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SOME NEW GRACEFUL GENERALIZED CLASSES OF LOBSTERS

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Abstract: The lobsters with central paths $H = x_0, x_1, \ldots, x_m$ to which we give graceful labelings satisfy the following properties.

(i) The vertex x_0 may be attached to one among the combinations $(e, 0, o)$, (e, o, e), (e, e, o), $(0, o, e)$, $(0, e, o)$.

(ii) The path $H \backslash \{x_0\}$ can be partitioned into sub paths P_i, $1 \leq i \leq k$, with the following properties.

(a) The vertices in P_i may be attached to at most four different combinations of odd, even, and pendant branches with some restriction on the length of P_i and conditions on the number of odd, even, and pendant branches.

(b) Each vertex in P_i is attached to an odd (or even) number of branches. If each vertex in P_i is attached to a an odd number of branches, then the length of P_i is 4.

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1. Introduction

A *graceful labeling* of a tree T with q edges is a bijection $f : V(T) \rightarrow \{0, 1, 2, \ldots,$

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$q\}$ such that $\{|f(u)-f(v)| : \{u,v\}$ is an edge of $T\} = \{1,2,\ldots,q\}$. A tree which has a graceful labeling is called a *graceful tree.* A *lobster* is a tree having a path from which every vertex has distance at most two. It is readily observed that a lobster L with diameter exceeding four has a unique path $H = x_0x_1 \ldots x_m$ such that besides the adjacencies in H, each x_j, $1 \le j \le m-1$, is at most adjacent to the centers of some stars $K_{1,s}$, $s \ge 0$, whereas the vertices x_0 and x_m are adjacent to the centers of at least one star $K_{1,s}$ with $s \ge 1$. This path H is called the *central path* of the lobster L. If some x_j of H is adjacent to the center of $K_{1,s}, s \ge 0$ then we call $K_{1,s}$ an *even branch* if s is nonzero even, an *odd branch* if s is odd, and a pendant branch if $s = 0$. The combination of branches incident on a vertex on the central path of a lobster is represented by a triple (x,y,z), where x, y, and z represent the number of odd, even, and pendant branches, respectively, incident on that vertex. Throughout the paper we use the symbols o and e to represent an odd number and a non-zero even number, respectively. For example, when we say that an x_j is attached to the combination $(o,0,e)$, we mean that x_j is attached to an odd number of odd branches, no even branch, and an even number of pendant branches.

In 1979, Bermond [1] conjectured that "all lobsters are graceful" which is a special case of the unsolved "the graceful tree conjecture" of Ringel and Kotzig (1964) [15], stating that "all trees are graceful". Bermond's conjecture is also open and all efforts so far have managed to give graceful labelings to only specific classes of lobsters. For details on the progress made so far one may refer to [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16]. In this paper we use extensively the graceful transformations discussed in [4, 5] and give graceful labelings to a generalized class of lobsters with the central path $H = x_0x_1x_2x_3 \ldots x_m$ in which the vertex x_0 may be attached to one among the combinations $(e,0,o)$, (e,o,e), (e,e,o), $(0,o,e)$, and $(0,e,o)$, and the path $H\backslash\{x_0\}$ can be partitioned into finite number of paths P_i, $1 \le i \le n$, such that the branches incident on the vertices of P_i may have one of the following features.

1. P_i consists of only one vertex and it may be attached to any of the combinations among $(o,0,o)$, (o,o,e), and (o,o,e).

2. P_i consists of two vertices and they may be attached to one of the combinations $(o,0,o)$, (o,e,o), (o,o,e), (e,o,o), $(0,o,o)$, (e,e,e), $(0,e,e)$, $(e,0,e)$.

3. P_i consists of four vertices and each vertex in P_i is attached to an odd number of branches in the following manner.

(i) A vertex in P_i may be attached to one of the combinations among $(o,0,e)$, (o,o,o), and (o,e,e).

(ii) If x_j and x_{j+1} be two vertices in P_i occurring consecutively in the central path, then they may be attached to any among the combinations $(o, 0, e)$, (o, e, e), (o, o, o), (e, o, e), $(0, o, e)$, (e, e, o), $(0, e, o)$, $(e, 0, o)$.

Now we state some existing terminologies results borrowed from [4, 5, 9, 10] to prove our main result.

Definition 1.1. For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a, and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from T and making b and u adjacent is termed as *the component $u(T)$ has been transferred or moved from a to b*. In this paper by the label of the component "$u(T)$" we mean the label of the vertex u. Let T be a tree and a and b be two vertices of T. By $a \rightarrow b$ *transfer* we mean that some components from a have been moved to b. If we consider successive transfers $a_1 \rightarrow a_2$, $a_2 \rightarrow a_3$, $a_3 \rightarrow a_4$, ... we simply write $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \ldots$ transfer. In the transfer $a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_{n-1} \rightarrow a_n$, each vertex a_i, $i = 1, 2, \ldots, n-1$ is called a vertex of transfer.

Lemma 1.2. (see [4]) *Let f be a graceful labeling of a tree T; let a and b be two vertices of T; let $u(T)$ and $v(T)$ be two components incident on a where $b \notin u(T) \cup v(T)$. Then the following hold:*

(i) if $f(u)+f(v) = f(a)+f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.

(ii) if $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

Definition 1.3. Let T be a labelled tree with a labeling f. We consider the vertices of T whose labels form the sequence $(a, b, a-1, b+1, a-2, b+2)$ (respectively, $(a, b, a+1, b-1, a+2, b-2)$). Let a be adjacent to some vertices having labels different from the above labels. The $a \longrightarrow b$ transfer is called *a transfer of the first type* if the labels of the transferred components constitute a set of consecutive integers. The $a \longrightarrow b$ transfer is called *a transfer of the second type* if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers. A sequence of eight transfers of the first type $a \rightarrow b \rightarrow a-1 \rightarrow b+1 \rightarrow a \rightarrow b \rightarrow a-1 \rightarrow b+1 \rightarrow a-2$ (respectively, $a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow a+2$), is called a *backward double 8 transfer of the first type* or $BD8TF$ a to $a-2$ (respectively, a to $a+2$). A sequence of five transfers of the first type $a \rightarrow b+1 \rightarrow a-1 \rightarrow b \rightarrow a-2 \rightarrow b+2$ (respectively,

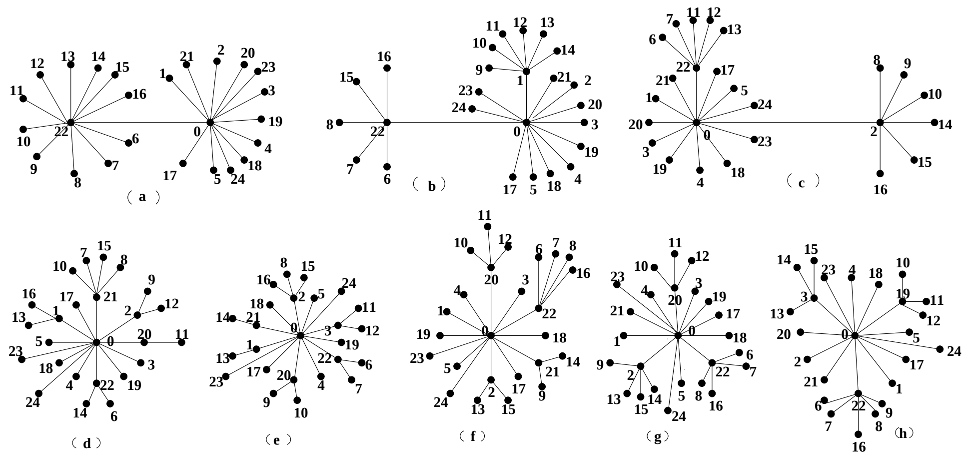

Figure 1: The graceful trees in (b), (c), d), (e), (f), (g), and (h) are obtained from the graceful tree in (a) by applying transfers of the first type $22 \to 1$, the transfer of second type $22 \to 2$, BD8TF 22 to 20, 5TF 22 to 3, 1JTF 22 to 20, 2JTF 22 to 20, and 4JTF 22 to 18, respectively.

