International Journal of Pure and Applied Mathematics
**Volume 91 No. 1 2014, 135-138**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
**doi:** http://dx.doi.org/10.12732/ijpam.v91i1.14
ijpam.eu

# MORE ON THE DIOPHANTINE EQUATION $4^x + 10^y = z^2$

Banyat Sroysang
Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University, Rangsit Center
Pathumthani, 12121, THAILAND

**Abstract:** In this paper, we show that the Diophantine equation $4^x+10^y = z^2$ has no non-negative integer solution where $x, y$ and $z$ are non-negative integers.

**AMS Subject Classification:** 11D61
**Key Words:** exponential Diophantine equation

## 1. Introduction

In 2011, Suvarnamani, Singta and Chotchaisthit [21] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where $x, y$ and $z$ are non-negative integers.

In 2012, Chotchaisthit [2] showed that the Diophantine equation $4^x+p^y = z^2$ has no non-negative integer solution where $x, y, z$ are non-negative integers and $p$ is a positive prime.

For related equations, we refer to [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

In this paper, we show that $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $3^x + 45^y = z^2$ where $x, y$ and $z$ are non-negative integers.

Received: January 14, 2014

## 2. Preliminaries

**Proposition 2.1.** [4] (**Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* $(a, b, x, y)$ *for the Diophantine equation* $a^x - b^y = 1$ *where* $a, b, x$ *and* $y$ *are integers such that* $\min\{a, b, x, y\} > 1$.

**Lemma 2.2.** [2, 21] *The Diophantine equation* $4^x + 1 = z^2$ *has no non-negative integer solution where* $x$ *and* $z$ *are non-negative integers.*

**Lemma 2.3.** *The Diophantine equation* $1 + 10^y = z^2$ *has no non-negative integer solution where* $y$ *and* $z$ *are non-negative integers.*

*Proof.* Suppose that there are non-negative integers $y$ and $z$ such that $1 + 10^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Thus, $y \geq 1$. Then $z^2 = 1 + 10^y \geq 1 + 10^1 = 10$ and then $z \geq 4$. Now, we consider on the equation $z^2 - 10^y = 1$. By Proposition 2.1, we have $y = 1$. We obtain that $z^2 = 11$. This is a contradiction. Hence, the Diophantine equation $1 + 10^y = z^2$ has no non-negative integer solution where $y$ and $z$ are non-negative integers. □

## 3. Main Results

**Theorem 3.1.** *The Diophantine equation* $4^x + 10^y = z^2$ *has no non-negative integer solution where* $x, y$ *and* $z$ *are non-negative integers.*

*Proof.* Suppose that there are non-negative integers $x, y$ and $z$ such that $4^x + 10^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. This implies that $z$ is even. Then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. We note that $4^x \equiv 1 \pmod 3$ and $10^x \equiv 1 \pmod 3$. This implies that $z^2 \equiv 2 \pmod 3$. This is a contradiction. Hence, the equation $4^x + 10^y = z^2$ has no non-negative integer solution where $x, y$ and $z$ are non-negative integers. □

**Corollary 3.2.** *Let* $k$ *be a positive integer. Then the Diophantine equation* $4^x + 10^y = w^{2k+2}$ *has no non-negative integer solution where* $x, y$ *and* $w$ *are non-negative integers.*

*Proof.* Suppose that there are non-negative integers $x, y$ and $w$ such that $4^x + 10^y = w^{2k+2}$. Let $z = w^{k+1}$. Thus, $4^x + 10^y = z^2$. This is a contradiction with Theorem 3.1. Hence, the equation $4^x + 10^y = w^{2k+2}$ has no non-negative integer solution where $x, y$ and $w$ are non-negative integers. □

**Corollary 3.3.** *The Diophantine equation* $16^u + 10^y = z^2$ *has no non-negative integer solution where* $u, y$ *and* $z$ *are non-negative integers.*

*Proof.* Suppose that there are non-negative integers $u, y$ and $z$ such that $16^u + 10^y = z^2$. Let $x = 2u$. Thus, $4^x + 10^y = z^2$. This is a contradiction with Theorem 3.1. Hence, the equation $16^u + 10^y = z^2$ has no non-negative integer solution where $u, y$ and $w$ are non-negative integers. □

## References

[1] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, **15** (2007), 145–148.

[2] S. Chotchaisthit, On the Diophantine equation $4^x + p^y = z^2$ where $p$ is a prime number, *Amer. J. Math. Sci.*, **1** (2012), 191–193.

[3] S. Chotchaisthit, On the Diophantine equation $p^x + (p+1)^y = z^2$ where $p$ is a Mersenne prime, *Int. J. Pure Appl. Math.*, **88** (2013), 169–172.

[4] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27** (2004), 167–195.

[5] J. F. T. Rabago, More on the Diophantine equation of type $p^x + q^y = z^2$, *Int. J. Math. Sci. Comp.*, **3** (2013), 15–16.

[6] J. F. T. Rabago, On an open problem by B. Sroysang, *Konuralp J. Math.*, **1** (2013), 30–32.

[7] J. F. T. Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. Math. Sci. Comp.*, **3** (2013), 28–29.

[8] B. Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 133–137.

[9] B. Sroysang, More on the Diophantine equation $2^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **88** (2013), 157–160.

[10] B. Sroysang, More on the Diophantine equation $2^x + 37^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 275–278.

[11] B. Sroysang, More on the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 601–604.

[12] B. Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 605–608.

[13] B. Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 111–114.

[14] B. Sroysang, On the Diophantine equation $5^x + 7^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 115–118.

[15] B. Sroysang, On the Diophantine equation $5^x + 23^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 119–122.

[16] B. Sroysang, On the Diophantine equation $7^x + 8^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 111–114.

[17] B. Sroysang, On the Diophantine equation $23^x + 32^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 231–234.

[18] B. Sroysang, On the Diophantine equation $31^x + 32^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 609–612.

[19] B. Sroysang, On the Diophantine equation $47^x + 49^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 279–282.

[20] B. Sroysang, On the Diophantine equation $89^x + 91^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 283–286.

[21] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Technol. RMUTT J.*, **1** (2011), 25–28.