ON THE GENERALIZED HYERS-ULAM STABILITY
FOR EULER-LAGRANGE TYPE FUNCTIONAL EQUATION

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Abstract: In this paper we give the general solution of the quadratic functional equation
\[ f(x + 3y) + f(y + 3z) + f(z + 3x) - 3f(x + y + z) = 7(f(x) + f(y) + f(z)), \]
and investigate its generalized Hyers-Ulam-Rassias stability.

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1. Introduction

In 1940, S.M. Ulam [14] raised the question concerning the stability of group homomorphisms:

Let \( G \) be a group and let \( G' \) be a metric group with metric \( d \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if \( f : G \rightarrow G' \) satisfies
\[ d(f(xy), f(x)f(y)) < \delta \quad \text{for all} \quad x, y \in G, \]
then there exists a homomorphism \( F : G \rightarrow G' \) with
\[ d(f(x), F(x)) < \varepsilon \quad \text{for all} \quad x \in G ? \]
In [2], Hyers considered the case of approximately additive mappings $f : X \rightarrow Y$, where $X$ and $Y$ are Banach spaces and $f$ satisfies
\[ \|f(x + y) - f(x) - f(y)\| \leq \varepsilon \]
for all $x, y \in X$. It was shown that the limit
\[ F(x) = \lim_{n \to \infty} 2^{-n} f(2^n x), \]
exists for all $x \in X$ and that $F : X \rightarrow Y$ is the unique additive mapping satisfying
\[ \|f(x) - F(x)\| \leq \varepsilon. \]

A generalization of Hyers theorem provided by Rassias in [4]. In 1982-1994, J. M. Rassias (see [5-12]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [13]. In 1994, a generalization of the Rassias' theorem was obtained by Gavruta [1] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

Consider the following functional equations:
\[
\begin{align*}
  f(x + 2y) + f(y + 2z) + f(z + 2x) \\
  = 2f(x + y + z) + 3(f(x) + f(y) + f(z)), \quad (I)
\end{align*}
\]
and
\[
 f(x + y) + f(y + z) + f(z + x) = f(x + y + z) + f(x) + f(y) + f(z). \quad (II).
\]

Recently, the author investigated in his paper Zivari the generalized Hyers-Ulam stability of the equation (I), and the functional equation (II) was solved by Pl. Kannappan in [3].

In this paper we consider the quadratic functional equation
\[
f(x + 3y) + f(y + 3z) + f(z + 3x) = 3f(x + y + z) + 7(f(x) + f(y) + f(z)),
\]
and determine the general solution of this functional equation and investigate its generalized Hyers-Ulam-Rassias stability.
2. The General Solution

We commence with the next result which is provide the general solution of the proposed functional equation.

**Theorem 1.** Let $X$ and $Y$ be real vector spaces. A function $f : X \to Y$ satisfies the functional equation

$$f(x + 3y) + f(y + 3z) + f(z + 3x) = 3f(x + y + z) + 7(f(x) + f(y) + f(z)), \quad (1)$$

for all $x, y, z \in X$ if and only if it satisfies

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (x, y \in X). \quad (2)$$

**Proof.** Assume that a function $f : X \to Y$ satisfies (1). Take $x = y = z$ in (1), we get

$$f(4x) - f(3x) = 7f(x), \quad (3)$$

for all $x \in X$, which implies that $f(0) = 0$. Letting $y = z = 0$ in (1), we have

$$f(3x) = 9f(x), \quad (†)$$

Combining Eq.(†) and (3), we get

$$f(4x) = 16f(x), \quad (‡)$$

Letting $z = 0$ in (1), we obtain

$$f(x + 3y) + f(y) + f(3x) - 3f(x + y) = 7(f(x) + f(y)).$$

Applying Eq.(‡), the above equation simplifies to

$$f(x + 3y) - 3f(x + y) = 6f(y) - 2f(x). \quad (4)$$

Replacing $x$ by $y$ and $y$ by $x$ in (4), so

$$f(y + 3x) - 3f(y + x) = 6f(x) - 2f(y). \quad (5)$$

Letting $y = z$ in (1), we get

$$f(x + 3y) + f(4y) + f(y + 3x) - 3f(x + 2y) = 7(f(x) + 2f(y)).$$
Using Eq. (‡), the above equation simplifies to
\[ f(x + 3y) + f(y + 3x) - 3f(x + 2y) = 7f(x) - 2f(y). \] (6)

Eliminating \( f(x + 3y) \) and \( f(y + 3x) \) from (6) by applying (4) and (5), we get
\[ 2f(x + y) + 2f(y) = f(x) + f(x + 2y). \]

Now the classical quadratic functional equation (2) follows if we replacing \( x \) by \( x - y \) in above equation.