$a \to b-1 \to a+1 \to b \to a+2 \to b-2$), is called *a 5 - transfer of the first type* or in brief *5TF* a to $b+2$ (respectively, a to $b-2$). A sequence of four transfers of the first type $a \to b+1 \to a-1 \to b+1 \to a-2$ (respectively, $a \to b-1 \to a+1 \to b-1 \to a+2$), is called *a 1 - jump transfer of the first type* or in brief *1JTF* a to $a-2$ (respectively, a to $a+2$). A sequence of two transfers of the first type $a \to b+1 \to a-2$ (respectively, $a \to b-1 \to a+2$), is called *a 2 - jump transfer of the first type* or in brief *2JTF* a to $a-2$ (respectively, a to $a+2$). A sequence of two transfers of the first type $a \to b+2 \to a-3$ (respectively, $a \to b-2 \to a+3$), is called *a 4 - jump transfer of the first type* or in brief *4JTF* a to $a-3$ (respectively, a to $a+3$).

Lemma 1.4. (see [5, 9, 10] *In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p$ (different from the above vertex labels), which satisfy $(n+1+i)+(n+p-i) = a+b$, $i \geq 0$ (respectively, $(n+i)+(n+p-1-i) = a+b$, $i \geq 0$). Then the following hold.*

- *(a) By making a transfer $a \to b$ of first type we can keep an odd number of components at a from the set A and move the rest to b, and the resultant tree thus formed will be graceful.*
- *(b) If A contains an even number of elements, then by making a sequence of*

transfers of the second type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots$ *(respectively,* $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots$*), an even number of elements from* A *can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.*

(c) *By a BD8TF* a *to* $b+1$ *(respectively,* $b-1$*), we can keep an even number of elements from* A *at* $a, b, a-1$*, and* $b+1$ *(respectively,* $a, b, a+1$*, and* $b-1$*), and move the rest to* $a-2$ *(respectively,* $a+2$*). By a 5TF* a *to* $a-2$ *(respectively,* $a+2$*), we can keep an even number of components at* a *and* $a-2$ *(respectively,* a *and* $a+2$*) and an odd number of components at the remaining vertices of the transfer and move the rest to* $b+2$ *(respectively,* $b-2$*). By a 1JTF* a *to* $b+1$ *(respectively,* $b-1$*), we can keep an even number of elements from* A *at* a*,* $a-1$*, and* $b+1$ *(respectively,* a*,* $a+1$*, and* $b-1$*) and no component at* b*, and move the rest to* $a-2$ *(respectively,* $a+2$*). By a 2JTF* a *to* $b+1$ *(respectively,* $b-1$*), we can keep an even number of components at* a *and* $b+1$ *(respectively,* $b-1$*) and no component at* b *and* $a-1$ *(respectively,* $a+1$*), and move the rest to* $a-2$ *(respectively,* $a+2$*). By making a 4JTF* a *to* $b+2$ *(respectively,* $b-2$*), we can keep an odd number (*≥ 3*) of components at* a *and* $b+2$ *(respectively,* $b-2$*) and no component at* b*,* $a-1$*,* $b+1$*, and* $a-2$ *(respectively,* b*,* $a+1$*,* $b-1$*, and* $a+2$*), and move the rest to* $a-3$ *(respectively,* $a+3$*). The resultant tree formed in each of the above cases is graceful.*

(d) *Consider the transfer* $R: a \to b \to a-1 \to b+1 \to \ldots \to z$ *(respectively,* $a \to b \to a+1 \to b-1 \to \ldots \to z$*), with* $z = a-p_1$ *or* $b+p_2$ *(respectively,* $a+r_1$ *or* $b-r_2$*), such that* R *is partitioned as* $R: T_1 \to T_2 \to T_3 \to \ldots \to T_n$*, where each* T_i*,* $1 \leq i \leq n$*, is either a transfer of the first type or any of the derived transfers. Construct a tree* T^* *from* T *by making the transfer* R *part wise, i.e. first the transfer* T_1*, then* T_2 *and so on. The tree* T^* *is graceful.*

(e) *Consider the transfer* $R': a \to b \to a-1 \to b+1 \to \ldots \to \ldots$ *(respectively,* $a \to b \to a+1 \to b-1 \to \ldots \to \ldots$*), such that* R' *is partitioned as* $R': {T'}_1 \to {T'}_2$*, where* T'_1 *is sequence of transfers consisting of the transfers of the first type and the derived transfers and* T'_2 *is a sequence of transfer of the second type. The tree* T^{**} *obtained from* T *by making the transfer* R' *is graceful.*

2. Results

Construction 2.1. We construct a lobster L with the central path $H = x_0x_1 \dots x_m$ as follows.

Suppose o_j, e_j, and p_j are the number of odd, even, and pendant branches incident on x_j, $0 \leq j \leq m$.

The vertex x_0 may be attached to: $(e, 0, o)$, (e, e, o) or $(0, e, o)$ with $e_0 \equiv (0 \; mod \; 4)$ or either $o_0 \geq 4$ or $p_0 \geq 3$, (e, o, e) or $(0, o, e)$ with $e_0 \equiv (3 \; mod \; 4)$ or $e_0 \geq 5$ and either $o_0 \geq 4$ or $p_0 \geq 4$.

The path $H \backslash \{x_0\}$ is partitioned into n parts as $P_i = x_{t_{i-1}+1}, x_{t_{i-1}+2}, \dots, x_{t_i}$, $t_0 = 0$, $t_n = m$. The value of $t_i - t_{i-1}$ and the combinations of branches incident on the vertices of P_i may satisfy one of the following conditions.

Case I: The sum of the number of branches incident on each vertex in P_i is even.

(1) Let $t_i - t_{i-1} = 1$. Then $P_i = \{x_{t_i}\}$ and the vertex x_{t_i} may be attached to one of the following combinations with the conditions specified.

(a) $(o, 0, o)$.

(b) (o, e, o) with either $e_{t_i} \equiv 0 \; (mod \; 4)$ or $o_{t_i} \geq 3$ or $p_{t_i} \geq 3$.

(c) (o, o, e) with $e_{t_i} \geq 3$ and either $e_{t_i} \equiv 3 \; (mod \; 4)$ or $o_{t_i} \geq 3$ or $p_{t_i} \geq 4$.

(2) Let $t_i - t_{i-1} = 2$ and the combinations of branches incident on $x_{t_{i-1}+1}$ and x_{t_i} may be according to one of the following.

(a) The vertices $x_{t_{i-1}+1}$ and x_{t_i} are attached to either (o, e, o) or $(o, 0, o)$ with one of the following conditions: (i) $e_{t_{i-1}+1} + e_{t_i} \equiv 0 \; (mod \; 4)$.
(ii) $e_{t_{i-1}+1} \equiv 0 \; (mod \; 4)$, $e_{t_i} \equiv 2 \; (mod \; 4)$, and either $p_{t_i} \geq 3$ or $o_{t_i} \geq 3$.
(iii) $e_{t_{i-1}+1} \equiv 2 \; (mod \; 4)$, $e_{t_i} \equiv 0 \; (mod \; 4)$, and either $p_{t_{i-1}+1} \geq 3$ or $o_{t_{i-1}+1} \geq 3$.

(b) Both the vertices $x_{t_{i-1}+1}$ and x_{t_i} are attached to (o, o, e) with one of the following conditions:
(i) $e_{t_{i-1}+1} + e_{t_i} \equiv 2 \; (mod \; 4)$.
(ii) $e_{t_{i-1}+1} \equiv 1 \; (mod \; 4)$, $e_{t_i} \equiv 3 \; (mod \; 4)$, and either $p_{t_i} \geq 4$ or $o_{t_i} \geq 3$.
(iii) $e_{t_{i-1}+1} \equiv 3 \; (mod \; 4)$, $e_{t_i} \equiv 1 \; (mod \; 4)$, and either $p_{t_{i-1}+1} \geq 4$ or $o_{t_{i-1}+1} \geq 3$.

(c) The vertex $x_{t_{i-1}+1}$ is attached to either (o, e, o) or $(o, 0, o)$ and x_{t_i} is attached to (o, o, e) with one of the following conditions:

(i) $e_{t_{i-1}+1} \equiv 0 \; (mod \; 4)$ and $e_{t_i} \equiv 3 \; (mod \; 4)$.

(ii) $e_{t_{i-1}+1} \equiv 2\ (mod\ 4),\ e_{t_i} \equiv 1\ (mod\ 4),\ e_{t_i} \geq 5.$

(iii) $e_{t_{i-1}+1} \equiv 0\ (mod\ 4),\ e_{t_i} \equiv 1\ (mod\ 4),\ e_{t_i} \geq 5$, and either $p_{t_i} \geq 4$ or $o_{t_i} \geq 3$.

(iv) $e_{t_{i-1}+1} \equiv 2\ (mod\ 4),\ e_{t_i} \equiv 3\ (mod\ 4)$, either $p_{t_{i-1}+1} \geq 3$ or $o_{t_{i-1}+1} \geq 3$.

