y)) = \bigvee_y(A(y)\odot \bigwedge_{B\in\tau}(B^*(y) \to B^*(x))$. Since $\bigwedge_{B\in\tau}(B^*(y) \to B^*) \in \tau_*$ and $\bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B^*(y) \to B^*)) \in \tau_*$, we have $A \in \tau_*$. Hence $\tau_* = \tau_{\mathcal{K}_{R^{-1*}_\tau}}$.

(2) and (3) are similarly proved as Theorem 9 (2) and (3), respectively.

(4) Since R^{-1} is Euclidean, then $\bigvee_{x\in X}(R_\tau(x,y)\odot R_\tau(x,z) \le R_\tau(y,z)$. Thus

$$\begin{array}{rl} \mathcal{K}_{R^{-1*}_\tau}(\mathcal{K}_{R^{-1*}_\tau}(A))(x) & = \bigwedge_{y\in X}(\mathcal{K}_{R^{-1*}_\tau}(A)(y) \to R^{-1*}_\tau(y,x)) \\ & = \bigwedge_{y\in X}(\bigwedge_{z\in X}(A(z) \to R^{-1*}_\tau(z,y)) \to R^{-1*}_\tau(y,x)) \\ & = \bigwedge_{y\in X}(R^{-1}_\tau(y,x) \to (\bigwedge_{z\in X}(A(z) \to R^{-1*}_\tau(z,y)))^*) \\ & = \bigwedge_{y\in X}(R^{-1}_\tau(y,x) \to \bigvee_{z\in X}(A(z) \odot R^{-1}_\tau(z,y))) \\ & \ge \bigvee_{z\in X}(A(z) \odot R^{-1}_\tau(z,x)). \end{array}$$

Since $\mathcal{K}_{R^{-1*}_\tau}(\mathcal{K}_{R^{-1*}_\tau}(A)) = \mathcal{K}^*_{R^{-1*}_\tau}(A)$, $\mathcal{K}_{R^{-1*}_\tau}(A) \in \tau_*$ and $\mathcal{K}_{R^{-1*}_\tau}(A) \le A^*$, then $\mathcal{K}_{R^{-1*}_\tau}(A) \le k_{\tau_*}(A)$.

Since $k_{\tau_*}(A) \le A^*$ iff $A \le k^*_{\tau_*}(A) = \mathcal{K}_{R^{-1*}_\tau}(k_{\tau_*}(A))$ because $k_{\tau_*}(A) \in \tau_* = \tau_{\mathcal{K}_{R^{-1*}_\tau}}$, then

$$\mathcal{K}_{R^{-1*}_\tau}(A) \ge \mathcal{K}_{R^{-1*}_\tau}(\mathcal{K}^{-1*}_{R_\tau}(k_{\tau_*}(A))) = \mathcal{K}^*_{R^{-1*}_\tau}(k_{\tau_*}(A)) = k_{\tau_*}(A).$$

□

Theorem 11. *Let $\mathcal{K}_{R^*} : L^X \to L^X$ be an L-join meet approximation operator on X defined as*

$$\mathcal{K}_{R^*}(A)(y) = \bigwedge_{x \in X} (A(x) \to R^*(x, y)).$$

(1) $A^* \leq \mathcal{K}_{R^*}(A)$ *iff* $A \leq \mathcal{K}_{R^{-1*}}(A^*)$ *and* $\tau_{\mathcal{K}_{R^{-1*}}} = (\tau_{\mathcal{K}_{R^*}})_*$.

