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τ_c-SETS IN A TREE

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Abstract: Let G be a simple graph. A subset S of $V(G)$ is called a clique transversal set of G if S intersects every clique of G. (That is every maximal complete subgraph of G). The minimum cardinality of a clique transversal set of G is called the clique traversal number of G and is denoted by $\tau_c(G)$. Any clique transversal set of G is a dominating set of G. Also $\gamma(G) \leq \tau_c(G)$. In this paper we characterize trees T those $\tau_c(T) = \gamma(T)$ and for which $\tau_c(T) = \gamma(T) + 1$. We also prove that $\tau_c(T) = n - \Delta(T)$ if and only if T is a wounded spider.

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Key Words: clique transversal sets, clique transversal number

1. Clique Transversals Sets in a Tree

Transversals play an important role in graphs. Representations for different social groups can be done using the graph model and the transversal theory. If a society has the structure of a tree, then a clique transversal set meets all the edges of the tree. Clearly it is a dominating set. A study of clique transversals sets has been made in [1], [2], [3], [4], [5], [7], [8], and [10] and clique transversals

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sets in a tree is made in this paper.

Theorem 1. *For a tree T, $\tau_c(T) \leq \frac{n}{2}$.*

Proof. Let (V_1, V_2) be the bipartite sets of T. Let $|V_1| = m$ and $|V_2| = n$. Let, without loss of generality, $m \leq n$, $\alpha_0(T) = min\{m, n\} = m$,

$$\begin{aligned}\tau_c(T) =& \alpha_0(T) = m\,2m = m + m\\ \leq& m + n\\ =& |V(T)|.\end{aligned}$$

Therefore $m \leq \frac{|V(T)|}{2}$, and $\tau_c(T) \leq \frac{|V(T)|}{2}$. □

Observation 2. *Let n be even. If $\beta_0(G) = \frac{n}{2}$, then $\tau_c(G) = \alpha_0(G) = n - \beta_0(G) = \frac{n}{2}$.*

Theorem 3. *Let T be a tree. $\tau_c(T) = \gamma(T)$ if and only if there exists a minimum dominating set say D of T such that $V(T) - D$ is independent.*

Proof. Suppose T has a minimum dominating set say D of T such that $V(T)-D$ is independent. Then D is also a clique transversal set of T. Therefore $\tau_c(T) \leq |D| = \gamma(T) \leq \tau_c(T)$. Therefore $\tau_c(T) = \gamma(T)$.Conversely, suppose $\tau_c(T) = \gamma(T)$. Let S_1 be a minimum dominating set of T. Since S_1 is a clique transversal set of T , $V - S_1$ is independent. Therefore there exists a minimum dominating set say S of T such that $V(T) - S$ is independent. □

Theorem 4. *Let T be a tree. $\tau_c(T) = \gamma(T) + k$, $k \geq 1$ if and only if there exists a minimum dominating set say D of T such that $\tau_c(\langle V(T) - D\rangle)$ is k.*

Proof. Suppose T has a minimum dominating set say D of T such that $\tau_c(\langle V(T) - D\rangle)$ is k. Let D_1 be a clique transversal set of $\langle V(T) - D\rangle$. Then $|D_1| = k$. $D \cup D_1$ is a clique transversal set of D. Therefore $\tau_c(T) \leq |D \cup D_1| = |D| + |D_1| = \gamma(T) + k$. □

Theorem 5. *Let T be a tree. $\tau_c(T) = \gamma(T) + 1$ if and only if for every minimum clique transversal set D of T, $\tau_c \langle (V(T) - D)\rangle = 1$.*

Proof. Suppose for every minimum clique transversal set D of T,

$$\tau_c \langle (V(T) - D)\rangle = 1.$$

Let $\{u\}$ be a clique transversal set of $\langle (V(T) - D)\rangle$. Then $D \cup \{u\}$ is also a clique transversal set of T. Therefore $\tau_c(T) \leq |D| + 1 = \gamma(T) + 1$.Therefore

