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ON (n,k)-QUASIPARANORMAL WEIGHTED COMPOSITION OPERATORS

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Abstract: In this paper we discuss the conditions for a composition operator and a weighted composition operator to be (n,k)-quasiparanormal and (n,k)-quasi-*-paranormal operator.

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1. Introduction

Throughout this paper, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In 1950, P. Halmos [7] introduced the extension of normal operators as hyponormal operators (defined by $T^*T \geq TT^*$). As an extension of hyponormal operators, some operators were introduced in recent years. Let n,k be positive integers. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be:

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- *class A* if $|T^2| \geq |T|^2$ (see [5, 13]).
- *k-quasiclass A* if $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$ ([6, 12]).
- *k-quasiparanormal* if $\|T^2(T^k(x))\|^{\frac{1}{2}}\|T^kx\|^{\frac{1}{2}} \geq \|T(T^kx)\|$for all $x \in \mathcal{H}$ (see [9]).
- *(n,k)-quasiparanormal* if $\|T^{1+n}(T^k(x))\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{1+n}} \geq \|T(T^kx)\|$ for all $x \in \mathcal{H}$ (see [14]).
- **-class A* if $|T^2| \geq |T^*|^2$ (see [3]).
- *k-quasi-*-class A* if $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$ ([10]).
- *k-quasi-*-paranormal* if $\|T^2(T^k(x))\|^{\frac{1}{2}}\|T^kx\|^{\frac{1}{2}} \geq \|T^*(T^kx)\|$ for all $x \in \mathcal{H}$ (see [11, 15]).
- *(n,k)-quasi-*-paranormal* if $\|T^{1+n}(T^k(x)\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{1+n}} \geq \|T^*(T^kx)\|$ for all $x \in \mathcal{H}$ (see [15]).

The relations among these various classes of operators are given by

$Class\ A \subseteq k - quasiclass\ A \subseteq k - quasiparanormal.$
$* - Class\ A \subseteq k - quasi - * - class\ A \subseteq k - quasi - * - paranormal.$

$$\begin{aligned} Hyponormal &\Rightarrow class\ A \Rightarrow quasi - class\ A \\ &\Rightarrow quasi - paranormal \Rightarrow k - quasi - paranormal. \end{aligned}$$

Also, for $n = 1$

$$(n, k) - quasiparanormal = k - quasiparanormal$$

and

$$(n, k) - quasi - * - paranormal = k - quasi - * - paranormal.$$

Let $L^2(\mu) = L^2(X, \mathcal{A}, \mu)$ where $(X, \mathcal{A}, \mu)$ is a σ-finite measure space. The relation of being almost everywhere, denoted by a.e., is an equivalence relation in $L^2(\mu)$. Let T be a measurable transformation on X. Then the *composition operator* C on the space $L^2(\mu)$ induced by T is given by

$$Cf = f \circ T \quad \text{for each } f \in L^2(\mu).$$

Let u be an essentially bounded function. Then the *weighted composition operator* W ($= W_{u,T}$) on the space $L^2(\mu)$ induced by u and T is given by

$$Wf = u \cdot f \circ T \quad \text{for each } f \in L^2(\mu)\,.$$

A transformation T is *measurable* if $T^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$. A measurable transformation T is said to be *non-singular* if

$$\mu(T^{-1}(A)) = 0 \quad \text{whenever } \mu(A) = 0 \quad \text{for } A \in \mathcal{A}\,.$$

If T is a measurable transformation then T^n is also a measurable transformation. If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to μ. Hence, by *Radon-Nikodym theorem* there exists a unique non-negative essentially bounded measurable function h_n such that

$$\mu(T^{-1})^n(A) = \int_A h_n d\mu \quad \text{for every } A \in \mathcal{A}$$

and h_n is called the nth order *Radon-Nikodym derivative* and is denoted by $\frac{d\mu(T^{-1})^n}{d\mu}$. It can be seen that $h_n = h \cdot h \circ T^{-1} \cdot h \circ T^{-2} \ldots h \circ T^{-(n-1)}$ and $h_n = h_{n-1} \cdot h \circ T^{-(n-1)}$. Throughout the paper , we assume that u is non-negative.