(2) *If R is an L-fuzzy preorder and we define* $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A) = \bigwedge\{A_i \mid A \leq A_i,\ A_i \in \tau_{\mathcal{K}_{R^*}}\}$ *and* $\mathcal{H}_{\tau_{\mathcal{K}_{R^{-1*}}}}(A) = \bigwedge\{A_i \mid A \leq A_i,\ A_i \in \tau_{\mathcal{K}_{R^{-1*}}} = (\tau_{\mathcal{K}_{R^*}})_*\}$, *then* $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}$ *and* $\mathcal{H}_{\tau_{\mathcal{K}_{R^{-1*}}}}$ *are L-upper approximation operators on X such that* $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}} = \mathcal{K}^*_{R^*}$ *and* $\mathcal{H}_{\tau_{\mathcal{K}_{R^{-1*}}}} = \mathcal{K}^*_{R^{-1*}}$ *with*

$$\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y))$$

$$\mathcal{H}_{(\tau_{\mathcal{K}_{R^*}})_*}(A)(y) = \mathcal{H}_{\tau_{\mathcal{K}_{R^{-1*}}}}(A)(y) = \bigvee_{x \in X} (A(x) \odot R^{-1}(x, y))$$

(3) *Define* $k_{\tau_{\mathcal{K}_{R^*}}}(A) = \bigvee\{A_i \mid A_i \leq A^*,\ A_i \in \tau_{\mathcal{K}_{R^*}}\}$. *Then* $k_{\tau_{\mathcal{K}_{R^*}}}$ *is an L-join meet approximation operator on X such that* $k_{\tau_{\mathcal{K}_{R^*}}}(k^*_{\tau_{\mathcal{K}_{R^*}}}(A)) = k_{\tau_{\mathcal{K}_{R^*}}}(A)$ *for all* $A \in L^X$. *Moreover,* $\tau_{k_{\tau_{\mathcal{K}_{R^*}}}} = (\tau_{\mathcal{K}_{R^*}})_* = \tau_{\mathcal{K}_{R^{-1*}}}$ *and* $k_{\tau_{\mathcal{K}_{R^*}}}(A) = (\mathcal{H}_{(\tau_{\mathcal{K}_{R^*}})_*}(A))^*$ *such that*

$$k_{\tau_{\mathcal{K}_{R^*}}}(A)(y) = \bigwedge_{x \in X} (A(x) \to R^{-1*}(x, y)).$$

(4) *Define* $k_{\tau_{\mathcal{K}_{R^{-1*}}}}(A) = \bigvee\{A_i \mid A_i \leq A^*,\ A_i \in \tau_{\mathcal{K}_{R^{-1*}}}\}$. *Then* $k_{\tau_{\mathcal{K}_{R^{-1*}}}}$ *is an L-join meet approximation operator on X such that* $k_{\tau_{\mathcal{K}_{R^{-1*}}}}(k^*_{\tau_{\mathcal{K}_{R^{-1*}}}}(A)) = k_{\tau_{\mathcal{K}_{R^{-1*}}}}(A)$ *for all* $A \in L^X$. *Moreover,* $\tau_{k_{\tau_{\mathcal{K}_{R^{-1*}}}}} = (\tau_{\mathcal{K}_{R^{-1*}}})_* = \tau_{\mathcal{K}_{R^*}}$ *and* $k_{\tau_{\mathcal{K}_{R^{-1*}}}}(A) = (\mathcal{H}_{(\tau_{\mathcal{K}_{R^{-1*}}})_*}(A))^*$ *such that*

$$k_{\tau_{\mathcal{K}_{R^{-1*}}}}(A)(y) = \bigwedge_{x \in X} (A(x) \to R^*(x, y)).$$

(5) *If R is Euclidean, then* $k_{\tau_{\mathcal{K}_{R^*}}} = \mathcal{K}_{R^*}$.

Proof. (1) Since $A^*(y) \le \mathcal{K}_{R^*}(A)(y) = \mathcal{K}_{R^*}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^*(x,y))$ iff $A(x) \le \bigwedge_{y\in X}(A^*(y) \to R^*(x,y)) = \mathcal{K}_{R^{-1*}}(A)(x)$, we have $\tau_{\mathcal{K}_{R^{-1*}}} = (\tau_{\mathcal{K}_{R^*}})_*$.

(2) Since R is an L-fuzzy preorder, $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$. From the definition of $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}$, $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A) \le \mathcal{K}^*_{R^*}(A)$.

