

**CONVERGENCE ANALYSIS OF EXTENDED KALMAN
FILTER IN A NOISY ENVIRONMENT THROUGH
DIFFERENCE EQUATIONS**

S. Elizabeth¹, R. Jothilakshmi² §

^{1,2}Department of Mathematics

Auxilium College

Vellore, 632 006, Tamilnadu, INDIA

Abstract: In this paper, the convergence aspects of the Extended Kalman Filter, when used as a deterministic observer for a nonlinear discrete-time systems, are addressed and analyzed. The conditions needed to ensure the boundedness of the error covariances which are related to the observability properties of the nonlinear systems are identified through difference equations. Furthermore, boundedness and stability conditions are provided in a noisy environment systems.

AMS Subject Classification: 39A10, 39A11, 39A99, 34C99

Key Words: boundedness, stability, extended Kalman filter, positive definite matrices

1. Introduction

Difference equations have many applications in several applied sciences (such as Biology, Ecology, Economics, Population dynamics, Genetics, etc), Engineering (Control Systems, Digital Signal Processing, Image Processing, etc.) and Medicine [1]. For this reason, there exist an increasing interest in studying difference equations and systems of difference equations.

Received: August 4, 2013

© 2014 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

Designing an observer for a nonlinear system is quite a challenge. Thus, as a first step, it is interesting to see how classical linearization techniques work with nonlinear systems and what their limitations are. All biological systems are non-linear, since growth or change cannot continue in a linear manner for very long time, without causing reduction in available resources. Baras [3] describe a method for constructing observers for dynamic systems as asymptotic limits of filters. They discuss the method as applied to the linear case, and a class of nonlinear systems with linear observations in the continuous-time domain. Essentially the extended Kalman filter (EKF) is used as their observer of both linear and nonlinear systems of equations. The extended Kalman filter (EKF) is a well-known standard linearization method for approximate nonlinear filtering [9]. The Kalman filter is one of the most widely used methods for tracking and estimation due to its simplicity, optimality, traceability and robustness. However the application of the Kalman filter to nonlinear systems can be difficult. The most common approach is to use the Extended Kalman filter which simply linearises all nonlinear models, so that the traditional Kalman filter is applied.

The paper is organized as follows. Section 2 describes the basic concepts. Section 3 provides the convergence aspects of the Extended Kalman Filter, when used as a deterministic observer for a nonlinear discrete-time system. Systems with nonlinear output maps are also considered. The conditions needed to ensure the boundedness of the error covariances which are related to the observability properties of the nonlinear systems are identified. Furthermore, the asymptotic convergence of the observation errors are analyzed in a noisy environment. Section 4 concludes the paper.

2. Preliminaries

We denote by $|\cdot|$, the Euclidean norm of a vector, and by $\|\cdot\|$ and $\|\cdot\|$, the induced norms on matrices and tensors. We also adopt the following notations. Consider two symmetric matrices P and Q , of the same dimension, the inequality $P \geq Q$ means that the difference $P - Q$ is non-negative definite matrix. Similarly, $P > Q$ means that $P - Q$ is positive definite matrix. A symmetric matrix Q is said to be bounded from above (below, respectively) if there is a number $q > 0$ such that $Q \leq qI$ ($qI \leq Q$). It is well known that, under stochastic controllability and observability assumptions, the Kalman filter for a time-varying system with artificial noises can be used as a global asymptotic observer for the deterministic system [2].

Consider a system of difference equations [12],

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, & x_0: \text{ unknown,} \\ y_k &= C_k x_k, \end{aligned} \tag{1}$$

where A_k is assumed as invertible matrix, $x_k \in \mathbb{R}^n$ is the input of the system state, $y_k \in \mathbb{R}^m$ is the measured output at a time instant k . Also consider the associated noisy system,

$$\begin{aligned} z_{k+1} &= A_k z_k + B_k u_k + N w_k, \\ \xi_k &= C_k z_k + R v_k, \end{aligned} \tag{2}$$

u_k, v_k, w_k are the Gaussian random vectors with zero mean covariances and mutually independent. The design parameters N and R are to be chosen as positive definite matrices. Filtering is a procedure of estimating hidden states based on observable data [4]. The Kalman filter equations for (2) are given as follows [5].