(d) The vertex $x_{t_{i-1}+1}$ is attached to either (o, o, e) and x_{t_i} is attached to (o, e, o) or or $(o, 0, o)$ with one of the following conditions:

(i) $e_{t_{i-1}+1} \equiv 3\ (mod\ 4)$ and $e_{t_i} \equiv 0\ (mod\ 4)$.

(ii) $e_{t_{i-1}+1} \equiv 1\ (mod\ 4),\ e_{t_{i-1}+1} \geq 5,\ e_{t_i} \equiv 2\ (mod\ 4)$.

(iii) $e_{t_{i-1}+1} \equiv 1\ (mod\ 4),\ e_{t_{i-1}+1} \geq 5,\ e_{t_i} \equiv 0\ (mod\ 4)$, and either $p_{t_{i-1}+1} \geq 4$ or $o_{t_{i-1}+1} \geq 3$.

(iv) $e_{t_{i-1}+1} \equiv 3\ (mod\ 4),\ e_{t_i} \equiv 2\ (mod\ 4)$, either $p_{t_i} \geq 3$ or $o_{t_i} \geq 3$.

(e) The vertices $x_{t_{i-1}+1}$ and x_{t_i} are attached to either (e, o, o) or $(0, o, o)$ with one of the following conditions:
(i) $e_{t_{i-1}+1} + e_{t_i} \equiv 0\ (mod\ 4)$.
(ii) $e_{t_{i-1}+1} + e_{t_i} \equiv 2\ (mod\ 4),\ e_{t_{i-1}+1} \geq 3,\ e_{t_i} \geq 3$ with either $o_{t_{i-1}+1} \geq 4$ or $o_{t_i} \geq 4$.
(iii) $e_{t_{i-1}+1} + e_{t_i} \equiv 2\ (mod\ 4),\ e_{t_{i-1}+1} \geq 3,\ e_{t_i} \geq 3$ with either $p_{t_{i-1}+1} \geq 3$ or $p_{t_i} \geq 3$.

(f) The vertices $x_{t_{i-1}+1}$ and x_{t_i} are attached to either (e, e, e) or $(0, e, e)$ with he following conditions:

(i) $e_{t_{i-1}+1} + e_{t_i} \equiv 2\ (mod\ 4)$ and $e_{t_{i-1}+1} + e_{t_i} \geq 6$.

(ii) $e_j \equiv 0\ (mod\ 4),\ e_j \geq 4,\ j = t_{i-1} + 1, t_i$, and $p_{t_{i-1}+1} + p_{t_i} \geq 6$.

(iii) $e_j \equiv 2\ (mod\ 4),\ j = t_{i-1} + 1, t_i$ with one of the following conditions.

(A) $e_{t_i} \geq 6$ with either $o_{t_i} \geq 4$ or $p_{t_i} \geq 4$.

(B) $e_{t_{i-1}+1} \geq 6$ with either $o_{t_{i-1}+1} \geq 4$ or $p_{t_{i-1}+1} \geq 4$.

(g) $x_{t_{i-1}+1}$ is attached to (e, e, e) or $(0, e, e)$ and x_{t_i} is attached to (e, o, o) or $(0, o, o)$ with one of the following conditions:

(i) $e_{t_{i-1}+1} + e_{t_i} \equiv 3\ (mod\ 4)$.

(ii) $e_{t_{i-1}+1} \equiv 0\ (mod\ 4),\ e_{t_i} \equiv 1\ (mod\ 4),\ e_{t_i} \geq 5$ with either $o_{t_i} \geq 4$ or $p_{t_i} \geq 3$.

(iii) $e_{t_{i-1}+1} \equiv 2\ (mod\ 4),\ e_{t_i} \equiv 3\ (mod\ 4)$ with either $o_{t_i} \geq 4$ or $p_{t_i} \geq 3$.

(h) $x_{t_{i-1}+1}$ is attached to (e, o, o) or $(0, o, o)$ and x_{t_i} is attached to (e, e, e) or $(0, e, e)$ with one of the following conditions:

(i) $e_{t_{i-1}+1} + e_{t_i} \equiv 3\ (mod\ 4)$.

(ii) $e_{t_{i-1}+1} \equiv 1\ (mod\ 4)$, $e_{t_i} \equiv 0\ (mod\ 4)$, $e_{t_{i-1}+1} \geq 5$ with either $o_{t_{i-1}+1} \geq 4$ or $p_{t_{i-1}+1} \geq 3$.

(iii) $e_{t_{i-1}+1} \equiv 3\ (mod\ 4)$, $e_{t_i} \equiv 2\ (mod\ 4)$ with either $o_{t_{i-1}+1} \geq 4$ or $p_{t_{i-1}+1} \geq 3$.

(k) Both the vertices $x_{t_{i-1}+1}$ and x_{t_i} are attached to $(e, 0, e)$ with at least one of the odd branches incident on each of $x_{t_{i-1}+1}$ and x_{t_i} contain three or more pendant vertices and $p_{t_{i-1}+1} + p_{t_i} \geq 6$.

Case II: In this case we have $t_i - t_{i-1} = 4$ and the sum of the number of branches incident on each vertex in P_i is odd and have one of the following.

(1) A vertex $x_j \in P_i$ is attached to one of the following combinations with the conditions specified.

(a) $(o, 0, e)$.

(b) (o, e, e) with either $e_{t_i} \equiv 0\ (mod\ 4)$ or $o_{t_i} \geq 3$ or $p_{t_i} \geq 4$.

(c) (o, o, o) with $e_{t_i} \geq 3$, $p_{t_i} \geq 3$ and either $e_{t_i} \equiv 3\ (mod\ 4)$ or $o_{t_i} \geq 3$ or $p_{t_i} \geq 5$.

(2) The vertices x_j and x_{j+1}, $t_{i-1} + 1 \leq j \leq t_{i-1} + 3$, are attached to one of the following combinations with the respective conditions specified.

(a) The vertices x_j and x_{j+1} are attached to either (o, e, e) or $(o, 0, e)$ with one of the following conditions:

(i) $e_j + e_{j+1} \equiv 0\ (mod\ 4)$.

(ii) $e_j \equiv 0\ (mod\ 4)$, $e_{j+1} \equiv 2\ (mod\ 4)$, and either $p_{j+1} \geq 4$ or $o_{j+1} \geq 3$.

(iii) $e_j \equiv 2\ (mod\ 4)$, $e_{j+1} \equiv 0\ (mod\ 4)$, and either $p_j \geq 4$ or $o_j \geq 3$.

(b) Both the vertices x_j and x_{j+1} are attached to (o, o, o) with $p_j \geq 3$ and $p_{j+1} \geq 3$ and one of the following conditions:

(i) $e_j + e_{j+1} \equiv 2\ (mod\ 4)$.

(ii) $e_j \equiv 1\ (mod\ 4)$, $e_{j+1} \equiv 3\ (mod\ 4)$, and either $p_{j+1} \geq 5$ or $o_{j+1} \geq 3$.

(iii) $e_j \equiv 3\ (mod\ 4)$, $e_{j+1} \equiv 1\ (mod\ 4)$, and either $p_j \geq 5$ or $o_j \geq 3$.

(c) The vertex x_j is attached to either (o, e, e) or $(o, 0, e)$ and x_{j+1} is attached to (o, o, o) with one of the following conditions:

(i) $e_j \equiv 0\ (mod\ 4)$ and $e_{j+1} \equiv 3\ (mod\ 4)$.

(ii) $e_j \equiv 2\ (mod\ 4)$, $e_{j+1} \equiv 1\ (mod\ 4)$, $e_{j+1} \geq 5$.

(iii) $e_j \equiv 0\ (mod\ 4)$, $e_{j+1} \equiv 1\ (mod\ 4)$, $e_{j+1} \geq 5$, and either $p_{j+1} \geq 5$ or $o_{j+1} \geq 3$.

(iv) $e_j \equiv 2\ (mod\ 4)$, $e_{j+1} \equiv 3\ (mod\ 4)$, either $p_j \geq 4$ or $o_j \geq 3$.

(d) The vertex x_j is attached to (o, o, o) and x_{j+1} is attached to either (o, e, e) or $(o, 0, e)$ with one of the following conditions:

(i) $e_j \equiv 3\ (mod\ 4)$ and $e_{j+1} \equiv 0\ (mod\ 4)$.

(ii) $e_j \equiv 1\ (mod\ 4)$, $e_j \geq 5$, $e_{j+1} \equiv 2\ (mod\ 4)$.