Since $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A) \in \tau_{\mathcal{K}_{R^*}}$, we have $\mathcal{K}_{R^*}(\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A)) = \mathcal{H}^*_{\tau_{\mathcal{K}_{R^*}}}(A)$. Since $\mathcal{K}^*_{R^*}$ is an increasing function by (K3), for $A \le \mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A)$, we have

$$\mathcal{K}^*_{R^*}(A) \le \mathcal{K}^*_{R^*}(\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A)) = \mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A).$$

$$\begin{aligned}\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}(A)(y) &= \mathcal{K}^*_{R^*}(A)(y)\\ &= (\textstyle\bigwedge_{x\in X}(A(x) \to R^*(x,y)))^*\\ &= \textstyle\bigvee_{x\in X}(A(x) \odot R(x,y)).\end{aligned}$$

Thus, $\mathcal{H}_{\tau_{\mathcal{K}_{R^*}}}$ is an L-upper approximation operator.

(3) By Theorem 9(3), $k_{\tau_{\mathcal{K}_{R^*}}}$ is an L-join meet approximation operator on X. Let $A \in (\tau_{\mathcal{K}_{R^*}})_*$. Then $A^* \in \tau_{\mathcal{K}_{R^*}}$. By the definition of $k_{\tau_{\mathcal{K}_{R^*}}}$, $k_{\tau_{\mathcal{K}_{R^*}}}(A) = A^*$. So, $A \in \tau_{k_{\tau_{\mathcal{K}_{R^*}}}}$. Thus $(\tau_{\mathcal{K}_{R^*}})_* \subset \tau_{k_{\tau_{\mathcal{K}_{R^*}}}}$. Conversely, it similarly proved.

$$\begin{aligned}(\mathcal{H}_{(\tau_{\mathcal{K}_{R^*}})_*}(A))^* &= (\textstyle\bigwedge\{A_i \mid A \le A_i,\ A_i \in (\tau_{\mathcal{K}_{R^*}})_*\})^*\\ &= \textstyle\bigvee\{A_i^* \mid A_i^* \le A^*,\ A_i^* \in \tau_{\mathcal{K}_{R^*}}\} = k_{\tau_{\mathcal{K}_{R^*}}}(A).\end{aligned}$$

By (1) an (2), since $\tau_{\mathcal{K}_{R^{-1*}}} = (\tau_{\mathcal{K}_{R^*}})_*$. and $\mathcal{H}_{\tau_{\mathcal{K}_{R^{-1*}}}} = \mathcal{K}^*_{R^{-1*}}$, we have

$$\begin{aligned}k_{\tau_{\mathcal{K}_{R^*}}}(A)(y) &= (\mathcal{H}_{(\tau_{\mathcal{K}_{R^*}})_*}(A)(y))^* = (\textstyle\bigvee_{x\in X}(A(x) \odot R^{-1}(x,y)))^*\\ &= \textstyle\bigwedge_{x\in X}(A(x) \to R^{-1}(x,y)).\end{aligned}$$

(4) It is similarly proved as (3).

(5) Since R is Euclidean, $\bigvee_{x\in X}(R(y,x) \odot R(z,x)) \le R(y,z)$. By Theorem 7(3), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$ and $\mathcal{K}_{R^*}(A) \le A^*$. By the definition of $k_{\tau_{\mathcal{K}_{R^*}}}$, $\mathcal{K}_{R^*}(A) \le k_{\tau_{\mathcal{K}_{R^*}}}(A)$. Since $k_{\tau_{\mathcal{K}_{R^*}}}(A) \le A^*$ iff $A \le k^*_{\tau_{\mathcal{K}_{R^*}}}(A) = \mathcal{K}_{R^*}(k_{\tau_{\mathcal{K}_{R^*}}}(A))$, by the definition of $k_{\tau_{\mathcal{K}_{R^*}}}$, then

$$\mathcal{K}_{R^*}(A) \ge \mathcal{K}_{R^*}(\mathcal{K}_{R^*}(k_{\tau_{\mathcal{K}_{R^*}}}(A))) = \mathcal{K}^*_{R^*}(k_{\tau_{\mathcal{K}_{R^*}}}(A)) = k_{\tau_{\mathcal{K}_{R^*}}}(A).$$

