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NEW $H^1(\Omega)$ CONFORMING FINITE ELEMENT SPACES

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Abstract: Discretization of Maxwell's eigenvalue problem with edge finite elements involves a simultaneous use of two discrete subspaces of $H^1(\Omega)$ and $H(\text{curl}\,,\Omega)$. In this paper, we introduce new scalar finite element spaces and edge finite element spaces, respectively. We also prove the unisolvence of degrees of freedom and analyze our spaces using the discrete de Rham diagram.

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1. Introduction

We consider the following Maxwell's eigenvalue problem: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \neq 0$ such that

$$\begin{cases} \mathbf{curl}\ (\mu^{-1}\mathbf{curl}\ \mathbf{u}) &= \lambda\varepsilon\mathbf{u}, & \text{in } \Omega, \\ \text{div}\ (\varepsilon\mathbf{u}) &= 0, & \text{in } \Omega, \\ \mathbf{u}\times\mathbf{n} &= 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a polygonal domain. The coefficients ε and μ are the electric permittivity and the magnetic permeability, respectively. The Sobolev space $H(\text{curl}\,,\Omega)$ plays a central role in the variational theory of Maxwell's equations(see [1], [2], [3]). Thus we need to derive edge finite elements suitable for

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discretizing the Maxwell system in this space. To analyze edge elements, we also need to use scalar finite elements in the space $H^1(\Omega)$. The relevant function spaces are related by the following exact sequence[4], [5]:

$$H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl}\,,\Omega) \xrightarrow{\nabla\times} H(\mathrm{div}\,,\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega). \tag{2}$$

In two space dimensions, the sequence reduces to

$$H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl}\,,\Omega) \xrightarrow{\nabla\times} L^2(\Omega). \tag{3}$$

In this paper, we shall deal with the two dimensional case. We construct finite element spaces $U_h \subset H^1(\Omega)$ and $\mathbf{V}_h \subset H(\mathrm{curl}\,,\Omega)$ which have the same relationship as the continuous spaces, in *section* 2 and *section* 3. In *section* 4, we also describe interpolation operators π_h and $\mathbf{r}_h$ that map from suitable subspaces $U \subset H^1(\Omega)$ and $\mathbf{V} \subset H(\mathrm{curl}\,,\Omega)$ into appropriate finite element spaces. Of central importance to the analysis is that the spaces and interpolation operators are linked by the commuting diagram called the discrete de Rham diagram.

2. New $H^1(\Omega)$ Conforming Finite Elements

In this section, we will introduce new scalar finite element spaces needed for discretizing the potential in Maxwell's equations. First, we assume a regular finite element mesh $\tau_h,\ h > 0$ of parallelograms with the maximum diameter h. Then each element $K \in \tau_h$ can be obtained from the reference element $\widehat{K}$, typically unit square $(0,1)^2$, via a diagonal affine map $F_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + b_K$, where B_K is an invertible diagonal matrix. Because of the simplicity of the mapping, although we define the elements on the reference domain $\widehat{K}$, the same definition can be used on a target element K in the mesh[6]. In order to define finite elements on parallelogram, we need the following tensor product polynomial space:

$$Q_{l,m}(\widehat{K}) = \{\text{polynomials of maximum degree } l \text{ in } \hat{x} \text{ and } m \text{ in } \hat{y}\}.$$

Definition 1. Let $k \geq 1$. The gradient conforming element is defined as follows:

$$\widehat{U}(\widehat{K}) = Q_{k+1,k+1}(\widehat{K}),$$

except constant multiple of the term $\hat{x}^{k+1}\hat{y}^{k+1}$.

Definition 2. On reference element, we define the following degrees of freedom:

$$\hat{p}(\hat{a}), \qquad \text{for the four vertices } \hat{a} \text{ of } \widehat{K}, \tag{4}$$

$$\int_{\hat{e}} \hat{p}\, \hat{q}\, d\hat{s}, \qquad \text{for each edge } \hat{e} \text{ of } \widehat{K}, \quad q \in P_{k-1}(\hat{e}), \tag{5}$$

$$\int_{\widehat{K}} \hat{p}\, \hat{\mathbf{q}}\, d\hat{\mathbf{x}}, \qquad \hat{\mathbf{q}} \in \Phi_k(\widehat{K}), \tag{6}$$

where $\Phi_k(\widehat{K})$ is $Q_{k-1,k-1}(\widehat{K})$ space except constant multiple of the term $\hat{x}^{k-1}\hat{y}^{k-1}$.