Proposition 1. *Change of Variable: Let X be a non-empty set and let $\mathcal{A}$ be a σ-algebra on X. Let μ and μT^{-1} be measures on $\mathcal{A}$ and let $h : X \to [0,\infty]$ be a measurable function. Then the following are equivalent:*

- *(i) μT^{-1} is absolutely continuous with respect to μ and h is Radon-Nikodym derivative of μT^{-1} with respect to μ.*
- *(ii) For every measurable function $f : X \to [0,\infty]$, the equality*

$$\int_X f d\mu T^{-1} = \int_X f h d\mu$$

holds.

The *conditional expectation operator* $E(\cdot \mid T^{-1}(\mathcal{A})) = E(f)$ is defined for each non-negative function f in L^p $(1 \leq p < \infty)$ and is uniquely determined by the following set of conditions:

- (i) $E(f)$ is $T^{-1}(\mathcal{A})$ measurable.

(ii) If B is any $T^{-1}(\mathcal{A})$ measurable set for which $\int_A f\,d\mu$ converges then we have

$$\int_A f\,d\mu = \int_A E(f)\,d\mu.$$

E is the projection operator onto the closure of the range of the composition operator C on $L^2(\mu)$.

Lemma 2. (see [8]) *Let P be the projection of $L^2(X, \mathcal{A}, \mu)$ onto $\overline{R(C)}$. Then*

(i) $C^*Cf = hf$ *and* $CC^*f = (h \circ T)Pf$ *for all* $f \in L^2(\mu)$.

(ii) $\overline{R(C)} = \{f \in L^2(\mu) : f$ *is* $T^{-1}(\mathcal{A})$ *measurable*$\}$.

(iii) *If f is $T^{-1}(\mathcal{A})$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).*

In [11], Senthilkumar has proved the conditions for composition and weighted composition operators to be k-quasiparanormal operator. In this paper we obtain the conditions for composition and weighted composition operators to be (n,k)-quasiparanormal and (n,k)-quasi-*-paranormal operators in terms of expectation operator and Radon-Nikodyn derivative $h (or h_n)$.

2. Compostion Operators

Let C be the composition operator and C^* be its adjoint which is given by $C^*f = h \cdot E(f) \circ T^{-1}$.

Proposition 3. *For every $n \in \mathbb{N}$,*

(i) $(C^*C)^n f = h^n f$.

(ii) $(CC^*)^n f = (h \circ T)^n P(f)$.

(iii) *E is the identity operator on $L^2(\mu)$ iff $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

Lemma 4. (see [14]) *An operator $T \in \mathcal{B}(\mathcal{H})$ is (n,k)-quasiparanormal if and only if*

$$T^{*k}T^{*(1+n)}T^{(1+n)}T^k - (1+n)\lambda^n T^{*k}T^*TT^k + n\lambda^{1+n}T^{*k}T^k \geq 0$$

for every $\lambda > 0$.

Theorem 5. *Let C be a composition operator on $L^2(\mu)$. Then C is (n,k)-quasiparanormal if and only if $h^{k+n+1} - (1+n)\lambda^n h^{k+1} + n\lambda^{1+n} h^k \geq 0$ a.e.*

Proof. Suppose C is (n,k)-quasiparanormal. Then for every $f \in L^2(\mu)$,

$$\langle (C^{*k}C^{*(1+n)}C^{(1+n)}C^k - (1+n)\lambda^n C^{*k}C^*CC^k + n\lambda^{1+n}C^{*k}C^k)f, f\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Therefore ,

$$\begin{aligned}
&\langle (C^{*k}C^{*(1+n)}C^{(1+n)}C^k - (1+n)\lambda^n C^{*k}C^*CC^k + n\lambda^{1+n}C^{*k}C^k)\chi_A, \chi_A\rangle \geq 0\\
&\Leftrightarrow \quad \int (h^{k+n+1}\chi_A - (1+n)\lambda^n h^{k+1}\chi_A + n\lambda^{1+n}h^k\chi_A)d\mu \geq 0\\
&\Leftrightarrow \quad \int_A (h^{k+n+1} - (1+n)\lambda^n h^{k+1} + n\lambda^{1+n}h^k)d\mu \geq 0\\
&\Leftrightarrow h^{k+n+1} - (1+n)\lambda^n h^{k+1} + n\lambda^{1+n}h^k \geq 0.
\end{aligned}$$
□

