

ON (n, k) -QUASIPARANORMAL WEIGHTED COMPOSITION OPERATORS

Anuradha Gupta¹, Neha Bhatia² §

¹Department of Mathematics
Delhi College of Arts and Commerce
University of Delhi
Delhi, 110023, INDIA

²Department of Mathematics
University of Delhi
Delhi, 110007, INDIA

Abstract: In this paper we discuss the conditions for a composition operator and a weighted composition operator to be (n, k) -quasiparanormal and (n, k) -quasi- $*$ -paranormal operator.

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1. Introduction

Throughout this paper, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} . In 1950, P. Halmos [7] introduced the extension of normal operators as hyponormal operators (defined by $T^*T \geq TT^*$). As an extension of hyponormal operators, some operators were introduced in recent years. Let n, k be positive integers. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be:

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§Correspondence author

- *class A* if $|T^2| \geq |T|^2$ (see [5, 13]).
- *k-quasiclass A* if $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$ ([6, 12]).
- *k-quasiparanormal* if $\|T^2(T^k(x))\|^{\frac{1}{2}}\|T^kx\|^{\frac{1}{2}} \geq \|T(T^kx)\|$ for all $x \in \mathcal{H}$ (see [9]).
- *(n,k)-quasiparanormal* if $\|T^{1+n}(T^k(x))\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{1+n}} \geq \|T(T^kx)\|$ for all $x \in \mathcal{H}$ (see [14]).
- **-class A* if $|T^2| \geq |T^*|^2$ (see [3]).
- *k-quasi-*-class A* if $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$ ([10]).
- *k-quasi-*-paranormal* if $\|T^2(T^k(x))\|^{\frac{1}{2}}\|T^kx\|^{\frac{1}{2}} \geq \|T^*(T^kx)\|$ for all $x \in \mathcal{H}$ (see [11, 15]).
- *(n,k)-quasi-*-paranormal* if $\|T^{1+n}(T^k(x))\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{1+n}} \geq \|T^*(T^kx)\|$ for all $x \in \mathcal{H}$ (see [15]).

The relations among these various classes of operators are given by

Class A \subseteq *k – quasiclass A* \subseteq *k – quasiparanormal*.

** – Class A* \subseteq *k – quasi – * – class A* \subseteq *k – quasi – * – paranormal*.

Hyponormal \Rightarrow *class A* \Rightarrow *quasi – class A*

\Rightarrow *quasi – paranormal* \Rightarrow *k – quasi – paranormal*.

Also, for $n = 1$

$$(n, k) - \text{quasiparanormal} = k - \text{quasiparanormal}$$

and

$$(n, k) - \text{quasi} - * - \text{paranormal} = k - \text{quasi} - * - \text{paranormal}.$$

Let $L^2(\mu) = L^2(X, \mathcal{A}, \mu)$ where (X, \mathcal{A}, μ) is a σ -finite measure space. The relation of being almost everywhere, denoted by a.e., is an equivalence relation in $L^2(\mu)$. Let T be a measurable transformation on X . Then the *composition operator* C on the space $L^2(\mu)$ induced by T is given by

$$Cf = f \circ T \quad \text{for each } f \in L^2(\mu).$$

Let u be an essentially bounded function. Then the *weighted composition operator* W ($= W_{u,T}$) on the space $L^2(\mu)$ induced by u and T is given by

$$Wf = u \cdot f \circ T \quad \text{for each } f \in L^2(\mu).$$

A transformation T is *measurable* if $T^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$. A measurable transformation T is said to be *non-singular* if

$$\mu(T^{-1}(A)) = 0 \quad \text{whenever } \mu(A) = 0 \quad \text{for } A \in \mathcal{A}.$$

If T is a measurable transformation then T^n is also a measurable transformation. If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to μ . Hence, by *Radon-Nikodym theorem* there exists a unique non-negative essentially bounded measurable function h_n such that

$$\mu(T^{-1})^n(A) = \int_A h_n d\mu \quad \text{for every } A \in \mathcal{A}$$

and h_n is called the *n*th order *Radon-Nikodym derivative* and is denoted by $\frac{d\mu(T^{-1})^n}{d\mu}$. It can be seen that $h_n = h \cdot h \circ T^{-1} \cdot h \circ T^{-2} \dots h \circ T^{-(n-1)}$ and $h_n = h_{n-1} \cdot h \circ T^{-(n-1)}$. Throughout the paper, we assume that u is non-negative.