Now Consider the measurement and time update of a noisy environment system of equations,

Measurement Update

$$\begin{aligned} \hat{x}_k &= \bar{x}_k + K_k (\xi_k - C_k \bar{x}_k), \\ P_k^{-1} &= \bar{P}_k^{-1} + C_k^T (R R^T)^{-1} C_k, \end{aligned} \tag{3}$$

Time Update

$$\begin{aligned} \bar{x}_{k+1} &= A_k \hat{x}_k + B_k u_k, \\ \bar{P}_{k+1} &= A_k P_k A_k^T + N N^T, \end{aligned} \tag{4}$$

$$K_k = P_k C_k^T (R R^T)^{-1} = \bar{P}_k C_k^T (C_k \bar{P}_k C_k^T + R R^T)^{-1}$$

where \bar{P}_k and P_k are the a priori and a posteriori covariances, and \bar{x}_k and \hat{x}_k be the priori and a posteriori estimates of the state at time k , respectively [10]. The filter is initiated with \bar{x}_0 and \bar{P}_0 ; \bar{P}_0 is used as a design parameter, assumed also positive definite.

To obtain an error dynamics, let's rewrite the Kalman filter in terms of the priori variables [5]. From (3) and (4) we use y_k instead of ξ_k ,

$$\bar{x}_{k+1} = A_k (I - K_k C_k) \bar{x}_k + B_k u_k + A_k K_k y_k, \quad (5)$$

$$\bar{P}_{k+1} = A_k (I - K_k C_k) \bar{P}_k A_k^T + N N^T. \quad (6)$$

If we define the error as $e_k = x_k - \bar{x}_k$, then the error dynamics is given as [8]

$$e_{k+1} = A_k (I - K_k C_k) e_k. \quad (7)$$

The associated Riccati difference equations for the error covariances are [6]

$$\bar{P}_{k+1} = A_k \left[\bar{P}_k^{-1} + C_k^T (R R^T)^{-1} C_k \right]^{-1} A_k^T + N N^T, \quad (8)$$

$$P_{k+1}^{-1} = [A_k P_k A_k^T + N N^T]^{-1} + C_k^T (R R^T)^{-1} C_k. \quad (9)$$

Note that $\bar{P}_0 > 0$ and $\text{rank } N = n$ implies $\bar{P}_k > 0$ and $P_k > 0$ for all $0 \leq k < \infty$ [7].

3. Main Results

Economic agents often possess limited information. In addition, economic information is often noisy, possibly because of measurement errors [11]. In this section, we study control problems under partial (or imperfect) information. Since we are interested in the asymptotic behaviour of the error e_k , it is necessary to obtain bounds for $\|\bar{P}_k\|$ and $\|P_k^{-1}\|$ [13, 14].

Lemma 3.1. *Consider the following noisy system:*

$$\begin{aligned} x_{k+1} &= A_k x_k + N w_k, \\ y_k &= C_k x_k + R v_k. \end{aligned} \quad (10)$$

Suppose that there are positive real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that the following conditions hold for some finite $M \geq 0$ and for all $k \geq M$:

$$\alpha_1 I \geq \sum_{i=k-M}^{k-1} \phi(k, i+1) N N^T \phi^T(k, i+1) \geq \alpha_2 I, \quad (11)$$

$$\beta_1 I \leq \sum_{i=k-M}^k \phi^T(i, k) C_i^T (R R^T)^{-1} C_i \phi(i, k) \leq \beta_2 I; \quad (12)$$

then

$$\frac{1}{\beta_2 + 1/\alpha_2} I \leq P_k \leq (\alpha_1 + 1/\beta_1) I,$$

where

$$\phi(k, i) = A_{k-1} A_{k-2} \dots A_i.$$

Remark 3.2. Under the above conditions, it can be shown that \bar{P}_k is bounded from above and below. Indeed, from (4),

$$\|\bar{P}_k\| \leq (\alpha_1 + 1/\beta_1) \|A\|^2 + \|N\|^2.$$

Also, from (3),

$$\bar{P}_k \geq P_k \geq \frac{1}{\beta_2 + 1/\alpha_2} I.$$

Therefore,

$$\frac{1}{\beta_2 + 1/\alpha_2} I \leq \bar{P}_k \leq (\alpha_1 + 1/\beta_1) \|A\|^2 + \|N\|^2 I.$$

It is obvious that P_k^{-1} and \bar{P}_k^{-1} are both bounded from above and below, which ensures the boundedness.