□

Theorem 12. *Let $R \in L^{X\times X}$ be an L-fuzzy relation and $\tau_{\mathcal{K}_{R^*}}$ the Alexandrov L-topology induced by $\mathcal{K}_{R^*}$. Then the following properties hold.*

(1) *If R is transitive, then $R \leq R_{\tau_{\mathcal{K}_{R^*}}}$.*

(2) *If $R_z \in \tau_{\mathcal{K}_{R^*}}$ for $z \in X$ where $R_z(x) = R(z,x)$ and $\bigwedge_{z\in X}(R(z,x) \to R(z,y)) \leq R(x,y)$, then $R_{\tau_{\mathcal{K}_{R^*}}} \leq R$.*

(3) *If R is an L-fuzzy preorder, then $R = R_{\tau_{\mathcal{K}_{R^*}}}$.*

(4) *If R is Euclidean, then $R^{-1} \leq R_{\tau_{\mathcal{K}_{R^*}}}$.*

(5) *If $R^*_z \in \tau_{\mathcal{K}_{R^*}}$ for $z \in X$ where $R^*_z(x) = R^*(z,x)$ and $\bigwedge_{z\in X}(R(z,x) \to R(z,y)) \leq R(x,y)$, then $R_{\tau_{\mathcal{K}_{R^*}}} \leq R^{-1}$.*

(6) *If R is reflexive and R is Euclidean, then $R^{-1} = R_{\tau_{\mathcal{K}_{R^*}}}$.*

(7) *If R is an L-fuzzy preorder, then $R^{-1} = R_{\tau_{(\mathcal{K}_{R^*})_*}}$.*

(8) *If R is reflexive and R is Euclidean, then $R = R_{\tau_{(\mathcal{K}_{R^*})_*}}$.*

Proof. (1) Since $R(x,y) \odot B(z) \odot R(z,x) \leq B(z) \odot R(z,y)$ iff $R(x,y) \leq B(z) \odot R(z,x) \to B(z) \odot R(z,y)$. Thus

$$\begin{aligned} R_{\tau_{\mathcal{K}_{R^*}}}(x,y) &= \textstyle\bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B(x) \to B(y)) \\ &= \textstyle\bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}((\mathcal{K}_{R^*}(B))^*(x) \to (\mathcal{K}_{R^*}(B))^*(y)) \\ &\geq \textstyle\bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}\bigvee_{z\in X}(B(z) \odot R(z,x) \to B(z) \odot R(z,y)) \\ &\geq R(x,y). \end{aligned}$$

(2) Since $R_z \in \tau_{\mathcal{K}_{R^*}}$ for $z \in X$, then

$$\begin{aligned} R_{\tau_{\mathcal{K}_{R^*}}}(x,y) &= \textstyle\bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B(x) \to B(y)) \\ &\leq \textstyle\bigwedge_{z\in X}(R_z(x) \to R_z(y)) \\ &= \textstyle\bigwedge_{z\in X}(R(z,x) \to R(z,y)) \leq R(x,y). \end{aligned}$$

(3) Since R is transitive, by (1), $R \leq R_{\tau_{\mathcal{K}_{R^*}}}$. Since R is an L-fuzzy preorder, we have $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$. Since $\mathcal{K}^*_{R^*}(\top_z)(x) = R(z,x)$, we have

$$\begin{aligned} R_{\tau_{\mathcal{K}_{R^*}}}(x,y) &= \textstyle\bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B(x) \to B(y)) \\ &\leq \textstyle\bigwedge_{z\in X}(\mathcal{K}^*_{R^*}(\top_z)(x) \to \mathcal{K}^*_{R^*}(\top_z)(y)) \\ &= \textstyle\bigwedge_{z\in X}(R(z,x) \to R(z,y)) \leq R(x,x) \to R(x,y) = R(x,y). \end{aligned}$$