Proposition 1. *Change of Variable: Let X be a non-empty set and let \mathcal{A} be a σ -algebra on X . Let μ and μT^{-1} be measures on \mathcal{A} and let $h : X \rightarrow [0, \infty]$ be a measurable function. Then the following are equivalent:*

- (i) μT^{-1} is absolutely continuous with respect to μ and h is Radon-Nikodym derivative of μT^{-1} with respect to μ .
- (ii) For every measurable function $f : X \rightarrow [0, \infty]$, the equality

$$\int_X f d\mu T^{-1} = \int_X f h d\mu$$

holds.

The *conditional expectation operator* $E(\cdot | T^{-1}(\mathcal{A})) = E(f)$ is defined for each non-negative function f in L^p ($1 \leq p < \infty$) and is uniquely determined by the following set of conditions:

- (i) $E(f)$ is $T^{-1}(\mathcal{A})$ measurable.

- (ii) If B is any $T^{-1}(\mathcal{A})$ measurable set for which $\int_A f d\mu$ converges then we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$

E is the projection operator onto the closure of the range of the composition operator C on $L^2(\mu)$.

Lemma 2. (see [8]) *Let P be the projection of $L^2(X, \mathcal{A}, \mu)$ onto $\overline{R(C)}$. Then*

- (i) $C^*Cf = hf$ and $CC^*f = (h \circ T)Pf$ for all $f \in L^2(\mu)$.
- (ii) $\overline{R(C)} = \{f \in L^2(\mu) : f \text{ is } T^{-1}(\mathcal{A}) \text{ measurable}\}$.
- (iii) *If f is $T^{-1}(\mathcal{A})$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).*

In [11], Senthilkumar has proved the conditions for composition and weighted composition operators to be k -quasiparanormal operator. In this paper we obtain the conditions for composition and weighted composition operators to be (n,k) -quasiparanormal and (n,k) -quasi- $*$ -paranormal operators in terms of expectation operator and Radon-Nikodym derivative h (or h_n).

2. Compostion Operators

Let C be the composition operator and C^* be its adjoint which is given by $C^*f = h \cdot E(f) \circ T^{-1}$.

Proposition 3. *For every $n \in \mathbb{N}$,*

- (i) $(C^*C)^n f = h^n f$.
- (ii) $(CC^*)^n f = (h \circ T)^n P(f)$.
- (iii) E is the identity operator on $L^2(\mu)$ iff $T^{-1}(\mathcal{A}) = \mathcal{A}$.

Lemma 4. (see [14]) *An operator $T \in \mathcal{B}(\mathcal{H})$ is (n,k) -quasiparanormal if and only if*

$$T^{*k}T^{*(1+n)}T^{(1+n)}T^k - (1+n)\lambda^n T^{*k}T^*TT^k + n\lambda^{1+n}T^{*k}T^k \geq 0$$

for every $\lambda > 0$.

Theorem 5. *Let C be a composition operator on $L^2(\mu)$. Then C is (n, k) -quasiparanormal if and only if $h^{k+n+1} - (1+n)\lambda^n h^{k+1} + n\lambda^{1+n} h^k \geq 0$ a.e.*

Proof. Suppose C is (n, k) -quasiparanormal. Then for every $f \in L^2(\mu)$,

$$\langle (C^{*k} C^{*(1+n)} C^{(1+n)} C^k - (1+n)\lambda^n C^{*k} C^* C C^k + n\lambda^{1+n} C^{*k} C^k) f, f \rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Therefore ,

$$\langle (C^{*k} C^{*(1+n)} C^{(1+n)} C^k - (1+n)\lambda^n C^{*k} C^* C C^k + n\lambda^{1+n} C^{*k} C^k) \chi_A, \chi_A \rangle \geq 0$$

$$\Leftrightarrow \int (h^{k+n+1} \chi_A - (1+n)\lambda^n h^{k+1} \chi_A + n\lambda^{1+n} h^k \chi_A) d\mu \geq 0$$

$$\Leftrightarrow \int_A (h^{k+n+1} - (1+n)\lambda^n h^{k+1} + n\lambda^{1+n} h^k) d\mu \geq 0$$

$$\Leftrightarrow h^{k+n+1} - (1+n)\lambda^n h^{k+1} + n\lambda^{1+n} h^k \geq 0. \quad \square$$