Lemma 6. (see [15]) *An operator $T \in \mathcal{B}(\mathcal{H})$ is (n,k)-quasi-*-paranormal if and only if*

$$T^{*k}T^{*(1+n)}T^{(1+n)}T^k - (1+n)\lambda^n T^{*k}TT^*T^k + n\lambda^{1+n}T^{*k}T^k \geq 0$$

for every $\lambda > 0$.

Theorem 7. *Let C be a composition operator on $L^2(\mu)$. Then C is (n,k)-quasi-*-paranormal if and only if $h^{k+n+1} - (1+n)\lambda^n h_k \cdot E(h)\circ T^{-k} + n\lambda^{1+n}h^k \geq 0$ a.e.*

Proof. Consider

$$\begin{aligned}
C^{*k}CC^*C^k f &= C^{*k}CC^*(f \circ T^k)\\
&= C^{*k}C(h \cdot E(f\circ T^k)\circ T^{-1})\\
&= C^{*k}C(h\circ T^{-1} \cdot f\circ T^{k-1})\\
&= C^{*k}(h\circ T^{-1} \cdot f\circ T^k)\circ T\\
&= h_k \cdot E(h)\circ T^{-k} \cdot f.
\end{aligned}$$

C is (n,k)-quasi-*-paranormal if and only if for every $f \in L^2(\mu)$ and $\lambda > 0$,

$$\begin{aligned}
&\langle (C^{*k}C^{*(1+n)}C^{(1+n)}C^k - (1+n)\lambda^n C^{*k}CC^*C^k + n\lambda^{1+n}C^{*k}C^k)f, f\rangle \geq 0\\
&\qquad \Leftrightarrow (h^{k+n+1} - (1+n)\lambda^n h_k \cdot E(h)\circ T^{-k} + n\lambda^{1+n}h^k)f \geq 0\\
&\qquad \Leftrightarrow h^{k+n+1} - (1+n)\lambda^n h_k \cdot E(h)\circ T^{-k} + n\lambda^{1+n}h^k \geq 0.
\end{aligned}$$
□

Example 8. Consider the space $l^2(w) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)(w)$ where $w = \langle m_n \rangle_{n=1}^{\infty}$ is a sequence of positive real numbers. μ is a measure given by $\mu(n) = m_n$. Let $T : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation. Then T^n is also a non-singular measurable transformation for $n \in \mathbb{N}$. Now,

$$h_k(s) = \frac{1}{m_s} \Sigma_{j \in T^{-k}(s)} m_j,$$
$$h^k(s) = \frac{1}{m_s^k} (\Sigma_{j \in T^{-1}(s)} m_j)^k,$$
$$E(f)(k) = \frac{\sum_{j \in T^{-1}T(k) f_j m_j}}{\sum_{j \in T^{-1}T(k) m_j}}$$

for all non-negative sequence $f = \langle f_n \rangle_{n=1}^{\infty}$ and $s, k \in \mathbb{N}$. By Theorem 5, C is (n,k)-quasiparanormal if and only if

$$\frac{1}{m_s^{n+k+1}} (\Sigma_{j \in T^{-1}(s)} m_j)^{n+k+1} - (1+n)\lambda^n \frac{1}{m_s^{k+1}} (\Sigma_{j \in T^{-1}(s)} m_j)^{k+1} + n\lambda^{n+1} \frac{1}{m_s^k} (\Sigma_{j \in T^{-1}(s)} m_j)^k \geq 0.$$

for all $f = \langle f_n \rangle_{n=1}^{\infty}$ and $s, k \in \mathbb{N}$. By Theorem 7, C is (n,k)-quasi-*-paranormal if and only if

$$\frac{1}{m_s^{n+k+1}} (\Sigma_{j \in T^{-1}(s)} m_j)^{n+k+1}$$
$$-(1+n)\lambda^n \frac{1}{m_s} \Sigma_{j \in T^{-k}(s)} m_j \frac{1}{m_{T^{-k}(s)}} \Sigma_{j \in T^{-(k+1)}(s)} m_j$$
$$+ n\lambda^{n+1} \frac{1}{m_s^k} (\Sigma_{j \in T^{-1}(s)} m_j)^k \geq 0.$$

