More on the Diophantine equation $3^x + 85^y = z^2$

Banyat Sroysang
Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University, Rangsit Center
Pathumthani, 12121, THAILAND

Abstract: In this paper, we show that $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $3^x + 85^y = z^2$ where $x, y$ and $z$ are non-negative integers. This result implies that $(1, 0, 2)$ is a solution $(x, u, v, z)$ for the Diophantine equation $3^x + 5^u 17^v = z^2$.

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1. Introduction

In [9], Sroysang showed that $(0, 1, 2), (3, 0, 3)$ and $(4, 2, 5)$ are only three solutions $(x, y, z)$ for the Diophantine equation $2^x + 3^y = z^2$ where $x, y$ and $z$ are non-negative integers.

In [8], Rabago solved the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where $x, y$ and $z$ are non-negative integers. The solutions are in $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively.

In [11, 12], Sroysang showed that $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for both two Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ where $x, y$ and $z$ are non-negative integers. We note that $5 \times 17 = 85$. In
this paper, we will solve the Diophantine equation $3^x + 85^y = z^2$ where $x, y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 2)$. Moreover, we obtain that $(1, 0, 0, 2)$ is a solution $(x, u, v, z)$ for the Diophantine equation $3^x + 5^u 17^v = z^2$.

For other results, we refer to [1, 2, 3, 4, 6, 7, 10, 13, 14, 15, 16, 17].

2. Preliminaries

Proposition 2.1. [5] (Catalan’s conjecture) $(3, 2, 2, 3)$ is a unique solution $(a, b, x, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [11] $(1, 2)$ is a unique solution $(x, z)$ for the Diophantine equation $3^x + 1 = z^2$ where $x$ and $z$ are non-negative integers.

Lemma 2.3. The Diophantine equation $1 + 85^y = z^2$ has no non-negative integer solution where $y$ and $z$ are non-negative integers.

Proof. Suppose that there are non-negative integers $y$ and $z$ such that $1 + 85^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. This implies that $y \geq 1$. Thus, $z^2 = 1 + 85^y \geq 1 + 85^1 = 86$. We obtain that $z \geq 10$. Now, we consider on the equation $z^2 - 85^y = 1$. By Proposition 2.1, we have $y = 1$. This implies that $z^2 = 86$. This is a contradiction. Hence, the Diophantine equation $1 + 85^y = z^2$ has no non-negative integer solution.

3. Main Results

Theorem 3.1. $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $3^x + 85^y = z^2$ where $x, y$ and $z$ are non-negative integers.

Proof. Let $x, y$ and $z$ be non-negative integers such that $3^x + 85^y = z^2$. Since $z$ is even, we obtain that $z^2 \equiv 0 \pmod{4}$. Note that $85^y \equiv 1 \pmod{4}$. This implies that $3^x \equiv 3 \pmod{4}$. By Lemma 2.3, we have $x \geq 1$. It follows that $x$ is odd. Now, we will divide the number $y$ into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 2$.

Case $y \geq 1$. Note that $85^y \equiv 0 \pmod{5}$. Since $3^y \equiv 2 \pmod{5}$ or $3^y \equiv 3 \pmod{5}$, we obtain that $z^2 \equiv 2 \pmod{5}$ or $z^2 \equiv 3 \pmod{5}$. This implies that $z$ is even. This is a contradiction.
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Hence, $(1,0,2)$ is a unique non-negative integer solution $(x,y,z)$ for the Diophantine equation $3^x + 85^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.

**Corollary 3.2.** The Diophantine equation $3^x + 85^y = w^4$ has no non-negative integer solution where $x$, $y$ and $w$ are non-negative integers.

**Proof.** Suppose that there are non-negative integers $x$, $y$ and $w$ such that $3^x + 85^y = w^4$. Let $z = w^2$. This implies that $3^x + 85^y = z^2$. By Theorem 3.1, we have $(x,y,z) = (1,0,2)$. This implies that $w^2 = z = 2$. This is a contradiction. Hence, the Diophantine equation $3^x + 85^y = w^4$ has no non-negative integer solution where $x$, $y$ and $w$ are non-negative integers.

**Corollary 3.3.** $(1,0,0,2)$ is a solution $(x,u,v,z)$ for the Diophantine equation $3^x + 5^u 17^v = z^2$.

**Proof.** This follows from Theorem 3.1 where $y = u = v$.

**References**