(iii) $e_j \equiv 1\ (mod\ 4)$, $e_j \geq 5$, $e_{j+1} \equiv 0\ (mod\ 4)$, and either $p_j \geq 5$ or $o_j \geq 3$.

(iv) $e_j \equiv 3\ (mod\ 4)$, $e_{j+1} \equiv 2\ (mod\ 4)$, either $p_{j+1} \geq 4$ or $o_{j+1} \geq 3$.

(e) The vertices x_j and x_{j+1} are attached to either (e, o, e) or $(0, o, e)$ with one of the following conditions:

(i) $e_j + e_{j+1} \equiv 0\ (mod\ 4)$.

(ii) $e_j + e_{j+1} \equiv 2\ (mod\ 4)$, $e_j \geq 3$, $e_{j+1} \geq 3$ with either $o_j \geq 4$ or $o_{j+1} \geq 4$.

(iii) $e_j + e_{j+1} \equiv 2\ (mod\ 4)$, $e_j \geq 3$, $e_{j+1} \geq 3$ with either $p_j \geq 4$ or $p_{j+1} \geq 4$.

(f) The vertices x_j and x_{j+1} are attached to either (e, e, o) or $(0, e, o)$ with he following conditions:

(i) $e_j + e_{j+1} \equiv 2\ (mod\ 4)$ and $e_j + e_{j+1} \geq 6$, and $p_j + p_{j+1} \geq 6$.

(ii) $e_j + e_{j+1} \equiv 0\ (mod\ 4)$, $e_j + e_{j+1} \geq 8$, and $p_j + p_{j+1} \geq 8$.

(iii) $e_j \equiv 2\ (mod\ 4)$, $e_{j+1} \equiv 2\ (mod\ 4)$ with one of the following conditions.

(A) $e_{j+1} \geq 6$ with either $o_{j+1} \geq 4$ or $p_{j+1} \geq 4$.

(B) $e_j \geq 6$ with either $o_j \geq 4$ or $p_j \geq 4$.

(g) x_j is attached to (e, e, e) or $(0, e, e)$ and x_{j+1} is attached to (e, o, o) or $(0, o, o)$ with one of the following conditions:

(i) $e_j + e_{j+1} \equiv 3\ (mod\ 4)$.

(ii) $e_j \equiv 0\ (mod\ 4)$, $e_{j+1} \equiv 1\ (mod\ 4)$, $e_{j+1} \geq 5$ with either $o_{j+1} \geq 4$ or $p_{j+1} \geq 3$.

(iii) $e_j \equiv 2\ (mod\ 4)$, $e_{j+1} \equiv 3\ (mod\ 4)$ with either $o_{j+1} \geq 4$ or $p_{j+1} \geq 3$.

(h) x_j is attached to (e, o, o) or $(0, o, o)$ and x_{j+1} is attached to (e, e, e) or $(0, e, e)$ with one of the following conditions:

(i) $e_j + e_{j+1} \equiv 3\ (mod\ 4)$.

(ii) $e_j \equiv 1\ (mod\ 4)$, $e_{j+1} \equiv 0\ (mod\ 4)$, $e_j \geq 5$ with either $o_j \geq 4$ or $p_j \geq 3$.

(iii) $e_j \equiv 3\ (mod\ 4)$, $e_{j+1} \equiv 2\ (mod\ 4)$ with either $o_j \geq 4$ or $p_j \geq 3$.

(k) Both the vertices x_j and x_{j+1} are attached to $(e, 0, o)$ with at least one of the odd branches incident on each of x_j and x_{j+1} contain three or more pendant vertices and $p_j + p_{j+1} \geq 8$.

$\Diamond$

Example 2.2. The lobster L in Figure 2 is a lobster of the type in Construction 2.1. Here $o_0 = 4$, $e_0 = 2$, $p_0 = 3$; $o_1 = 3$, $e_1 = 0$, $p_1 = 3$;

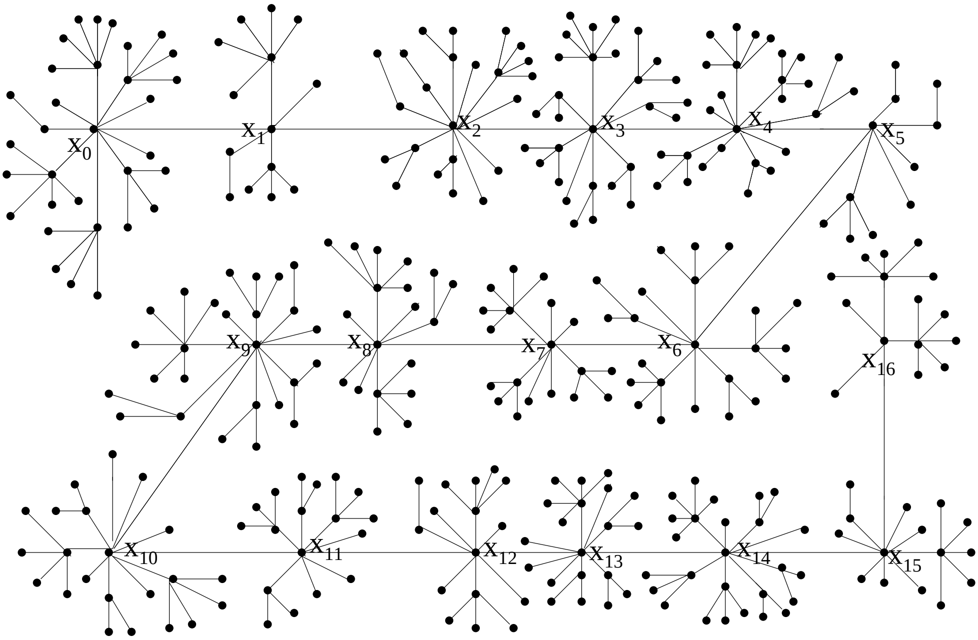

Figure 2: A lobster of the type in Construction 2.1

$o_2 = 2,\ e_2 = 4,\ p_2 = 4$; $o_3 = 2,\ e_3 = 5,\ p_3 = 1$; $o_4 = 3,\ e_4 = 3,\ p_4 = 3$; $o_5 = 3,\ e_5 = 0,\ p_5 = 2$; $o_6 = 1,\ e_6 = 4,\ p_6 = 2$; $o_7 = 3,\ e_7 = 0,\ p_7 = 4$; $o_8 = 1,\ e_8 = 2,\ p_8 = 5$; $o_9 = 2,\ e_9 = 4,\ p_9 = 3$; $o_{10} = 0,\ e_{10} = 4,\ p_{10} = 5$; $o_{11} = 2,\ e_{11} = 2,\ p_{11} = 3$; $o_{12} = 2, e_{12} = 1,\ p_{12} = 4$; $o_{13} = 1,\ e_{13} = 3,\ p_{13} = 4$; $o_{14} = 3,\ e_{14} = 3,\ p_{14} = 2$; $o_{15} = 2,\ e_{15} = 0,\ p_{15} = 6$; $o_{16} = 2,\ e_{16} = 0,\ p_{16} = 2$. Thus, x_0 is attached to (e, e, o), x_1 is attached to $(o, 0, o)$, x_2 is attached to (e, e, e), x_3 is attached to (e, o, o), x_4 is attached to (o, o, o), x_5 is attached to $(o, 0, e)$, x_6 is attached to (o, e, e), x_7 is attached to $(o, 0e)$, x_8 is attached to (o, e, o), x_9 is attached to (e, e, o), x_{10} is attached to $(0, e, o)$, x_{11} is attached to (e, e, o), x_{12} is attached to (e, o, e), each of x_{13} and x_{14} is attached to (o, o, e), and each of x_{15} and x_{16} is attached to $(e, 0, e)$.

Theorem 2.3. *The lobster L in Construction 2.1 is graceful.*

Proof. Suppose $o_0 + e_0 + p_0 = 2\lambda_0 + 1$, for $i = 1, 2, \ldots, m$, $o_i + e_i + p_i = \lambda_i$, and $|E(L)| = q$. Since x_0 is attached to an odd number of branches and the number of vertices of the central path on which odd number of branches are incident is even, $2\lambda_0 + 1 + \sum_{i=1}^{m} \lambda_i$ is odd, say $2k + 1$. Next we proceed as per the following steps.

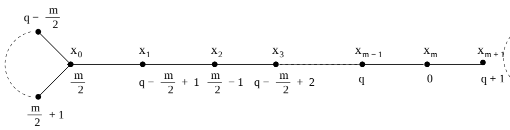

Figure 3: The tree $G(L)$ corresponding to the lobster L for the case m is even.