Lemma 6. (see [15]) *An operator $T \in \mathcal{B}(\mathcal{H})$ is (n, k) -quasi- $*$ -paranormal if and only if*

$$T^{*k} T^{*(1+n)} T^{(1+n)} T^k - (1+n)\lambda^n T^{*k} T T^* T^k + n\lambda^{1+n} T^{*k} T^k \geq 0$$

for every $\lambda > 0$.

Theorem 7. *Let C be a composition operator on $L^2(\mu)$. Then C is (n, k) -quasi- $*$ -paranormal if and only if $h^{k+n+1} - (1+n)\lambda^n h_k \cdot E(h) \circ T^{-k} + n\lambda^{1+n} h^k \geq 0$ a.e.*

Proof. Consider

$$\begin{aligned} C^{*k} C C^* C^k f &= C^{*k} C C^*(f \circ T^k) \\ &= C^{*k} C(h \cdot E(f \circ T^k) \circ T^{-1}) \\ &= C^{*k} C(h \circ T^{-1} \cdot f \circ T^{k-1}) \\ &= C^{*k} (h \circ T^{-1} \cdot f \circ T^k) \circ T \\ &= h_k \cdot E(h) \circ T^{-k} \cdot f. \end{aligned}$$

C is (n, k) -quasi- $*$ -paranormal if and only if for every $f \in L^2(\mu)$ and $\lambda > 0$,

$$\langle (C^{*k} C^{*(1+n)} C^{(1+n)} C^k - (1+n)\lambda^n C^{*k} C C^* C^k + n\lambda^{1+n} C^{*k} C^k) f, f \rangle \geq 0$$

$$\Leftrightarrow (h^{k+n+1} - (1+n)\lambda^n h_k \cdot E(h) \circ T^{-k} + n\lambda^{1+n} h^k) f \geq 0$$

$$\Leftrightarrow h^{k+n+1} - (1+n)\lambda^n h_k \cdot E(h) \circ T^{-k} + n\lambda^{1+n} h^k \geq 0. \quad \square$$

Example 8. Consider the space $l^2(w) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)(w)$ where $w = \langle m_n \rangle_{n=1}^{\infty}$ is a sequence of positive real numbers. μ is a measure given by $\mu(n) = m_n$. Let $T : \mathbb{N} \rightarrow \mathbb{N}$ be a non-singular measurable transformation. Then T^n is also a non-singular measurable transformation for $n \in \mathbb{N}$. Now,

$$\begin{aligned} h_k(s) &= \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j, \\ h^k(s) &= \frac{1}{m_s^k} (\sum_{j \in T^{-1}(s)} m_j)^k, \\ E(f)(k) &= \frac{\sum_{j \in T^{-1}T(k)} f_j m_j}{\sum_{j \in T^{-1}T(k)} m_j} \end{aligned}$$

for all non-negative sequence $f = \langle f_n \rangle_{n=1}^{\infty}$ and $s, k \in \mathbb{N}$. By Theorem 5, C is (n, k) -quasiparanormal if and only if

$$\begin{aligned} \frac{1}{m_s^{n+k+1}} (\sum_{j \in T^{-1}(s)} m_j)^{n+k+1} - (1+n)\lambda^n \frac{1}{m_s^{k+1}} (\sum_{j \in T^{-1}(s)} m_j)^{k+1} \\ + n\lambda^{n+1} \frac{1}{m_s^k} (\sum_{j \in T^{-1}(s)} m_j)^k \geq 0. \end{aligned}$$

for all $f = \langle f_n \rangle_{n=1}^{\infty}$ and $s, k \in \mathbb{N}$. By Theorem 7, C is (n, k) -quasi- $*$ -paranormal if and only if

$$\begin{aligned} \frac{1}{m_s^{n+k+1}} (\sum_{j \in T^{-1}(s)} m_j)^{n+k+1} \\ - (1+n)\lambda^n \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j \frac{1}{m_{T^{-k}(s)}} \sum_{j \in T^{-(k+1)}(s)} m_j \\ + n\lambda^{n+1} \frac{1}{m_s^k} (\sum_{j \in T^{-1}(s)} m_j)^k \geq 0. \end{aligned}$$