3. Weighted Composition Operators

Let W be the weighted composition operator on $L^2(\mu)$. Let W^* be its adjoint which is given by $W^* f = h \cdot E(u \cdot f) \circ T^{-1}$ for $f \in L^2(\mu)$. For a positive integer n, $u_n = u \cdot (u \circ T) \cdot (u \circ T)^2 \ldots (u \circ T)^{(n-1)}$. For $f \in L^2(\mu)$, $W^n f = u_n \cdot f \circ T^{-n}$ and $W^{*n} f = h_n \cdot E(u_n \cdot f) \circ T^{-n}$.

Proposition 9. (see [4]) *For $u \geq 0$,*

(1) $W^* W f = hE[(u^2)] \circ T^{-1} f.$

(2) $WW^*f = u(h\circ T)E(uf)$.

Theorem 10. *Let W be a weighted compostion operator on $L^2(\mu)$. Then W is (n,k)-quasiparanormal if and only if $h_{k+n+1}\cdot E(u^2_{k+n+1})\circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1}\cdot E(u^2_{k+1})\circ T^{-1} + n\lambda^{n+1}h_k\cdot E(u^2_k) \geq 0$ a.e.*

Proof. Suppose W is (n,k)-quasiparanormal. Then for $f\in L^2(\mu)$ and for $\lambda > 0$,

$$\langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k - (1+n)\lambda^n W^{*k}W^*WW^k + n\lambda^{1+n}W^{*k}W^k)f, f\rangle \geq 0.$$

Let $f=\chi_A$ with $\mu(A)<\infty$. Then

$$\begin{aligned}
&\langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k - (1+n)\lambda^n W^{*k}W^*WW^k\\
&+ n\lambda^{1+n}W^{*k}W^k)\chi_A, \chi_A\rangle \geq 0\\
&\Leftrightarrow \langle (h_{k+n+1}\cdot E(u^2_{k+n+1})\circ T^{-(n+k+1)} - (1+n)\lambda^n h_{k+1}\cdot E(u^2_{k+1})\circ T^{-(k+1)}\\
&+ n\lambda^{n+1}h_k\cdot E(u^2_k)\circ T^{-k})\chi_A, \chi_A\rangle \geq 0\\
&\Leftrightarrow \int (h_{k+n+1}\cdot E(u^2_{k+n+1})\circ T^{-(n+k+1)} - (1+n)\lambda^n h_{k+1}\cdot E(u^2_{k+1})\circ T^{-(k+1)}\\
&+ n\lambda^{n+1}h_k\cdot E(u^2_k)\circ T^{-k})\chi_A d\mu \geq 0\\
&\Leftrightarrow \int_A h_{k+n+1}.E(u^2_{k+n+1})\circ T^{-(n+k+1)} - (1+n)\lambda^n h_{k+1}.E(u^2_{k+1})\circ T^{-(k+1)}\\
&+ n\lambda^{n+1}h_k\cdot E(u^2_k)\circ T^{-k}d\mu \geq 0\\
&\Leftrightarrow h_{k+n+1}\cdot E(u^2_{k+n+1})\circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1}\cdot E(u^2_{k+1})\circ T^{-1}\\
&+ n\lambda^{n+1}h_k\cdot E(u^2_k) \geq 0.
\end{aligned}$$

□

Corollary 11. *If W be a weighted compostion operator on $L^2(\mu)$ and $T^{-1}(\mathcal{A}) = \mathcal{A}$. Then W is (n,k)-quasiparanormal if and only if $h_{k+n+1}\cdot(u^2_{k+n+1})\circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1}\cdot u^2_{k+1}\circ T^{-1} + n\lambda^{n+1}h_k\cdot u^2_k \geq 0$ a.e.*