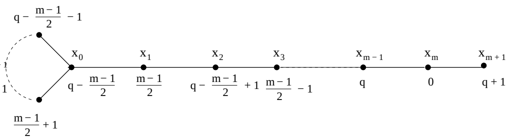

Figure 4: The tree $G(L)$ corresponding to the lobster L for the case m is odd.

Step 1: We first form the graceful tree $G(L)$ as shown in Figures 3 and 4 with $|E(G(L))| = q+1$, i.e. we attach a new pendant vertex x_{m+1} to the vertex x_m, the degree of each vertex x_i, $1 \le i \le m$, is two, and x_0 is attached to $q-m$ pendant vertices. We consider the following graceful labeling f of $G(L)$.

If m is even:

$$f(v) = \begin{cases} \frac{m}{2} - i, \; v = \; x_{2i}, \; i = \; 0, 1, 2, \dots, \frac{m}{2} \\ q - \frac{m}{2} + 1 + i, \; v = \; x_{2i+1}, \; i = \; 0, 1, 2, \dots, \frac{m}{2} \\ r, \; r = \; \frac{m}{2} + 1, \frac{m}{2} + 2, \dots, q - \frac{m}{2} \\ \text{for the } q-m \text{ pendant vertices adjacent to } x_0. \end{cases}$$

If m is odd:

$$f(v) = \begin{cases} \frac{m-1}{2} - i, \; v = \; x_{2i+1}, \; i = \; 0, 1, 2, \dots, \frac{m-1}{2} \\ q - \frac{m-1}{2} + i, \; v = \; x_{2i}, \; i = \; 0, 1, 2, \dots, \frac{m+1}{2} \\ r, \; r = \; \frac{m-1}{2} + 1, \frac{m-1}{2} + 2, \dots, q - \frac{m-1}{2} - 1 \\ \text{for the } q-m \text{ pendant vertices adjacent to } x_0. \end{cases}$$

Let A_0 be the set of all pendant vertices adjacent to x_0 in $G(L)$. The set A_0 can be written as $A_0 = \; \{a_1, a_2, \dots, a_{q-m}\}$, where, for $1 \le i \le q-m$,

$$a_i = \begin{cases} \begin{cases} q - \frac{m+i-1}{2} & \text{if } i \text{ is odd} \\ \frac{m+i}{2} & \text{if } i \text{ is even} \end{cases} & \text{if } m \text{ is even} \\ \\ \begin{cases} \frac{m+i}{2} & \text{if } i \text{ is odd} \\ q - \frac{m+i-1}{2} & \text{if } i \text{ is even} \end{cases} & \text{if } m \text{ is odd} \end{cases}$$

Further, the sums of consecutive elements of A_0 are alternately, q and $q+1$ with $a_1 + a_2 = \; f(x_0) + f(x_1)$. Define an integer r_0 as

$$2r_0 = \begin{cases} o_0 + e_0 & \text{if } x_0 \text{ is attached to } (e, 0, o); \\ & \text{or } \; (e, e, o) \text{ or } (0, e, o) \text{ with } e_0 \equiv (0 \; mod 4) \text{ or } e_0 \equiv (2 \; mod 4) \text{ and } o_0 \ge 4; \\ & \text{or } \; (e, o, e) \text{ or } (0, o, e) \text{ with } e_0 \equiv (3 \; mod 4) \text{ or } e_0 \equiv (1 \; mod 4) \text{ and } o_0 \ge 4 \\ o_0 + e_0 + 2 & \text{if } x_0 \text{ is attached to } (e, e, o) \text{ or } (0, e, o) \text{ with } e_0 \equiv (2 \; mod 4) \text{ and } p_0 \ge 3; \\ & \text{or } \; (e, o, e) \text{ or } (0, o, e) \text{ with } e_0 \equiv (1 \; mod 4) \text{ and } p_0 \ge 4 \end{cases}$$

Next we carry out a transfer $x_0 \longrightarrow x_1$ of the first type retaining $2r_0$ elements of A_0, namely $a_1, a_2, \ldots, a_{2r_0}$ at x_0 and move the set of remaining vertices, say A_1 to x_1. By Lemma 1.2, the new tree thus formed is graceful.

Step 2: For $t_{i-1} + 1 \leq j \leq t_i$, we choose integers $r_j,\ r_j \geq 0$, depending on the length of P_i and the combination of branches incident on the vertices x_j of P_i and given as in Table 1.

Step 3: Observe that the set $A_1 = \{a_{2r_0+1}, a_{2r_0+2}, \ldots, a_{q-m}\}$ and the labels of the vertices x_1 and x_2 of the transfer $T_1 : x_1 \longrightarrow x_2 \longrightarrow \ldots \longrightarrow x_{m-1} \longrightarrow x_m \longrightarrow x_{m+1}$ correspond to the set A and the vertex labels a and b of Lemma 1.4. Carry out the transfer T_1 consisting of m transfers of the first type keeping $2r_i + 1$ elements of A_i at x_i, for $1 \leq i \leq m$, where A_{i+1}, is obtained from A_i by deleting the elements kept at x_i. By Lemma 1.4, the resultant tree thus formed is graceful. Next make a transfer $x_{m+1} \longrightarrow x_m$, i.e. $q + 1 \longrightarrow 0$ of the first type bringing back each element of A_{m+1} to x_m and remove the vertex x_{m+1} and get a new graceful tree, say G_1.

Step 4: Next carry out the transfer, $T_2 : x_m \to x_{m-1} \to x_{m-2} \to \ldots \to x_1 \to x_0$. The labels of the vertices x_m and x_{m-1} and the set A_{m+1} correspond to the vertex labels a and b and the set A of Lemma 1.4. First consider the transfer $x_m \to x_{m-1} \to \ldots \to x_{t_{n-1}}$. If for $t_{n-1} + 1 \leq j \leq t_n,\ \lambda_j$ is odd, i.e. $\lambda_j - (2r_j + 1)$ is even, then $t_n - t_{n-1} \equiv 0\ (mod\ 4)$, and carry out $\frac{t_n - t_{n-1}}{4}$ successive BD8TFs, else carry out $t_n - t_{n-1}$ successive transfers of the first type to keep $\lambda_j - (2r_j + 1)$ vertices of A_m at x_j. Let $B_{t_{n-1}}$ be the set of vertices of A_{m+1} that have been transferred to $x_{t_{n-1}}$. The resultant tree, say G_4, thus formed is graceful by Lemma 1.4. Applying Lemma 1.4 one can repeat the above process n times keeping $\lambda_j - (2r_j + 1)$ vertices from B_{t_i} at each vertex $x_j,\ t_{i-1} + 1 \leq j \leq t_i, 1 \leq i \leq n$, where $B_{t_{i-1}}$ is obtained from B_{t_i} by deleting the vertices kept at $x_j,\ t_{i-1} + 1 \leq j \leq t_i$, and eventually form a new graceful tree, say G_2.

Step 5: B_0 is the set of vertices transferred to the vertex x_0 in Step 4. The set B_0 is of the form $B_0 = \{a_{s+1}, a_{s+2}, \ldots, a_{q-m}\}$. Now carry out the transfer $x_0 \to a_1$ of the first type keeping $2\lambda_0 + 1 - 2r_0$ vertices from B_0 and transferring the set C_1 of the remaining vertices to a_1. The resultant tree thus formed, say G_3 is graceful by Lemma 1.4.

Step 6: Finally, we carry out the transfer $T_3 : a_1 \to a_2 \to a_3 \to a_4 \ldots \to a_s$, where $s = 2r_0 + \sum_{j=0}^{m}(2r_j + 1)$.

