

MORE ON THE DIOPHANTINE EQUATION $4^x + 10^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

In 2011, Suvarnamani, Singta and Chotchaisthit [21] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers.

In 2012, Chotchaisthit [2] showed that the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where x, y, z are non-negative integers and p is a positive prime.

For related equations, we refer to [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

In this paper, we show that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 45^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [4] (Catalan's conjecture) $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [2, 21] The Diophantine equation $4^x + 1 = z^2$ has no non-negative integer solution where x and z are non-negative integers.

Lemma 2.3. The Diophantine equation $1 + 10^y = z^2$ has no non-negative integer solution where y and z are non-negative integers.

Proof. Suppose that there are non-negative integers y and z such that $1 + 10^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Thus, $y \geq 1$. Then $z^2 = 1 + 10^y \geq 1 + 10^1 = 10$ and then $z \geq 4$. Now, we consider on the equation $z^2 - 10^y = 1$. By Proposition 2.1, we have $y = 1$. We obtain that $z^2 = 11$. This is a contradiction. Hence, the Diophantine equation $1 + 10^y = z^2$ has no non-negative integer solution where y and z are non-negative integers. \square

3. Main Results

Theorem 3.1. The Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

Proof. Suppose that there are non-negative integers x, y and z such that $4^x + 10^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. This implies that z is even. Then $z^2 \equiv 0 \pmod{3}$ or $z^2 \equiv 1 \pmod{3}$. We note that $4^x \equiv 1 \pmod{3}$ and $10^y \equiv 1 \pmod{3}$. This implies that $z^2 \equiv 2 \pmod{3}$. This is a contradiction. Hence, the equation $4^x + 10^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. \square

Corollary 3.2. Let k be a positive integer. Then the Diophantine equation $4^x + 10^y = w^{2k+2}$ has no non-negative integer solution where x, y and w are non-negative integers.

Proof. Suppose that there are non-negative integers x, y and w such that $4^x + 10^y = w^{2k+2}$. Let $z = w^{k+1}$. Thus, $4^x + 10^y = z^2$. This is a contradiction with Theorem 3.1. Hence, the equation $4^x + 10^y = w^{2k+2}$ has no non-negative integer solution where x, y and w are non-negative integers. \square

Corollary 3.3. *The Diophantine equation $16^u + 10^y = z^2$ has no non-negative integer solution where u, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers u, y and z such that $16^u + 10^y = z^2$. Let $x = 2u$. Thus, $4^x + 10^y = z^2$. This is a contradiction with Theorem 3.1. Hence, the equation $16^u + 10^y = z^2$ has no non-negative integer solution where u, y and w are non-negative integers. \square

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