(4) Since R is Euclidean, then $R(y,x) \odot B(z) \odot R(z,x) \leq B(z) \odot R(z,y)$ iff $R(y,x) \leq B(z) \odot R(z,x) \to B(z) \odot R(z,y)$. Thus

$$\begin{aligned} R_{\tau_{\mathcal{K}_{R^*}}}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B(x) \to B(y)) \\ &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}((\mathcal{K}_{R^*}(B))^*(x) \to (\mathcal{K}_{R^*}(B))^*(y)) \\ &\geq \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}} \bigvee_{z\in X}(B(z) \odot R(z,x) \to B(z) \odot R(z,y)) \\ &\geq R^{-1}(x,y). \end{aligned}$$

(5) Since $R^*_z \in \tau_{\mathcal{K}_{R^*}}$, we have

$$\begin{aligned} R_{\tau_{\mathcal{K}_{R^*}}}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B(x) \to B(y)) \\ &\leq \bigwedge_{z\in X}(R^*(z,x) \to R^*(z,y)) \\ &= \bigwedge_{z\in X}(R(z,y) \to R(z,x)) \leq R(y,x) = R^{-1}(x,y). \end{aligned}$$

(6) Since R is Euclidean, by (4), $R^{-1} \leq R_{\tau_{\mathcal{K}_{R^*}}}$. Since R is reflexive and Euclidean, by Theorem 7(3), we have $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$. Since $\mathcal{K}_{R^*}(\top_z) = R^*(z,-) \in \tau_{\mathcal{K}_{R^*}}$, we have

$$\begin{aligned} R_{\tau_{\mathcal{K}_{R^*}}}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B(x) \to B(y)) \\ &\leq \bigwedge_{z\in X}(\mathcal{K}_{R^*}(\top_z)(x) \to \mathcal{K}_{R^*}(\top_z)(y)) \\ &= \bigwedge_{z\in X}(R^*(z,x) \to R^*(z,y)) = \bigwedge_{z\in X}(R(z,y) \to R(z,x)) \\ &\leq R(y,y) \to R(y,x) = R^{-1}(x,y). \end{aligned}$$

(7) Since R is transitive, we have

$$\begin{aligned} R_{(\tau_{\mathcal{K}_{R^*}})_*}(x,y) &= \bigwedge_{B\in(\tau_{\mathcal{K}_{R^*}})_*}(B(x) \to B(y)) = \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B^*(x) \to B^*(y)) \\ &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(\mathcal{K}_{R^*}(B)(x) \to \mathcal{K}_{R^*}(B)(y)) \\ &\geq \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}} \bigwedge_{z\in X}((B(z) \to R^*(z,x)) \to (B(z) \to R^*(z,y))) \\ &\geq \bigwedge_{z\in X}(R^*(z,x) \to R^*(z,y)) = \bigwedge_{z\in X}(R(z,y) \to R(z,x)) \\ &\geq R^{-1}(x,y). \end{aligned}$$

Since R is an L-fuzzy preorder, by Theorem 7(2), $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ for all $A \in L^X$, $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(\top_z)) = \mathcal{K}_{R^*}(\top_z)$ for all $z \in X$. So, $\mathcal{K}^*_{R^*}(\top_z) \in \tau_{\mathcal{K}_{R^*}}$. Thus,

$$\begin{aligned} R_{(\tau_{\mathcal{K}_{R^*}})_*}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B^*(x) \to B^*(y)) \\ &\leq \bigwedge_{z\in X}(\mathcal{K}_{R^*}(\top_z)(x) \to \mathcal{K}_{R^*}(\top_z)(y)) \\ &= \bigwedge_{z\in X}(R^*(z,x) \to R^*(z,y)) = \bigwedge_{z\in X}(R(z,y) \to R(z,x)) \\ &\leq R(y,y) \to R(y,x) = R^{-1}(x,y). \end{aligned}$$