Consider the transfer T_3 of the vertices whose labels are in C_1. The manner in which we have moved the vertices of A_1 in step 3, we notice that the first $2r_0$ vertices in T_3 are incident on x_0, the next $\sum_{j=1}^{t_1}(2r_j + 1)$ vertices in T_3

Cases	the value of r_l ($x_l \in P_l$), $l = \begin{cases} t_{l-1}+1 & \text{for Case (I)} \\ j & \text{for Case (II)} \end{cases}$
$(I)(1)(a)$ and $(II)(1)(a)$	$2r_l + 1 = o_l$
$(I)(1)(b)$ and $(II)(1)(b)$	$2r_l + 1 = \begin{cases} o_l + e_l \text{ if } e_l \equiv 0 \ (mod\ 4) \text{ or } o_l \geq 3 \\ o_l + e_l + 2 \text{ if } e_l \equiv 2 \ (mod\ 4) \text{ and } o_l = 1 \end{cases}$
$(I)(1)(c)$ and $(II)(1)(c)$	$2r_l + 1 = \begin{cases} o_l + e_l + 1 \text{ if } e_l \equiv 3 \ (mod\ 4) \text{ or } o_l \geq 3 \\ o_l + e_l + 3 \text{ if } e_l \equiv 1 \ (mod\ 4) \text{ and } o_l = 1 \end{cases}$
(I)(2)(a) and (II)(2)(a)	$2r_l + 1 = \begin{cases} o_l + e_l \text{ if } e_l + e_{l+1} \equiv 0 \ (mod\ 4) \text{ or } o_l \geq 3 \\ o_l + e_l + 2 \text{ if } e_l + e_{l+1} \equiv 2 \ (mod\ 4) \text{ and } o_l = 1 \end{cases}$
(I)(2)(b), (II)(2)(b), (I)(2)(f), and (II)(2)(f)	$2(r_l + r_{l+1} + 1) = \begin{cases} o_l + o_{l+1} + e_l + e_{l+1} + 2, \text{ for Cases (i), or (ii) with } o_{l+1} \geq 3 \text{ or (iii) with } o_l \geq 3 \\ o_l + o_{l+1} + e_l + e_{l+1} + 4, \text{ for Cases (ii) with } o_{l+1} = 1 \text{ and } p_{l+1} \geq 3 \\ \text{or (iii) with } o_l = 1 \text{ and } p_l \geq 3 \end{cases}$
(I)(2)(c), (I)(2)(d), (II)(2)(c) and (II)(2)(d)	$2(r_l + r_{l+1} + 1) = \begin{cases} o_l + o_{l+1} + e_l + e_{l+1} + 1, \text{ for Cases (i), (ii), (iii) with } o_l \geq 3 \text{ and (iv) with } o_{l+1} \geq 3 \\ o_l + o_{l+1} + e_l + e_{l+1} + 3, \text{ for Cases (iii) with } o_{l+1} = 1 \text{ and } p_{l+1} \geq 4 \\ \text{or (ivi) with } o_l = 1 \text{ and } p_l \geq 3 \end{cases}$
(I)(2)(e) and (II)(2)(e)	$2(r_l + r_{l+1} + 1) = \begin{cases} o_l + o_{l+1} + e_l + e_{l+1}, \text{ for Cases (i) and (ii).} \\ o_l + o_{l+1} + e_l + e_{l+1} + 2, \text{ for Case (iii).} \end{cases}$
$(I)(2)(g)$ and $(II)(2)(g)$	$2r_l + 1 = \begin{cases} o_l + e_l + 1 \text{ if } e_l \equiv 2 \ (mod\ 4) \text{ or } o_l \geq 4 \\ o_l + e_l + 3 \text{ if } e_l \equiv 0 \ (mod\ 4) \text{ and } o_l = 2 \end{cases}$ and $2r_{l+1} + 1 = \begin{cases} o_{l+1} + e_{l+1} \text{ if } e_{l+1} \equiv 1 \ (mod\ 4) \text{ or } o_{l+1} \geq 4 \\ o_{l+1} + e_{l+1} + 2 \text{ if } e_{l+1} \equiv 3 \ (mod\ 4) \text{ and } o_{l+1} \leq 2 \end{cases}$
$(I)(2)(h)$ and $(II)(2)(h)$	$2r_l + 1 = \begin{cases} o_l + e_l \text{ if } e_l \equiv 1 \ (mod\ 4) \text{ or } o_l \geq 4 \\ o_l + e_l + 2 \text{ if } e_l \equiv 3 \ (mod\ 4) \text{ and } o_l \leq 2 \end{cases}$ and $2r_{l+1} + 1 = \begin{cases} o_{l+1} + e_{l+1} + 1 \text{ if } e_{l+1} \equiv 2 \ (mod\ 4) \text{ or } o_{l+1} \geq 4 \\ o_{l+1} + e_{l+1} + 3 \text{ if } e_{l+1} \equiv 0 \ (mod\ 4) \text{ and } o_{l+1} \leq 2 \end{cases}$
$(I)(2)(k)$ and $(II)(2)(k)$	$2(r_l + r_{l+1} + 1) = o_l + o_{l+1} + 4$

Table 1

are incident on $x_j,\ 1 \leq j \leq t_1$, the next $\sum_{j=\ t_1+1}^{t_2}(2r_j+1)$ vertices in T_3 are incident on $x_j,\ t_1+1 \leq j \leq t_2$, and so on. Further, we observe that the set C_1 and the vertices a_1 and a_2 satisfy the properties of the set A and the vertex labels a and b in Lemma 1.4. We partition the transfer $T_3 : T_3^{(0)} \rightarrow T_3^{(1)} \rightarrow T_3^{(2)} \rightarrow T_3^{(3)} \rightarrow \ldots \rightarrow T_3^{(n)}$ and accomplish the transfer T_3 by successively carrying out the transfers $T_3^{(0)}, T_3^{(1)}, T_3^{(2)}, \ldots, T_3^{(n)}$ in order. $T_3^{(0)}$ is the transfer $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \ldots \rightarrow a_{2r_0} \rightarrow a_{2r_0+1}$. For $i = 1, 2, \ldots, n-1$, $T_3^{(i)}$ is the transfer $a_{s_{i-1}+1} \rightarrow a_{s_{i-1}+2} \rightarrow \ldots \rightarrow a_{s_i} \rightarrow a_{s_i+1}$, where $s_0 = 2r_0$, $s_i = 2r_0 + \sum_{j=1}^{t_i}(2r_j+1)$ and the vertices $a_{s_{i-1}+1}, a_{s_{i-1}+2}, \ldots, a_{s_i}$ are incident on the path $P_i : x_{t_{i-1}+1}x_{t_{i-1}+2}\ldots x_{t_i}$. $T_3^{(n)}$ is the transfer $a_{s_{n-1}+1} \rightarrow a_{s_{n-1}+2} \rightarrow \ldots \rightarrow a_{s_n-1} \rightarrow a_{s_n}$, where $s_n = s$ and the vertices $a_{s_{n-1}+1}, a_{s_{n-1}+2}, \ldots, a_{s_n}$ are incident on the path $P_n : x_{t_{n-1}+1}x_{t_{n-1}+2}\ldots x_{t_n}$. Each transfer $T_3^{(i)}$, $i = 0, 1, 2, \ldots, n$ consists of sequence of transfers of the first type and one or more of the derived transfers. By Lemma 1.4 the tree obtained after accomplishing the transfer T_3 will be graceful.

We first carry out the transfer $T_3^{(0)} : a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots \rightarrow a_{2r_0} \rightarrow a_{2r_0+1}$. If x_0 is attached to $(e, 0, o)$ then $T_3^{(0)}$ consists of $2r_0$ successive transfers of the first type. If x_0 is attached to (e,e,o) or $(0,e,o)$ with $e_0 \equiv 0\ (mod\ 4)$ then $T_3^{(0)}$ consists of $\frac{e_0}{4}$ BD8TF followed by o_0 successive transfers of the first type. If x_0 is attached to (e,e,o) or $(0,e,o)$ with $e_0 \equiv 2\ (mod\ 4)$ and $o_0 \geq 4$ (respectively, $p_0 \leq 3$), then $T_3^{(0)}$ consists of $\frac{e_0-2}{4}$ successive BD8TF, followed by one 5TF (respectively, 2JTF), and finally $o_0 - 3$ (respectively, o_0) successive transfers of the first type. If x_0 is attached to (e,o,e) or $(0,o,e)$ with $e_0 \equiv 3\ (mod\ 4)$ then $T_3^{(0)}$ consists of one 1JTF, followed by $\frac{e_0-3}{4}$ BD8TF followed by o_0 successive transfers of the first type. If x_0 is attached to (e,o,e) or $(0,o,e)$ with $e_0 \equiv 1\ (mod\ 4)$, $e_0 \geq 5$ and $o_0 \geq 4$ (respectively, $p_0 \geq 4$) then $T_3^{(0)}$ consists of one 1JTF, followed by $\frac{e_0-5}{4}$ BD8TF, followed by one 5TF (respectively, 2JTF), and finally $o_0 - 3$ (respectively, o_0) successive transfers of the first type.

Next we carry out the transfer $T_3^{(1)} : a_{2r_0+1} \rightarrow a_{2r_0+2} \rightarrow a_{2r_0+3} \rightarrow \ldots \rightarrow a_{s_1}$, where a_{s_1} is the $[2r_0 + \sum_{j=1}^{t_1}(2r_j+1)+1]^{th}$ vertex of the T_3.