Theorem 12. *Let W be a weighted compostion operator on $L^2(\mu)$. Then W is (n,k)-quasi-*-paranormal if and only if $h_{k+n+1}.E(u^2_{k+n+1})\circ T^{-(n+1)} - (1+n)\lambda^n h_k\cdot h\circ T^{-(k-1)}\cdot E(u^2_{k+1}) + n\lambda^{n+1}h_k\cdot E(u^2_k) \geq 0$ a.e.*

Proof. Suppose W is (n,k)-quasi-*-paranormal. Then for $f\in L^2(\mu)$ and for every $\lambda > 0$,

$$\begin{aligned}
&\langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k\\
&\qquad - (1+n)\lambda^n W^{*k}WW^*W^k + n\lambda^{1+n}W^{*k}W^k)f, f\rangle \geq 0.
\end{aligned}$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then

$$\begin{aligned}
&\langle (W^{*k}W^{*(1+n)}W^{(1+n)}W^k - (1+n)\lambda^n W^{*k}WW^*W^k \\
&\quad + n\lambda^{1+n}W^{*k}W^k)\chi_A, \chi_A\rangle \geq 0 \\
&\quad \Leftrightarrow \langle (h_{k+n+1}.E(u^2_{k+n+1}) \circ T^{-(n+k+1)} \\
&\quad - (1+n)\lambda^n h_k.E(u_{k+1}(h\circ T)E(u_{k+1})) \circ T^{-k} \\
&\quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k})\chi_A, \chi_A\rangle \geq 0 \\
&\quad \Leftrightarrow \int (h_{k+n+1}.E(u^2_{k+n+1}) \circ T^{-(n+k+1)} \\
&\quad - (1+n)\lambda^n h_k \cdot E(u_{k+1}(h\circ T)E(u_{k+1})) \circ T^{-k} \\
&\quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k})\chi_A d\mu \geq 0 \\
&\quad \Leftrightarrow \int_A h_{k+n+1}.E(u^2_{k+n+1}) \circ T^{-(n+k+1)} \\
&\quad - (1+n)\lambda^n h_k.E(u_{k+1}(h\circ T)E(u_{k+1})) \circ T^{-k} \\
&\quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \circ T^{-k} d\mu \geq 0 \\
&\quad \Leftrightarrow h_{k+n+1} \cdot E(u^2_{k+n+1}) \circ T^{-(n+1)} \\
&\quad - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot E(u^2_{k+1}) \\
&\quad + n\lambda^{n+1}h_k \cdot E(u_k^2) \geq 0.
\end{aligned}$$

□

Corollary 13. *Let W is a weighted compostion operator on $L^2(\mu)$ and $T^{-1}(\mathcal{A}) = \mathcal{A}$. Then W is (n,k)-quasi-*-paranormal if and only if $h_{k+n+1} \cdot u^2_{k+n+1} \circ T^{-(n+1)} - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot u^2_{k+1} + n\lambda^{n+1}h_k \cdot u_k^2 \geq 0$ a.e.*

The operator transform $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ introduced in [1] by Aluthge is the Aluthge tranform of T. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closed to being a normal operator. More generally we may have family of operators $\{T_r : 0 < r \leq 1\}$ where $T_r = |T|^r U|T|^{1-r}$. For a composition operator C , the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{h}f$ and $Uf = \frac{1}{\sqrt{h\circ T}}f \circ T$. In [2] Lambert has given general Aluthge transformation for composition operator as $C_r = |C|^r U|C|^{1-r}$ and $C_r f = (\frac{h}{h\circ T})^{\frac{r}{2}} f \circ T$. That is C_r is the weighted composition operators with weights $\pi = (\frac{h}{h\circ T})^{\frac{r}{2}}$ where $0 < r < 1$. Since C_r is weighted composition operator it is easy to show that $|C_r|f = \sqrt{h[E(\pi)^2 \circ T^{-1}]}f$ and $|C_r^*|f = \nu E[\nu f]$ where $\nu = \frac{\pi\sqrt{h\circ T}}{[E(\pi\sqrt{h\circ T})^2]^{\frac{1}{4}}}$. Also we have

1. $C_r^k f = \pi_k(f \circ T^k)$.
2. $C_r^{*k} = h^k(E_{\pi_k} f) \circ T^{-k}$.
3. $C_r^{*k} C_r^k f = h^k E(\pi_k^2) \circ T^{-k} f$.