(8) Since $\bigvee_{x\in X}(R(z,y)\odot R(x,y)) \le R(z,x)$ for all $x,y,z \in X$, then $R(z,y)\odot R(x,y) \le R(z,x)$ iff $R(x,y) \le R(z,y) \to R(z,x)$. Thus

$$\begin{aligned} R_{(\tau_{\mathcal{K}_{R^*}})_*}(x,y) &= \bigwedge_{B\in(\tau_{\mathcal{K}_{R^*}})_*}(B(x)\to B(y)) = \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B^*(x)\to B^*(y)) \\ &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(\mathcal{K}_{R^*}(B)(x)\to \mathcal{K}_{R^*}(B)(y)) \\ &\ge \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(\bigwedge_{z\in X}(B(z)\to R^*(z,x))\to \bigwedge_{w\in X}(B(w)\to R^*(w,y))) \\ &\ge \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}\bigwedge_{z\in X}((B(z)\to R^*(z,x))\to (B(z)\to R^*(z,y))) \\ &\ge \bigwedge_{z\in X}(R^*(z,x)\to R^*(z,y)) \\ &= \bigwedge_{z\in X}(R(z,y)\to R(z,x)) \ge R(x,y). \end{aligned}$$

Since R is reflexive and Euclidean, by Theorem 7(3), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$ for all $A\in L^X$, $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(\top_z)) = \mathcal{K}^*_{R^*}(\top_z)$ for all $z\in X$. So, $\mathcal{K}_{R^*}(\top_z)\in \tau_{\mathcal{K}_{R^*}}$. Thus,

$$\begin{aligned} R_{(\tau_{\mathcal{K}_{R^*}})_*}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{K}_{R^*}}}(B^*(x)\to B^*(y)) \\ &\le \bigwedge_{z\in X}(\mathcal{K}^*_{R^*}(\top_z)(x)\to \mathcal{K}^*_{R^*}(\top_z)(y)) \\ &= \bigwedge_{z\in X}(R(z,x)\to R(z,y)) \le R(x,x)\to R(x,y) = R(x,y). \end{aligned}$$

□

Example 13. Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with the law of double negation defined by

$$x\odot y = (x+y-1)\vee 0,\;\; x\to y = (1-x+y)\wedge 1,\; x^* = 1-x.$$

Let $X=\{a,b,c\}$ and $A\in L^X$ as follows:

$$A(a)=1, A(b)=0.1, A(c)=0.4.$$

Define $R\in L^{X\times X}$ as follows

$$R = \begin{pmatrix} 1 & 0.2 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.6 & 0.5 & 1 \end{pmatrix}.$$

(1) Since $0.4 = R(a,c)\odot R(c,b) \not\le R(a,b) = 0.2$ and $\mathcal{K}_{R^*}(A) = (0,0.8,0.1)$, we have

$$\mathcal{K}_{R^*}(A) = (0,0.8,0.1) \ne \mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = (0,0.6,0.1).$$

For $R_x = (1,0.2,0.9), R_y = (0.8,1,0.7), R_z = (0.6,0.5,1)$, since

$$\mathcal{K}_{R^*}(R_x) = (0,0.6,0.1) \ne R^*_x,\;\; \mathcal{K}_{R^*}(R_y) = (0.2,0,0.3) = R^*_y$$

$$\mathcal{K}_{R^*}(R_z) = (0.4, 0.5, 0) = R^*_z,$$

Hence $R_y, R_z \in \tau_{\mathcal{K}_{R^*}}$ but $R_x \notin \tau_{\mathcal{K}_{R^*}}$ and $\bigwedge_{z\in X}(R(z,c) \to R(z,b)) = 0.3 \neq R(c,b) = 0.5$ from:

$$(\bigwedge_{z\in X}(R(z,x) \to R(z,y))) = \begin{pmatrix} 1 & 0.2 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.6 & 0.3 & 1 \end{pmatrix}$$

(2) Since $0.6 = R(a,c) \odot R(b,c) \not\leq R(a,b) = 0.2$, R is not Euclidean. For $\mathcal{K}_{R^*}(A) = (0, 0.8, 0.1)$, we have

$$\mathcal{K}^*_{R^*}(A) = (1, 0.2, 0.9) \neq \mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = (0.4, 0.2, 0.5).$$