Now we state a theorem on the convergence of the error, on which an extension is made to the nonlinear systems.

Lemma 3.3. Assume for some $a > 0$ there exists a function V such that:

1. $V : n^+ \times B_a \rightarrow \mathbb{R}^+$; $V(k, 0) = 0$ is positive definite and continuous with respect to the second argument;

2. $\Delta V(k, e_k) = V(k+1, e_{k+1}) - V(k, e_k) \leq -\mu(|e_k|)$, where μ is of class K . Then the origin of (7) is asymptotically stable.

Theorem 3.4. Consider the system (1) and the Kalman filter equations (3) and (4) for the associated system (2). Suppose that A_k is invertible for all k and that assumption (1) holds. Suppose that $\|A\| = \sup \{\|A_k\| : k = 0, 1, \dots, \}$ and $\|C\| = \sup \{\|C_k\| : k = 0, 1, \dots, \}$ are bounded. Then the Kalman filter for the noisy system (2) is a global asymptotic observer for the deterministic system (1), as long as N has rank n , R and \bar{Q}_0 are positive definite matrices.

Proof. Let $P_K^- = (Q_k^-)^{-1}$. From (4) we have,

$$A_k^{-1}Q_{k+1}^-A_k^{-T} = Q_k + A_k^{-1}NN^T A_k^{-T}.$$

Inverting the above equation

$$A_k^T P_{k+1}^- A_k = Q_k^{-1} - Q_k^{-1}(Q_k^{-1} + A_k^T(NN^T)^{-1}A_k)^{-1}Q_k^{-1}.$$

If we note

$$\begin{aligned} Q_k &= (I - K_k C_k)Q_k^- \quad \text{or} \\ Q_k^{-1} &= (Q_K^-)^{-1}(I - K_k C_k)^{-1} \end{aligned}$$

then

$$\begin{aligned} A_k^T P_{k+1}^- A_k &= \\ \{P_k^- - P_k^- (I - K_k C_k)^{-1}(Q_k^{-1} + A_k^T(NN^T)^{-1}A_k)^{-1}P_k^-\} (I - K_k C_k)^{-1}. \end{aligned} \quad (13)$$

Thus, from (7) and (13) we have,

$$\begin{aligned} e_{k+1}^T P_{k+1}^- e_{k+1} &= e_k^T (I - K_k C_k)^T A_k^T P_{k+1}^- A_k (I - K_k C_k) e_k \\ &= e_k^T (I - K_k C_k)^T \{P_k^- - P_k^- (I - K_k C_k)^{-1}(Q_k^{-1} \\ &\quad + A_k^T(NN^T)^{-1}A_k)^{-1}P_k^-\} e_k. \end{aligned}$$

Since $Q_k = (I - K_k C_k)Q_k^-$ is symmetric, then $(I - K_k C_k)^T = P_k^- (I - K_k C_k)Q_k^-$.
Therefore,

$$\begin{aligned} e_{k+1}^T P_{k+1}^- e_{k+1} &= e_k^T \{P_k^- (I - K_k C_k) - P_k^- (Q_k^{-1} + A_k^T(NN^T)^{-1}A_k)^{-1}P_k^-\} e_k \\ &= e_k^T P_k^- e_k - e_k^T \{P_k^- K_k C_k + P_k^- (Q_k^{-1} \\ &\quad + A_k^T(NN^T)^{-1}A_k)^{-1}P_k^-\} e_k. \end{aligned}$$

Now if we let $V(k, e_k) = e_k^T P_k^- e_k$ then V satisfies the conditions given in lemma (3.3). Moreover, noting that

$$P_k^- K_k C_k = C_k^T (C_k Q_k^- C_k^T + RR^T)^{-1} C_k.$$

$$\begin{aligned} \Delta V(k, e_k) &= e_{k+1}^T P_{k+1}^- e_{k+1} - e_k^T P_k^- e_k \\ &= -e_k^T \{C_k^T (C_k Q_k^- C_k^T + RR^T)^{-1} C_k + P_k^- (Q_k^{-1} \\ &\quad + A_k^T(NN^T)^{-1}A_k)^{-1}P_k^-\} e_k \\ &\leq -e_k^T P_k^- (Q_k^{-1} + A_k^T(NN^T)^{-1}A_k)^{-1}P_k^- e_k. \end{aligned}$$