Corollary 14. *Let $C_r \in L^2(\mu)$. Then C_r is (n,k)-quasiparanormal if and only if $h_{k+n+1} \cdot E(\pi_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot E(\pi_{k+1}^2) \circ T^{-1} + n\lambda^{n+1} h_k \cdot E(\pi_k^2) \geq 0$ a.e.*

Proof. Since C_r is the weighted composition operator with $\pi = (\frac{h}{h \circ T})^{\frac{r}{2}}$ it follows from theorem 10 that C_r is (n,k)-quasiparanormal if and only if $h_{k+n+1} \cdot E(\pi_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_{k+1} \cdot E(\pi_{k+1}^2) \circ T^{-1} + n\lambda^{n+1} h_k \cdot E(\pi_k^2) \geq 0$ a.e. $\square$

Corollary 15. *Let $C_r \in L^2(\mu)$. Then W is (n,k)-quasi-*-paranormal if and only if $h_{k+n+1}.E(\pi_{k+n+1}^2) \circ T^{-(n+1)} - (1+n)\lambda^n h_k \cdot h \circ T^{-(k-1)} \cdot E(\pi_{k+1}^2) + n\lambda^{n+1} h_k \cdot E(\pi_k^2) \geq 0$ a.e.*

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Th, **13** (1990), 307–315.

[2] C. Burnap, I. Jung, A. Lambert, *Separating Partial Normality classes with Composition operators*, J. Operator Theory, **53**, No. 2 (2005), 381–387.

[3] C. Burnap, I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., **337**, No. 1 (2008), 686–694, doi: 10.1016/j.jmaa.2007.02.082

[4] J. Campbell, J. Jamison, *On some classes of weighted composition operators*, Glasgow Math J., **32** (1990), 82–94, doi: 10.1017/S0017089500009095.

[5] T. Furuta, M. Ito, T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math., **1** (1998), 389–403.

[6] F. Gao, X. Fang, *On k-quasiclass A operators*, J. Inequal. Appl. 2009, art. ID 921634, 1–10.

[7] P.R. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math., **2** (1950), 125-134.

[8] A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc., **18** (1986), 395–400.

[9] S.Mecheri, *Bishop's property* (β) *and Riesz idempotent for k-quasi-paranormal operators*, Banach J. Math. Anal., **6** (2012), 147–154.

[10] S. Mecheri, *Isolated points of spectrum of k-quasi-*-class A operators*, Studia Math., **208** (2012), 87-96, doi: 10.4064/sm208-1-6

[11] D. Senthilkumar, P. M. Naik, R.Santhi, *Weighted Composition of k-Quasi-paranormal operators*, Int. J. Math. Arch.,**3**, No. 2 (2012), 739–746.

[12] K. Tanahashi, I.H. Jeon, I.H. Kim, A. Uchiyama, *Quasinilpotent part of class A or (p, k)-quasihyponormal operators*, Oper. Theory Adv. Appl. 187, Birkhauser, 2008, 199–210, doi: 10.1007/978-3-7643-8893-5-13.

[13] A. Uchiyama, *An example of non-reducing eigenspace of a paranormal operator*, Nihonkai Math. J., **14** (2003), 121–123.

[14] J.T. Yuan, G.X. Ji, *On (n,k)-quasiparanormal operators*, Studia Math., **209** (2012), 289–301, doi: 10.4064/sm209-3-6.

[15] Q. Zeng, H. Zhong, *On (n,k)-quasi-*-paranormal*, arXiv:1209.5050.