Our first theorem proves the unisolvence of the new element. Since the number of degrees of freedom and the dimension of $\widehat{U}(\widehat{K})$ are both $k^2 + 4k + 3$ and thus it suffices to show the following result.

Theorem 3. *If* $\hat{p} \in \widehat{U}(\widehat{K})$ *and all the degrees of freedom* (4) – (6) *of* $\hat{p}$ *vanish, then* $\hat{p} = 0$.

Proof. We use the fact that the vertex degrees of freedom vanish on each edge $\hat{e}$ of $\widehat{K}$. For example, on the edge $\hat{y} = 0$ we have

$$\hat{p} = \hat{x}(1 - \hat{x})r$$

for some $r \in P_{k-1}(\hat{e})$. Choosing $\hat{q} = r$ in the degrees of freedom (5) for this edge shows that $r = 0$. Now using the fact that $\hat{p} = 0$ on each edge $\hat{e}$ of $\widehat{K}$, we have

$$\hat{p} = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})r$$

for some $r \in \Phi_k(\widehat{K})$. Choosing $\hat{q} = r$ in the degrees of freedom (6) shows that $r = 0$ and hence $\hat{p} = 0$ on $\widehat{K}$, as required. □

The finite element on a general element K can be obtained by using the diagonal affine map F_K via

$$p \circ F_K = \hat{p} \tag{7}$$

in the usual way. Then we have the following space:

$$U_h = \{p_h \in H^1(\Omega) \mid \ p_h|_K \in U(K) \text{ for all } K \in \tau_h\}. \tag{8}$$

3. New Curl Conforming Finite Elements

In this section, we introduce new edge finite element spaces needed for discretizing the electric field in Maxwell's equations.

Definition 4. For $k \geq 1$, the curl conforming element is defined as follows:

$$\widehat{\mathbf{V}}(\widehat{K}) = Q_{k,k+1}(\widehat{K}) \times Q_{k+1,k}(\widehat{K}),$$

where $(\hat{x}^k\hat{y}^{k+1},\ 0)$ and $(0,\ \hat{x}^{k+1}\hat{y}^k)$ are replaced by the single element $(\hat{x}^k\hat{y}^{k+1},\ -\hat{x}^{k+1}\hat{y}^k)$.

Then we see that the dimensions of $\widehat{\mathbf{V}}(\widehat{K})$ is $2k^2+6k+3$. From the *definition* 4, we give an following example for $k = 1$. The space $\widehat{\mathbf{V}}(\widehat{K})$ consists of all vector polynomials $(\hat{u}_1,\ \hat{u}_2)$ where

$$\begin{aligned} \hat{u}_1 &= a_1 + a_2\hat{x} + a_3\hat{y} + a_4\hat{x}\hat{y} + a_5\hat{y}^2 + c\hat{x}\hat{y}^2, \\ \hat{u}_2 &= b_1 + b_2\hat{x} + b_3\hat{y} + b_4\hat{x}\hat{y} + b_5\hat{x}^2 - c\hat{x}^2\hat{y}. \end{aligned}$$

Now we need to define the degrees of freedom. For this purpose, we define

$$\Psi_k(\widehat{K}) = Q_{k,k-1}(\widehat{K}) \times Q_{k-1,k}(\widehat{K}),$$

where $(\hat{x}^k\hat{y}^{k-1},\ 0)$ and $(0,\ \hat{x}^{k-1}\hat{y}^k)$ are replaced by the element $(\hat{x}^k\hat{y}^{k-1},\ -\hat{x}^{k-1}\hat{y}^k)$.

Definition 5. For any $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \widehat{\mathbf{V}}(\widehat{K})$, we define the following degrees of freedom:

$$\int_{\hat{e}} \hat{\mathbf{u}} \times \hat{\mathbf{n}}\ \hat{q}\ d\hat{s}, \qquad q \in P_k(\hat{e}), \quad \text{for each edge } \hat{e} \text{ of } \widehat{K}, \tag{9}$$

$$\int_{\widehat{K}} \hat{\mathbf{u}} \cdot \hat{q}\ d\hat{\mathbf{x}}, \qquad \hat{q} \in \Psi_k(\widehat{K}). \tag{10}$$

In order to see that these choices of $\widehat{\mathbf{V}}(\widehat{K})$ and degrees of freedom determine a finite element subspace of $H(\mathrm{curl}, \widehat{K})$, we need to show that the degrees of freedom are unisolvent.