Conversely, suppose that a function \( f : X \rightarrow Y \) satisfies (2). Replacing \( x \) with \( x + 2y \) and all cyclic permutations of the variables in (2), then we have
\[ f(x + 3y) + f(x + y) = 2f(x + 2y) + 2f(y), \]
\[ f(y + 3z) + f(y + z) = 2f(y + 2z) + 2f(z), \]
\[ f(z + 3x) + f(z + x) = 2f(z + 2x) + 2f(x). \] (7)

Eliminating \( f(x + 2y) \), \( f(y + 2z) \) and \( f(z + 2x) \) in the sum of all equations in (7), by applying Theorem 2.1 of [15], then we get
\[ f(x + 3y) + f(y + 3z) + f(z + 3x) - 4f(x + y + z) = - (f(x + y) + f(y + z) + f(z + x)) + 8(f(x) + f(y) + f(z)). \] (8)

Combining Eq. (II) and (8), then the functional equation (1) follows, and the proof is complete. \( \square \)

For convenience, we use the following abbreviations:
\[ Df(x, y, z) = f(x + 3y) + f(y + 3z) + f(z + 3x) \]
\[ -3f(x + y + z) - 7(f(x) + f(y) + f(z)). \]

Theorem 2. Suppose \( X \) is a real vector space and \( Y \) is a Banach space. Let \( \varphi : X^3 \rightarrow [0, \infty) \) be a function such that
\[ \lim_{n \rightarrow \infty} 9^{-n}\varphi(3^n x, 3^n y, 3^n z) = 0 \quad (x, y, z \in X), \] (9)
and \( \sum_{n=0}^{\infty} 9^{-n}\varphi(3^n x, 3^n y, 3^n z) \) be convergent. Let \( f : X \rightarrow Y \) be a mapping satisfying \( f(0) = 0 \) and
\[ \|Df(x, y, z)\| \leq \varphi(x, y, z), \] (10)
for all \( x, y, z \in X \), then there exists a unique function \( F : X \rightarrow Y \) which satisfies (1) and
\[ \|f(x) - F(x)\| \leq \frac{1}{9} \sum_{n=0}^{\infty} 9^{-n}\varphi(3^n x, 0, 0) \quad (x \in X). \] (11)
\textbf{Proof.} Letting $y = z = 0$ in (10), we get
\[ \|f(3x) - 9f(x)\| \leq \varphi(x, 0, 0). \]
Dividing the above inequality by 9, we obtain
\[ \|\frac{f(3x)}{9} - f(x)\| \leq \frac{1}{9} \varphi(x, 0, 0). \tag{12} \]
One can use the induction on $n$ to show that
\[ \|\frac{f(3^n x)}{9^n} - f(x)\| \leq \frac{1}{9} \sum_{k=0}^{n-1} 9^{-k} \varphi(3^k x, 0, 0), \tag{13} \]
for all and all $x \in X$. Replacing $x$ by $3^m x$ in (13), we have
\[ \|\frac{f(3^{n+m} x)}{9^{n+m}} - \frac{f(3^m x)}{9^m}\| \leq \frac{1}{9} \sum_{k=m}^{n+m-1} 9^{-k} \varphi(3^k x, 0, 0) \quad (x \in X). \]
It follows that the sequence $\left\{\frac{1}{9^n} f(3^n x)\right\}$ is Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{\frac{1}{9^n} f(3^n x)\right\}$ is convergent. Set
\[ F(x) := \lim_{n \to \infty} \frac{1}{9^n} f(3^n x), \quad (x \in X). \]
Then by the definition of $F$, we can see that (11) holds. To show that $F$ satisfies in (1), we set $(x, y, z) = (3^n x, 3^m y, 3^n z)$ in (10), and divide the result by $9^n$, hence
\[ \frac{1}{9^n} \|Df(3^n x, 3^m y, 3^n z)\| \leq \frac{\varphi(3^n x, 3^m y, 3^n z)}{9^n}. \]
Take the limit as $n \to \infty$, so
\[ \|DF(x, y, z)\| \leq 0, \]
for all $x, y, z \in X$. Therefore, $F$ satisfies (1). The uniqueness of $F$ follows from Theorem 1. \hfill \Box

\textbf{Corollary 3.} Let $f : X \to Y$ be a function with $f(0) = 0$ and
\[ \|Df(x, y, z)\| \leq \varepsilon, \]
for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique function $F : X \to Y$ which satisfies (1) and
\[ \|f(x) - F(x)\| \leq \frac{\varepsilon}{8} \quad (x \in X). \]
Proof. Apply Theorem 2, for $\varphi(x, y, z) = \varepsilon$. 

Corollary 4. Let $f : X \rightarrow Y$ be a function with $f(0) = 0$ and

$$\|Df(x, y, z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

with $p < 2$ and for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique quadratic function $F : X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{|9 - 3p|} \|x\|^p \quad (x \in X).$$

Proof. Apply Theorem 2, for $\varphi(x, y, z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$. 

References


