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MORE ON THE DIOPHANTINE EQUATION $8^x + 59^y = z^2$

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Abstract: In this paper, we prove that $(1, 0, 3)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $8^x + 59^y = z^2$ where x, y and z are non-negative integers.

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1. Introduction

In 2012, Sroysang [9] proved that $(1, 0, 3)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers.

In 2013, Rabago [6] showed that $(1, 0, 3)$, $(1, 1, 5)$, $(2, 1, 9)$ and $(3, 1, 23)$ are only four solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers.

In the same year, Sroysang [14] proved that the Diophantine equation $7^x + 8^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. For related papers, we refer to [1, 2, 3, 4, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18].

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In this paper, we prove that $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x+59^y=z^2$ where x,y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [5] (**Catalan's conjecture**) $(3,2,2,3)$ *is a unique solution* (a,b,x,y) *for the Diophantine equation* $a^x-b^y=1$ *where* a,b,x *and* y *are integers such that* $\min\{a,b,x,y\}>1$.

Lemma 2.2. [9] $(1,3)$ *is a unique solution* (x,z) *for the Diophantine equation* $8^x+1=z^2$ *where* x *and* z *are non-negative integers.*

Lemma 2.3. *The Diophantine equation* $1+59^y=z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers y and z such that $1+59^y=z^2$. If $y=0$, then $z^2=2$ which is impossible. Thus, $y\geq 1$. Note that $z^2=1+59^y\geq 1+59^1=60$. Thus, $z\geq 8$. Now, we consider on the equation $z^2-59^y=1$. By Proposition 2.1, we obtain that $y=1$. Then $z^2=60$. This is a contradiction. Hence, the equation $1+59^y=z^2$ has no non-negative integer solution. □

3. Results

Theorem 3.1. $(1,0,3)$ *is a unique solution* (x,y,z) *for the Diophantine equation* $8^x+59^y=z^2$ *where* x,y *and* z *are non-negative integers.*

Proof. Let x,y and z be non-negative integers such that $8^x+59^y=z^2$. By Lemma 2.3, we obtain that $x\geq 1$. Then z is odd. It follows that $z=2t+1$ for some non-negative integer t. Then $8^x+59^y=4(t^2+t)+1$. It follows that $59^y\equiv 1 \pmod 4$. Thus, y is even. Then $y=2k$ for some non-negative integer k. Now, we will divide the number k into two cases.

Case $k=0$. Then $y=0$. By Lemma 2.2, we obtain that $x=1$ and $z=3$.

Case $k\geq 1$. . Note that $z^2-59^{2k}=2^{3x}$. It follows that $(z5-59^k)(z+59^k)=2^{3x}$. Then $z-59^k=2^u$ and $z+59^k=2^{3x-u}$ where u is a non-negative integer. Then $2(59^k)=2^{3x-u}-2^u=2^u(2^{3x-2u}-1)$. We have two subcases.

Subcase $u=0$. Then $z-59^k=1$. It follows that z is even. This is a contradiction.

Subcase $u = 1$. We obtain that $2^{3x-2} - 1 = 59^k$. Then $2^{3x-2} - 59^k = 1$. If $x = 1$, then $k = 0$ and then $y = 0$. This implies that $x \geq 2$. By Proposition 2.1, we have $k = 1$. It follows that $2^{3x-2} = 60$. This is impossible.

Hence, $(1, 0, 3)$ is a unique solution (x, y, z) for the equation $8^x + 59^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation* $8^x + 59^y = w^4$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and w such that $8^x + 59^y = w^4$. Let $z = w^2$. We obtain that $8^x + 59^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 3)$. It follows that $w^2 = z = 3$. This is a contradiction. Hence, the equation $8^x + 59^y = w^4$ has no non-negative integer solution. □

Corollary 3.3. $(3, 0, 3)$ *is a solution* (u, y, z) *for the Diophantine equation* $2^u + 59^y = z^2$.

Proof. This follows from Theorem 3.1 where $u = 3z$. □

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