Theorem 6. *For* $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \widehat{\mathbf{V}}(\widehat{K})$, *the condition* (9) – (10) *uniquely determines* $\hat{\mathbf{u}}$.

Proof. Since the number of conditions, $4(k+1)+[2k(k+1)-1] = 2k^2+6k+3$ equals the dimension of $\widehat{\mathbf{V}}(\widehat{K})$, it suffices to show that if (9) – (10) are all zero then $\hat{\mathbf{u}} = 0$. Since $\hat{\mathbf{u}} \times \hat{\mathbf{n}} \in P_k(\hat{e})$ for each edge $\hat{e}$, $\hat{\mathbf{u}} \times \hat{\mathbf{n}} = 0$ from condition (9). This implies that

$$\hat{u}_1 = \hat{y}(1-\hat{y})\hat{v}_1, \quad \hat{u}_2 = \hat{x}(1-\hat{x})\hat{v}_2,$$

where $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2) \in \Psi_k(\widehat{K})$. From condition (10), we obtain the desired result. □

Since we are working in $H(\mathrm{curl}\,, \widehat{K})$, vector functions must be transformed in a more careful way to conserve their properties. Suppose $\hat{\mathbf{u}} \in H(\mathrm{curl}\,, \widehat{K})$. Then $\hat{\mathbf{u}}$ is transformed to a function $\mathbf{u}$ defined on K in $H(\mathrm{curl}\,, K)$ via the following formula:

$$\mathbf{u} \circ F_K = (dF_K)^{-T}\hat{\mathbf{u}}, \tag{11}$$

where dF_K is the Jacobian matrix of the mapping F_K. Then we define the following spaces, using the equation (11):

$$\mathbf{V}_h = \{\mathbf{u}_h \in H(\mathrm{curl}\,, \Omega) \mid \ \mathbf{u}_h|_K \in \mathbf{V}(K) \text{ for all } K \in \tau_h\}. \tag{12}$$

4. Analysis for the New Finite Element Spaces

To analyze our spaces, we first define interpolation operators π_h and $\mathbf{r}_h$, respectively. Let $p \in H^{\frac{3}{2}+\delta}(K)$, $\delta > 0$. Then the restriction on the regularity of p allows us to use the Sobolev Embedding Theorem and hence to ensure vertex values are well defined. We define an interpolation operator $\pi_K : H^{\frac{3}{2}+\delta}(K) \to U(K)$ satisfying

$$(p - \pi_K p)(a), \qquad \text{for the four vertices } a \text{ of } K, \tag{13}$$

$$\int_e (p - \pi_K p) q \; ds, \qquad \text{for each edge } e \text{ of } K, \quad q \in P_{k-1}(e), \tag{14}$$

$$\int_K (p - \pi_K p)\mathbf{q} \; d\mathbf{x}, \qquad \mathbf{q} \in \Phi_k(K). \tag{15}$$

Via the local interpolation operator, we then define the global interpolation operator $\pi_h : H^{\frac{3}{2}+\delta}(\Omega) \to U_h$ by

$$(\pi_h p)|_K = \pi_K p$$

for all $K \in \tau_h$. Then we have the following theorem for the accuracy properties of the interpolant[7].

Theorem 7. *Let τ_h be a regular family of meshes of Ω. Then there exists a constant C independent of h and p such that*

$$\| p - \pi_h p \|_{H^1(\Omega)} \leq C h^{s-1} \| p \|_{H^s(\Omega)}, \quad \frac{3}{2} + \delta \leq s \leq k+1.$$

Now we define an interpolation $\mathbf{r}_K : \mathbf{H}^{k+1}(K) \to \mathbf{V}(K)$ by

$$\int_e (\mathbf{u} - \mathbf{r}_K \mathbf{u}) \times \mathbf{n}\, q\, ds = 0, \quad q \in P_k(e), \quad \text{for each edge } e \text{ of } K, \tag{16}$$

$$\int_K (\mathbf{u} - \mathbf{r}_K \mathbf{u}) \cdot \mathbf{q}\, d\mathbf{x} = 0, \quad \mathbf{q} \in \Psi_k(K). \tag{17}$$

Then a global projection operator $\mathbf{r}_h : \mathbf{H}^{k+1}(\Omega) \to \mathbf{V}_h$ is defined piecewise

$$(\mathbf{r}_h \mathbf{u})|_K = \mathbf{r}_K \mathbf{u}$$

for all $K \in \tau_h$.