3. Weighted Composition Operators

Let W be the weighted composition operator on $L^2(\mu)$. Let W^* be its adjoint which is given by $W^*f = h \cdot E(u \cdot f) \circ T^{-1}$ for $f \in L^2(\mu)$. For a positive integer n , $u_n = u \cdot (u \circ T) \cdot (u \circ T)^2 \dots (u \circ T)^{(n-1)}$. For $f \in L^2(\mu)$, $W^n f = u_n \cdot f \circ T^{-n}$ and $W^{*n} f = h_n \cdot E(u_n \cdot f) \circ T^{-n}$.

Proposition 9. (see [4]) For $u \geq 0$,

$$(1) \quad W^*Wf = hE[(u^2)] \circ T^{-1}f.$$

$$(2) \quad WW^*f = u(h \circ T)E(uf).$$

Theorem 10. *Let W be a weighted composition operator on $L^2(\mu)$. Then W is (n, k) -quasiparanormal if and only if $h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot E(u_{k+1}^2) \circ T^{-1} + n\lambda^{n+1} h_k \cdot E(u_k^2) \geq 0$ a.e.*

Proof. Suppose W is (n, k) -quasiparanormal. Then for $f \in L^2(\mu)$ and for $\lambda > 0$,

$$\langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k - (1+n)\lambda^n W^{*k}W^*WW^k + n\lambda^{1+n}W^{*k}W^k)f, f \rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then

$$\begin{aligned} & \langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k - (1+n)\lambda^n W^{*k}W^*WW^k \\ & \quad + n\lambda^{1+n}W^{*k}W^k)\chi_A, \chi_A \rangle \geq 0 \\ \Leftrightarrow & \langle (h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+k+1)} - (1+n)\lambda^n h_{k+1} \cdot E(u_{k+1}^2) \circ T^{-(k+1)} \\ & \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k})\chi_A, \chi_A \rangle \geq 0 \\ \Leftrightarrow & \int (h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+k+1)} - (1+n)\lambda^n h_{k+1} \cdot E(u_{k+1}^2) \circ T^{-(k+1)} \\ & \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k})\chi_A d\mu \geq 0 \\ \Leftrightarrow & \int_A h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+k+1)} - (1+n)\lambda^n h_{k+1} \cdot E(u_{k+1}^2) \circ T^{-(k+1)} \\ & \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k} d\mu \geq 0 \\ \Leftrightarrow & h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot E(u_{k+1}^2) \circ T^{-1} \\ & \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \geq 0. \quad \square \end{aligned}$$

Corollary 11. *If W be a weighted composition operator on $L^2(\mu)$ and $T^{-1}(\mathcal{A}) = \mathcal{A}$. Then W is (n, k) -quasiparanormal if and only if $h_{k+n+1} \cdot (u_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot u_{k+1}^2 \circ T^{-1} + n\lambda^{n+1} h_k \cdot u_k^2 \geq 0$ a.e.*

Theorem 12. *Let W be a weighted composition operator on $L^2(\mu)$. Then W is (n, k) -quasi- $*$ -paranormal if and only if $h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot E(u_{k+1}^2) + n\lambda^{n+1} h_k \cdot E(u_k^2) \geq 0$ a.e.*

Proof. Suppose W is (n, k) -quasi- $*$ -paranormal. Then for $f \in L^2(\mu)$ and for every $\lambda > 0$,

$$\begin{aligned} & \langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k \\ & \quad - (1+n)\lambda^n W^{*k}WW^*W^k + n\lambda^{1+n}W^{*k}W^k)f, f \rangle \geq 0. \end{aligned}$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then