Case (I): In this case the combination of branches incident on the vertices of P_1 is as per Case (I).

Case (1): Let $t_1 = 1$ and x_1 be attached to one of the combinations (a) to (c) in (1). Here $T_3^{(1)}$ is the transfer $a_{2r_0+1} \rightarrow a_{2r_0+2} \rightarrow \ldots \rightarrow a_{2r_0+2r_1+1} \rightarrow a_{2r_0+2r_1+2}$

Case (a): Here $T_3^{(1)}$ consists of $2r_1+1$ successive transfers of the first kind.

Case (b): If $e_1 \equiv 0 (mod\ 4)$, then $T_3^{(1)}$ consists of $\frac{e_1}{4}$ successive BD8TF followed by the o_1 successive transfers of the first kind. If $o_1 \geq 3$ and $e_1 \equiv 2(mod\ 4)$, then $T_3^{(1)}$ consists of one 5TF, followed by $\frac{e_1-2}{4}$ successive BD8TF, and finally $o_1 - 3$ successive transfers of the first kind. If $o_1 = 1$ and $e_1 \equiv 2(mod\ 4)$, then $T_3^{(1)}$ consists of one 2JTF, followed by $\frac{e_1-2}{4}$ successive BD8TF, and finally one transfer of the first kind.

Case (c): If $e_1 \equiv 3(mod\ 4)$, then $T_3^{(1)}$ consists of one 1JTF, followed by $\frac{e_1-3}{4}$ successive BD8TF, and finally o_1 successive transfers of the first kind. If $o_1 \geq 3$ and $e_1 \equiv 1(mod\ 4)$, then $T_3^{(1)}$ consists of one 1JTF, followed by one 5TF, followed by $\frac{e_1-5}{4}$ successive BD8TF, and finally $o_1 - 3$ successive transfers of the first kind. If $o_1 = 1$ and $e_1 \equiv 1(mod\ 4)$, then $T_3^{(1)}$ consists of one 1JTF, followed by one 2JTF, followed by $\frac{e_1-5}{4}$ successive BD8TF, and finally one transfer of the first kind.

Case (2): Suppose x_1 and x_2 are attached to one of the combinations in (2).

Case (a): If Case(i) holds then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2}{4}$ BD8TF, and finally o_2 successive transfers of the first type. If Case(ii) holds with $o_2 \geq 3$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-1}{4}$ BD8TF, followed by 1 5TF, and finally $o_2 - 3$ successive transfers of the first type. If Case(ii) holds with $o_2 = 3$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-1}{4}$ BD8TF, followed by one 2JTF, and finally one transfer of the first type. If Case(iii) holds with $o_1 \geq 3$ then $T_3^{(1)}$ consists of one 5TF, followed by $o_1 - 3$ successive transfers of the first type, followed by $\frac{e_1+e_2-1}{4}$ BD8TF, and finally o_2 successive transfers of the first type. If Case(iii) holds with $o_1 = 3$ then $T_3^{(1)}$ consists of one transfer of the first type, followed by one 2JTF, followed by $\frac{e_1+e_2-1}{4}$ BD8TF, and finally o_2 successive transfers of the first type.

Case (b): If Case(i) holds with $e_j \equiv 3(mod\ 4)$, $j = 1, 2$, $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-1}{4}$ BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type. If Case(i) holds with $e_j \equiv 1(mod\ 4)$, $j = 1, 2$, then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-1}{4}$ successive BD8TF, followed by one 2JTF, followed by $\frac{e_2-1}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case (ii) holds with $o_2 \geq 3$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-1}{4}$ successive BD8TF, followed by one 2JTF, followed by $\frac{e_2-3}{4}$ successive BD8TF, followed by one 5TF, and finally $o_2 - 3$ successive transfers of the first type. If Case (ii) holds with $o_2 = 1$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-1}{4}$ successive BD8TF, followed by two successive 2JTF, followed by $\frac{e_2-3}{4}$ successive BD8TF, and finally one transfer of the first type. If Case (iii) holds with $o_1 \geq 3$ then $T_3^{(1)}$ consists of one 5TF, followed by $o_1 - 3$ successive transfers of the first type, followed by $\frac{e_1-3}{4}$ successive BD8TF, followed by one 2JTF, followed by $\frac{e_2-1}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case (iii) holds with $o_1 = 1$ then $T_3^{(1)}$ consists of one transfer of the first type, followed by followed by $\frac{e_1-3}{4}$ successive BD8TF, followed by two successive 2JTF, followed by $\frac{e_2-1}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type.

Case (c): If Cases(i)and (ii) hold $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-3}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type. If Case(iii) holds with $o_2 \geq 3$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 1JTF, followed by one 5TF, and finally $o_2 - 3$ successive transfers of the first type. If Case(iii) holds with $o_2 = 1$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-5}{4}$ successive

BD8TF, followed by one 1JTF, followed by one 2JTF, and finally one transfer of the first type. If Case(iv) holds with $o_1 \geq 3$ then $T_3^{(1)}$ consists of $o_1 - 3$ successive transfers of the first type, followed by one 5TF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type. If Case(iv) holds with $o_1 = 1$ then $T_3^{(1)}$ consists of one transfer of the first type, followed by one 2JTF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type.

Case (d): If Cases(i)and (ii) hold then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-3}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case(iii) holds with $o_1 \geq 3$ then $T_3^{(1)}$ consists of $o_1 - 3$ successive transfers of the first type, followed by one 1JTF, followed by one 5TF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case(iii) holds with $o_1 = 1$ then $T_3^{(1)}$ consists of one transfer of the first type, followed by one 1JTF, followed by one 2JTF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case(iv) holds with $o_2 \geq 3$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 5TF, and finally $o_2 - 3$ successive transfers of the first type. If Case(iv) holds with $o_2 = 1$ then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 2JTF, and finally one transfer of the first type.

Case (e): If Case(i) holds then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case (ii) holds with $o_1 \geq 4$, then $T_3^{(1)}$ consists of one 5TF, followed by $o_1 - 3$ successive transfers of the first type, followed by $\frac{e_1+e_2-2}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case (ii) holds with $o_2 \geq 4$, then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-2}{4}$ successive BD8TF, followed by $o_2 - 3$ successive transfers of the first type, and finally one 5TF.

Case (f): If Case(i) holds then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-6}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type. If Case(ii) holds then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-4}{4}$ successive BD8TF, followed by one 1JTF, followed by one 2JFT, followed by one 1JTF, followed by $\frac{e_2-4}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case(iii)(A) holds with $p_2\ ge4$ (respectively,

o_2 $ge4$) then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, $\frac{e_1+e_2-8}{4}$ successive BD8TF, followed by one 1JTF, followed by one 2JFT (respectively, 5TF), and finally o_2 (respectively, o_2-3) successive transfers of the first type. If Case(iii)(B) holds with p_1 $ge4$ (respectively, o_1 $ge4$) then $T_3^{(1)}$ consists of o_1 (respectively, o_1-3) successive transfers of the first type, followed by one 2JFT (respectively, 5TF), followed by one 1JTF, followed by $\frac{e_1+e_2-8}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type.

Case (g): If Case(i) holds then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-3}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type. If Case (ii) or (iii) holds the $o_2 \geq 4$ (respectively, $p_2 \geq 3$) then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 5TF (respectively, 2JTF), and finally o_2-3 (respectively, o_2) successive transfers of the first type.

Case (h): If Case(i) holds then $T_3^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2-3}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type. If Case (ii) or (iii) holds the $o_1 \geq 4$ (respectively, $p_1 \geq 3$) then $T_3^{(1)}$ consists of o_1-3 (respectively, o_1) successive transfers of the first type, followed by one 5TF (respectively, 2JTF), followed by $\frac{e_1+e_2-5}{4}$ successive BD8TF, followed by one 1JTF, and finally o_2 successive transfers of the first type.

Case (k): Here $T_3^{(1)}$ consists of o_1-1 successive transfers of the first type, followed by one 4JTF, and finally o_2-1 successive transfers of the first type.

Case (II): In this case $P_1 = x_1x_2x_3x_4$ and the combinations of branches incident on the vertices of P_1 is as per Case (II). If x_1 is attached to one of the combinations mentioned in (1) then we first carry out the transfer $S_1^{(1)}$: $a_{2r_0+1} \rightarrow a_{2r_0+2} \rightarrow \ldots \rightarrow a_{2r_0+2r_1+1} \rightarrow a_{2r_0+2r_1+2}$, which is same as the transfer $T_3^{(1)}$ itself described in the respective cases in (I)(1). If x_1 and x_2 are attached to the combinations mentioned (2) then we first carry out the transfer $S_1^{(1)}$: $a_{2r_0+1} \rightarrow a_{2r_0+2} \rightarrow \ldots \rightarrow a_{2r_0+2r_1+2r_2+2} \rightarrow a_{2r_0+2r_1+2r_2+3}$, which is same as the transfer $T_3^{(1)}$ itself described in the respective cases in (I)(2).