$\tau_c(T) \leq \gamma(T)$ or $\tau_c(T) = \gamma(T) + 1$. Suppose $\tau_c(T) \leq \gamma(T)$. But $\gamma(T) \leq \tau_c(T)$. Therefore $\gamma(T) = \tau_c(T)$. There exists a minimum clique transversal set D of T such that $V(T) - D$ is independent, a contradiction to the hypothesis. Therefore $\tau_c(T) = \gamma(T) + 1$. Conversely, let $\tau_c(T) = \gamma(T) + 1$. Let S_1 be a minimum clique transversal set of T. Then S_1 is a dominating set of T. Therefore $|S_1| = \tau_c(T) = \gamma(T) + 1$. Therefore $\tau_c \langle V(T) - S_1 \rangle = 1$. □

Theorem 6. *$\tau_c(T) = n - \Delta(T)$ if and only if T is a wounded spider.*

Proof. Suppose T is a wounded spider. Then $\gamma(T) = n - \Delta(T)$. Also there exists a minimum dominating set of T such that $V(T) - D$ is independent. Therefore $\tau_c(T) = \gamma(T) = n - \Delta(T)$. Conversely, suppose $\tau_c(T) = n - \Delta(T)$. $\tau_c(T) = \alpha_0(T)$ (Since T is triangle free). Therefore $\alpha_0(T) = n - \Delta(T)$. Therefore $\beta_0(T) = \Delta(T)$. Let v be a vertex of maximum degree $\Delta(T)$ in T. Let $u_1, u_2, \ldots, u_{\Delta(T)}$ be the vertices of T adjacent to v. Since T is a tree, $\{u_1, u_2, \ldots, u_{\Delta(T)}\}$ is an independent set of T. Since $\beta_0(T) = \Delta(T)$, $\{u_1, u_2, \ldots, u_{\Delta(T)}\}$ is a maximum independent set of T. If $V(T) - N[v] = \phi$, then T is a star which is a wounded spider. Suppose $V(T) - N[v] \neq \phi$. Let $w \in V(T) - N[v]$. w is not adjacent with v. If w is not adjacent with any u_i, $1 \leq i \leq \Delta(T)$, then $\{u_1, u_2, \ldots, u_{\Delta(T)}, w\}$ is an independent set of T of cardinality $\beta_0(T) + 1$, a contradiction. Therefore w is adjacent with some u_i, $1 \leq i \leq \Delta(T)$. Suppose w is adjacent with u_i and u_j, $i \neq j$, $1 \leq i, j \leq \Delta(T)$. Then w, u_i, v, u_j form a cycle, a contradiction. Therefore w is adjacent with exactly one u_i, $1 \leq i \leq \Delta(T)$. Suppose there exists w_1, $w_2 \in V(T) - N[v]$ such that w_i and w_2 are adjacent with u_i, $1 \leq i \leq \Delta(T)$, then w_i and w_2 are not adjacent and $\{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{\Delta(T)}, w_1, w_2\}$ is an independent set of T of cardinality $\beta_0 + 1$, a contradiction. Therefore any u_i has at most one vertex w_i in $V(T) - N[v]$, adjacent with it. Therefore T is a wounded spider. □

Corollary 7. *If $\tau_c(T) = n - \Delta(T)$, then $\tau_c(T) = \gamma(T)$.*

Proof. Suppose $\tau_c(T) = n - \Delta(T)$. Then T is a wounded spider. Therefore $\gamma(T) = n - \Delta(T)$. That is $\tau_c(T) = \gamma(T)$. □

2. A Constructive Characterization of Trees

The Characterizations: For a constructive characterization of trees T for which $\alpha(T) = \lceil \frac{n(T)}{2} \rceil$, (see [11]) introduce the following operation.

Operation: Let w be an arbitary vertex of a tree T_w and let v be a vertex of the complete graph k_2. Let T be obtained from $T_w \cup k_2$ by adding the edge uw.

We now define the families τ_1 and τ_2 as follows:

$T \in \tau_1$ if and only if $T = K_2$ or T is obtained from K_2 by a finite sequence of operations above.

$T \in \tau_2$ if and only if $T = K_1$ or T is obtained from K_1 by a finite sequence of operations above.

Theorem 8. *Let T be a tree of even order n. Then $\tau_c(T) = \frac{n}{2}$ if and only if $T \in \tau_1$.*

Proof. Follows from the above operation. □

Theorem 9. *Let T be a tree of odd order n. Then $\tau_c(T) = \frac{n-1}{2}$ if and only if $T \in \tau_1$.*

Proof. Follows from the above operation. □

Remark 10. P_4, H_1^+, where H is a non trivial tree and wounded spider with only one leg wounded belong to τ_1.

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