For $R^*_x = (0, 0.8, 0.1), R^*_y = (0.2, 0, 0.3), R^*_z = (0.4, 0.5, 0)$, since

$$\mathcal{K}_{R^*}(R^*_x) = (0.4, 0.2, 0.5) \neq R_x, \ \ \mathcal{K}_{R^*}(R^*_y) = (0.8, 1, 0.7) = R_y$$

$$\mathcal{K}_{R^*}(R^*_z) = (0.6, 0.5, 0.7) \neq R_z,$$

Hence $R^*_y \in \tau_{\mathcal{K}_{R^*}}$ but $R^*_x, R^*_z \notin \tau_{\mathcal{K}_{R^*}}$ and $\bigwedge_{z\in X}(R(z,c) \to R(z,b)) = 0.3 \neq R(c,b) = 0.5$ from:

$$(\bigwedge_{z\in X}(R(z,x) \to R(z,y))) = \begin{pmatrix} 1 & 0.2 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.6 & 0.3 & 1 \end{pmatrix}$$

(3) Put $R^2(x,y) = \bigvee_{z\in X}(R(x,z) \odot R(z,y))$, we obtain a relation R^2 as

$$R^2 = \begin{pmatrix} 1 & 0.4 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.6 & 0.5 & 1 \end{pmatrix}.$$

Since $R^2(x,y) \odot R^2(y,z) \leq R^2(x,z)$ and $R^2(x,x) = 1$ for all $x, y, z \in X$, R^2 is an L-fuzzy preorder. By Theorems 7(2) and 8(2), we have

$$\mathcal{K}_{R^{2*}}(A) = \mathcal{K}_{R^{2*}}(\mathcal{K}^*_{R^{2*}}(A)),$$

$$\tau_{\mathcal{K}_{R^{2*}}} = \{\bigvee_{x\in X}(A(x) \odot R^2(x,-)) \mid A \in L^X\}.$$

Since

$$\begin{array}{ll} R_{\tau_{\mathcal{K}_{R^{2*}}}}(x,y) & = \bigwedge_{A\in L^X}(\bigvee_{z\in X}(A(z)\odot R^2(z,x)) \to \bigvee_{w\in X}(A(w)\odot R^2(w,y))) \\ & \geq \bigwedge_{A\in L^X}(\bigwedge_{z\in X}((A(z)\odot R^2(z,x)) \to (A(z)\odot R^2(z,y))) \\ & \geq R^2(x,y), \\ R_{\tau_{\mathcal{K}_{R^{2*}}}}(x,y) & = \bigwedge_{A\in L^X}(\bigvee_{z\in X}(A(z)\odot R^2(z,x)) \to \bigvee_{w\in X}(A(w)\odot R^2(w,y))) \\ & \leq \bigwedge_{p\in X}(\bigvee_{z\in X}(\top_p(z)\odot R^2(z,x)) \to \bigvee_{w\in X}(\top_p(w)\odot R^2(w,y))) \\ & \leq \bigwedge_{p\in X}(R^2(p,x) \to R^2(p,y)) \leq R^2(x,x) \to R^2(x,y) = R^2(x,y), \end{array}$$

we have $R^2(x,y) = R_{\tau_{\mathcal{K}_{R^{2*}}}}$.

For $R^2_x = (1, 0.4, 0.9), R^2_y = (0.8, 1, 0.7), R^2_z = (0.6, 0.5, 1)$, since

$$\mathcal{K}_{R^{2*}}(R^2_x) = (0, 0.6, 0.1) = R^{2*}_x,\;\; \mathcal{K}_{R^{2*}}(R^2_y) = (0.2, 0, 0.3) = R^{2*}_y$$

$$\mathcal{K}_{R^{2*}}(R^2_z) = (0.4, 0.5, 0) = R^{2*}_z,$$

Hence $R^2_x, R^2_y, R^2_z \in \tau_{\mathcal{K}_{R^{2*}}}$ and $\bigwedge_{z\in X}(R^2(z,a) \to R^2(z,b)) = R^2(a,b)$ for all $a, b \in X$.