Now define an integer k_1 and $t^{(1)}$ as $k_1 = 1$ and $t^{(1)} = 2r_1+2$ if we have the case (a) and $k_1 = 2$ and $t^{(1)} = 2r_1+2r_2+3$ if we have the case (b). Let the set of elements of C_1 that have come to $a_{t^{(1)}}$ be $C_1^{(1)}$. We observe that the set $C_1^{(1)}$ and the vertices $a_{t^{(1)}}$ and $a_{t^{(1)}+1}$ satisfy the properties of the set A and the vertices a and b of Lemma 1.4. Now suppose x_{k_1+1} is (respectively,

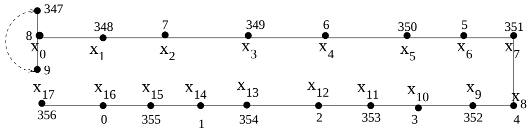

Figure 5: The tree $G(L)$ with a graceful labeling.

x_{k_1+1} and x_{k_1+2} are) attached to one of the combinations in (a) (respectively, (b)), then we repeat the procedure in case (a) (respectively, (b)) and make the transfer $a_{t^{(1)}} \to a_{t^{(1)}+1} \to \ldots \to a_{t^{(2)}}$, where $a_{t^{(2)}}$ is the $[\sum_{j=1}^{k_1+1}(2r_j+1)+1]^{th}$ (respectively, $[\sum_{j=1}^{k_1+2}(2r_j+1)+1]^{th}$) vertex in T_3 and keep desired number of vertices from $C_1^{(1)}$ at each vertex of the transfer. Next, we define $k_2 = k_1+1$ (respectively, k_1+2). If $k_2 = 4$, then we have reached the vertex a_{s_1+1}; Otherwise, we repeat the process by using Lemma 1.4 till we reach the vertex a_{s_1} in T_3.

In the similar manner we carry out the transfers $T_3^{(2)}$, $T_3^{(3)}$, ..., $T_3^{(n)}$ successively in order by repeating the procedure in which we have accomplished the transfer $T_3^{(1)}$ and complete the transfer $T_3 : T_3^{(0)} \to T_3^{(1)} \to T_3^{(2)} \to T_3^{(3)} \to \ldots \to T_3^{(n)}$. We observe that the resultant tree thus formed is the lobster L and by Lemma 1.4 L is graceful. □

Example. The lobster L in Example 2.2 (Figure 2) is a lobster of the type in Theorem 2.3. Here $q = 355$. We first form the graceful tree $G(L)$ as in Figure 5. Figure 6 represents the tree obtained after Step 3. Figure 7 represents the tree obtained after Step 4. Figure 8 represents the lobster L with a graceful labeling after Step 6.

References

[1] J. C. Bermond, Radio antennae and French windmills, *Graph Theory and Combinatorics, In Research Notes in Maths*, **Vol.34**, (1979), 18 - 39.

[2] W. C. Chen, H. I. Lu, Y. N. Yeh, Operations of interlaced trees and graceful trees, *Southeast Asian Bulletin of Mathematics* **Vol. 4** (1997), 337 - 348.

[3] J. A. Gallian, *A dynamic survey of graph labeling - DS6, Fifteenth Edition*, Electronic Journal of Combinatorics, USA, (2012). url: *http://www.combinatorics.org/Surveys/.*

[4] P. Hrnčiar, A. Havier, All trees of diameter five are graceful, *Discrete Mathematics*, **Vol. 233** (2001), 133 - 150.

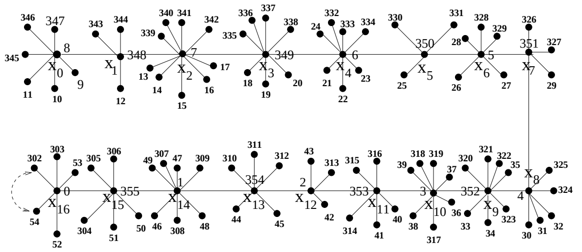

Figure 6: The graceful tree obtained after Step 3.

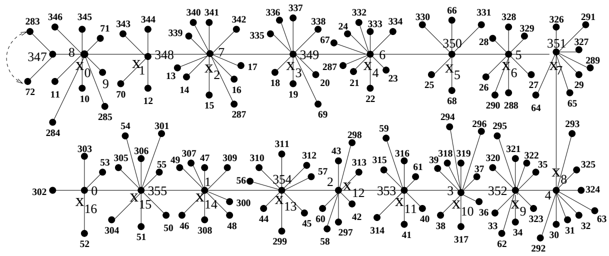

Figure 7: The graceful tree obtained after Step 4.

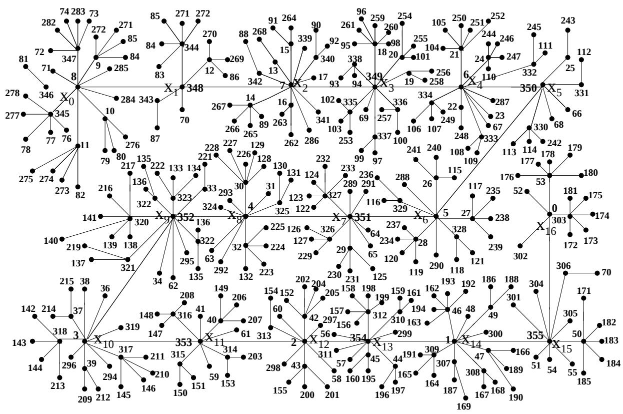

Figure 8: The lobster L with the graceful labeling after Step 6.

[5] D. Mishra and A. C. Panda, Some New Transformations And Their Applications Involving Graceful Tree Labeling, *International Journal of Mathematical Sciences and Engineering Aplications*, **Vol. 7**, No. 1 (2013), 239 - 254.

[6] D. Mishra, A. C. Panda, and R. V. Dash, A Class of Graceful lobsters with even number of branches incident on the central path, *International Journal of Mathematical Sciences and Applications*, **Vol. 1**, No. 2 (May,

2011), 463 - 472.

[7] D. Mishra and P. Panigrahi, Some New Classes of Graceful Lobsters Obtained by Applying Inverse and Component Moving Transformations, *International Journal of Mathematics Trends and Technology (IJMTT)*, **Vol.1**, Issue - 2, July-Aug, 2011, 6 - 16.

[8] D. Mishra and P. Panigrahi, Some New classes of Graceful Lobsters Obtained from Diameter Four Trees, *Mathematica Bohemica*, **Vol. 135**, No. 3 (2010), 257 - 278.

[9] D. Mishra and P. Panigrahi, A New Class of Graceful lobsters obtained from Diameter Four Trees, *Utilitus Mathematica*, **Vol. 80**, (2009), 183 - 209.

[10] D. Mishra and P. Panigrahi, Some Graceful Lobsters with All Three Types of Branches Incident on the Vertices of the Central Path, *Computers and Mathematics with Applications*, **Vol. 56**, (2008), 1382 - 1394.

[11] D. Mishra and P. Panigrahi, Graceful lobsters obtained by Component Moving of Diameter Four Trees, *ARS Combinatoria*, **Vol. 81** (2006), 129 - 146.

[12] D. Morgan, All lobsters with perfect matchings are graceful, *Preprint.* url: *http://www.cs.ualberta.ca / ˜ davidm/tiger.ps* .

[13] H. K. Ng, Gracefulness of a class of lobsters, *Notices AMS*, **Vol. 7**, (1986), 825-05-294.

[14] P. Panigrahi, D. Mishra, Larger Gracefu and Interlaced Lobsters Obtained by Joining Smaller Ones, *Southeast Asian Bulletin of Mathematics*, **Vol. 33**, (2009), 509 - 525.

[15] G. Ringel, *Proceedings of Symposium Smolenice*, Prague Publishing House of Czechoslovak Academy of Science, Chezoslovakia, (1964), 162.

[16] J. G. Wang, D. J. Jin, X. G. Lu, D. Zhang, The gracefulness of a class of lobster trees, *Mathematical and Computer Modelling*, **Vol. 20**, No. 9, (1994), 105 - 110.