$$\begin{aligned}
& \langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k - (1+n)\lambda^n W^{*k}WW^*W^k \\
& \quad + n\lambda^{1+n}W^{*k}W^k)\chi_A, \chi_A \rangle \geq 0 \\
& \Leftrightarrow \langle (h_{k+n+1}.E(u_{k+n+1}^2) \circ T^{-(n+k+1)} \\
& \quad - (1+n)\lambda^n h_k.E(u_{k+1}(h \circ T)E(u_{k+1})) \circ T^{-k} \\
& \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k})\chi_A, \chi_A \rangle \geq 0 \\
& \Leftrightarrow \int (h_{k+n+1}.E(u_{k+n+1}^2) \circ T^{-(n+k+1)} \\
& \quad - (1+n)\lambda^n h_k \cdot E(u_{k+1}(h \circ T)E(u_{k+1})) \circ T^{-k} \\
& \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k})\chi_A d\mu \geq 0 \\
& \Leftrightarrow \int_A h_{k+n+1}.E(u_{k+n+1}^2) \circ T^{-(n+k+1)} \\
& \quad - (1+n)\lambda^n h_k.E(u_{k+1}(h \circ T)E(u_{k+1})) \circ T^{-k} \\
& \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k} d\mu \geq 0 \\
& \Leftrightarrow h_{k+n+1} \cdot E(u_{k+n+1}^2) \circ T^{-(n+1)} \\
& \quad - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot E(u_{k+1}^2) \\
& \quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \geq 0. \quad \square
\end{aligned}$$

Corollary 13. *Let W is a weighted composition operator on $L^2(\mu)$ and $T^{-1}(\mathcal{A}) = \mathcal{A}$. Then W is (n,k) -quasi- $*$ -paranormal if and only if $h_{k+n+1} \cdot u_{k+n+1}^2 \circ T^{-(n+1)} - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot u_{k+1}^2 + n\lambda^{n+1}h_k \cdot u_k^2 \geq 0$ a.e.*

The operator transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ introduced in [1] by Aluthge is the Aluthge transform of T . The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closed to being a normal operator. More generally we may have family of operators $\{T_r : 0 < r \leq 1\}$ where $T_r = |T|^r U |T|^{1-r}$. For a composition operator C , the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{h}f$ and $Uf = \frac{1}{\sqrt{h \circ T}}f \circ T$. In [2] Lambert has given general Aluthge transformation for composition operator as $C_r = |C|^r U |C|^{1-r}$ and $C_r f = (\frac{h}{h \circ T})^{\frac{r}{2}} f \circ T$. That is C_r is the weighted composition operators with weights $\pi = (\frac{h}{h \circ T})^{\frac{r}{2}}$ where $0 < r < 1$. Since C_r is weighted composition operator it is easy to show that $|C_r|f = \sqrt{h[E(\pi)^2 \circ T^{-1}]}f$ and $|C_r^*|f = \nu E[\nu f]$ where $\nu = \frac{\pi \sqrt{h \circ T}}{[E(\pi \sqrt{h \circ T})^2]^{\frac{1}{4}}}$. Also we have

1. $C_r^k f = \pi_k(f \circ T^k)$.
2. $C_r^{*k} = h^k(E_{\pi_k} f) \circ T^{-k}$.
3. $C_r^{*k} C_r^k f = h^k E(\pi_k^2) \circ T^{-k} f$.

Corollary 14. *Let $C_r \in L^2(\mu)$. Then C_r is (n, k) -quasiparanormal if and only if $h_{k+n+1} \cdot E(\pi_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot E(\pi_{k+1}^2) \circ T^{-1} + n\lambda^{n+1} h_k \cdot E(\pi_k^2) \geq 0$ a.e.*

Proof. Since C_r is the weighted composition operator with $\pi = (\frac{h}{h \circ T})^{\frac{r}{2}}$ it follows from theorem 10 that C_r is (n, k) -quasiparanormal if and only if $h_{k+n+1} \cdot E(\pi_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot E(\pi_{k+1}^2) \circ T^{-1} + n\lambda^{n+1} h_k \cdot E(\pi_k^2) \geq 0$ a.e. \square

Corollary 15. *Let $C_r \in L^2(\mu)$. Then W is (n, k) -quasi- $*$ -paranormal if and only if $h_{k+n+1} \cdot E(\pi_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot E(\pi_{k+1}^2) + n\lambda^{n+1} h_k \cdot E(\pi_k^2) \geq 0$ a.e.*

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