(4) Put $R^{[2]}(x,y) = \bigvee_{z\in X}(R(x,z)\odot R(y,z))$, we obtain an L-fuzzy relation $R^{[2]}$ as

$$R^{[2]} = \begin{pmatrix} 1 & 0.8 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.9 & 0.7 & 1 \end{pmatrix}.$$

Since $R^{[2]}(x,z)\odot R^{[2]}(y,z) \leq R^{[2]}(x,y)$ and $R^{[2]}(x,x) = 1$ for all $x, y, z \in X$, $R^{[2]}$ is reflexive and Euclidean. By Theorems 7(2) and 8(2), we have

$$\mathcal{K}^*_{R^{[2]*}}(A) = \mathcal{K}_{R^{[2]*}}(\mathcal{K}_{R^{[2]*}}(A)),$$

$$\tau_{\mathcal{K}_{R^{[2]*}}} = \{\bigwedge_{x\in X}(A(x) \to R^{[2]*}(x,-)) \mid A \in L^X\}.$$

Since

$$\begin{aligned}
R_{\tau_{\mathcal{K}_{R^{[2]*}}}}(x,y) &= \bigwedge_{A\in L^X}(\bigwedge_{z\in X}(A(z)\to R^{[2]*}(z,x)) \\
&\quad \to \bigwedge_{w\in X}(A(w)\to R^{[2]*}(w,y))) \\
&\geq \bigwedge_{A\in L^X}(\bigwedge_{z\in X}((A(x)\to R^{[2]*}(z,x)) \\
&\quad \to (A(z)\to R^{[2]}(z,y))) \\
&\geq \bigwedge_{z\in X}(R^{[2]*}(z,x)\to R^{[2]*}(z,y)) = \bigwedge_{z\in X}(R^{[2]}(z,y) \\
&\quad \to R^{[2]}(z,x)) \\
&\geq R^{[2]}(y,x) = R^{[2]-1}(x,y), \\
R_{\tau_{\mathcal{K}_{R^{[2]*}}}}(x,y) &= \bigwedge_{A\in L^X}(\bigwedge_{z\in X}(A(z)\to R^{[2]*}(z,x)) \\
&\quad \to \bigwedge_{w\in X}(A(w)\to R^{[2]*}(w,y))) \\
&\leq \bigwedge_{p\in X}(\bigwedge_{z\in X}(\top_p(z) \\
&\quad \to R^{[2]*}(z,x)) \to \bigvee_{w\in X}(\top_p(w)\to R^{[2]*}(w,y))) \\
&\leq \bigwedge_{p\in X}(R^{[2]*}(p,x)\to R^{[2]*}(p,y)) \\
&= \bigwedge_{p\in X}(R^{[2]}(p,y)\to R^{[2]}(p,x)) \leq R^{[2]}(y,x) = R^{[2]-1}(x,y),
\end{aligned}$$

we have $R^{[2]-1}(x,y) = R_{\tau_{\mathcal{K}_{R^{(2)*}}}}$.

For $R^{[2]*}_x = (0, 0.2, 0.1), R^{[2]*}_y = (0.2, 0, 0.3), R^{[2]*}_z = (0.1, 0.3, 0)$, since

$$\mathcal{K}_{R^{[2]*}}(R^{[2]*}_x) = (1, 0.8, 0.9) = R^{[2]}_x, \;\; \mathcal{K}_{R^{[2]*}}(R^{[2]*}_y) = (0.8, 1, 0.7) = R^{[2]}_y$$

$$\mathcal{K}_{R^{[2]*}}(R^{[2]*}_z) = (0.9, 0.7, 1) = R^{[2]}_z,$$

Hence $R^{[2]*}_x, R^{[2]*}_y, R^{[2]*}_z \in \tau_{\mathcal{K}_{R^{[2]*}}}$ and $\bigwedge_{z\in X}(R^{[2]}(z,a)\to R^{[2]}(z,b)) = R^{[2]}(a,b)$ for all $a, b \in X$.

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