Since

$$\begin{aligned} \|Q_k^{-1} + A + k^T(NN^T)^{-1}A_k\| &\leq \|Q_k^{-1}\| + \|N^{-1}A_k\|^2 \\ &\leq p + \|N^{-1}\|^2\|A\|^2 = r \\ e_k^T P_k^- (Q_k^{-1} + A_k^T(NN^T)^{-1}A_k)^{-1} P_k^- e_k &\geq \frac{1}{r} |P_k^- e_k|^2. \end{aligned}$$

If we use $|P_k^- e_k| \geq \frac{1}{q}|e_k|$, the above equation will be of the form,

$$\Delta V(k, e_k) \leq \frac{-1}{rq^2} |e_k|^2 \leq \frac{-1}{rq^2 p_1} V(k, e_k),$$

where we used the bounds given in assumption (1) and $p_1 = p + \|R^{-1}\|^2\|C\|^2$. Therefore by Lemma (3.3), e_k converges to zero asymptotically. \square

4. Conclusion

In this paper we have argued that the principle difficulty for identifying the error in the noisy environment. We applied the Riccati difference equation techniques to the Extended Kalman filter in a noisy environment which consistently predict the new state and observation of the system. Also we extend the arrival of convergence and stability concepts of the Extended Kalman filter through difference equations.

Acknowledgments

This research is funded by the University Grants Commission (F. No. 42 - 20/2013 (SR) UGC), New Delhi, India.

References

- [1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.
- [2] Alfredo, Costanzo, The polynomial extended Kalman filter as an exponential observer for nonlinear discrete-time systems, In: *Proceedings of the 47-th IEEE Conference on Decision and Control Cancun*, Mexico (2008), 9-11.
- [3] J.S. Baras, A. Bensoussan, N.R. James, Dynamic observers as asymptotic limits of recursive filters: Special cases, *SIAM J. Appl. Math.*, **48**, No. 5 (2008), 1147-1158.
- [4] D.E. Catlin, Estimation, control and the discrete Kalman filter, *Applied Mathematical Sciences*, Springer-Verlag (1989), 71-84.
- [5] S.T. Chung, J.W. Grizzle, Sampled-data observer error linearization, *Automatica*, **26**, No. 6 (2001), 997-1007.
- [6] S. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, New York, 1996.
- [7] E.M. Elsayed, On the solution of some difference equations, *European Journal of Pure and Applied Mathematics*, **4**, No. 3 (2011), 287-303.
- [8] Eric J. Kostelich, James A. Yorke, Noise reduction: Finding the simplest dynamical system consistent with the data, *Physica D: Nonlinear Phenomena* (1990), 183-196.
- [9] S.J. Julier, J.K. Uhlmann, H.F. Durrant-Whyte, A new approach for filtering nonlinear systems, In: *The Proceedings of the American Control Conference*, Seattle, Washington (1995), 1628-1632.
- [10] K.H. Kim, G.I. Jee, C.G. Park, J.G. Lee, The stability analysis of the adaptive fading extended Kalman filter using the innovation covariance, *International Journal of Control, Automation and Systems*, **7** (2009), 49-56.
- [11] R. Lopez-Valcarce, S. Dasgupta, Stable estimates in equation error identification, In: *Proceedings of the 36-th IEEE Conference on DEcision and Control*, **2** (1997), 1678-1679.

- [12] J.C. Peyton Jones, S.A. Billings, Recursive algorithm for computing the frequency response of a class of non-linear difference equation models, *International Journal of Control* (2007), 1925-1940.
- [13] Michel Verhaegen, Patrick Dewilde, Subspace model identification Part 1. The output-error state-space model identification class of algorithms, *International Journal of Control* (1991), 1187-1210.
- [14] Y. Song, J.W. Grizzle, The extended Kalman filter as a local asymptotic observer for discrete-time nonlinear systems, *J. Math. Systems Estimate and Control*, **5** (1995), 59-78.