Using interpolation operators π_h and $\mathbf{r}_h$, we show that the scalar space U_h and curl conforming space $\mathbf{V}_h$ are connected in an intimate way from the following de Rham diagram commutes[8], [9], [10]:

$$\begin{array}{ccc} U & \xrightarrow{\nabla} & \mathbf{V} \\ {\scriptstyle \pi_h}\downarrow & & \downarrow{\scriptstyle \mathbf{r}_h} \\ U_h & \xrightarrow[\nabla]{} & \mathbf{V}_h \end{array}$$

Theorem 8. *If U_h is defined by* (8) *and $\mathbf{V}_h$ by* (12)*, then $\nabla U_h \subset \mathbf{V}_h$. In addition, if p is sufficiently smooth such that $\mathbf{r}_h \nabla p$ and $\pi_h p$ are defined, then we have $\nabla \pi_h p = \mathbf{r}_h \nabla p$.*

Proof. Clearly, if $p_h \in U_h$ then we see directly that $\nabla p_h \in \mathbf{V}_h$. To prove the commuting property, we map to the reference element and show that all degrees of freedom $(9)-(10)$ vanish for $\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}$. Then we conclude that $\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p} = 0$. For the edge degrees of freedom (9), if $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2)$ is unit outward normal to $\hat{e} = [\hat{a}, \hat{b}]$ and $\hat{q} \in P_k(\hat{e})$ then using the corresponding unit tangent vector $\hat{\boldsymbol{\tau}} = (-\hat{n}_2, \hat{n}_1)$ and integration by parts we have

$$\begin{aligned} & \int_{\hat{e}} (\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}) \times \hat{\mathbf{n}} \hat{q}\; d\hat{s} \\ &= \int_{\hat{e}} (\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \widehat{\nabla} \hat{p}) \times \hat{\mathbf{n}} \hat{q}\; d\hat{s} \end{aligned}$$

$$= -\int_{\hat{e}} (\widehat{\nabla}\pi_{\widehat{K}}\hat{p} - \widehat{\nabla}\hat{p}) \cdot \hat{\boldsymbol{\tau}}\hat{q}\; d\hat{s}$$
$$= -\int_{\hat{e}} \frac{\partial}{\partial \hat{s}}(\pi_{\widehat{K}}\hat{p} - \hat{p})\hat{q}\; d\hat{s}$$
$$= -(\pi_{\widehat{K}}\hat{p} - \hat{p})(\hat{b}) + (\pi_{\widehat{K}}\hat{p} - \hat{p})(\hat{a}) + \int_{\hat{e}} \frac{\partial \hat{q}}{\partial \hat{s}}(\pi_{\widehat{K}}\hat{p} - \hat{p})\; d\hat{s}.$$

Since $\frac{\partial \hat{q}}{\partial \hat{s}} \in P_{k-1}(\hat{e})$ and using the vertex interpolation property and the degrees of freedom (5) for $\pi_{\widehat{K}}$, we conclude that the right-hand side above vanishes. For the edge degrees of freedom (10), if $\hat{\mathbf{q}} \in \Psi_k(\widehat{K})$ then we have

$$\int_{\widehat{K}} (\widehat{\nabla}\pi_{\widehat{K}}\hat{p} - \mathbf{r}_{\widehat{K}}\widehat{\nabla}\hat{p}) \cdot \hat{\mathbf{q}}\; d\hat{\mathbf{x}}$$
$$= \int_{\widehat{K}} \widehat{\nabla}(\pi_{\widehat{K}}\hat{p} - \hat{p}) \cdot \hat{\mathbf{q}}\; d\hat{\mathbf{x}}$$
$$= \int_{\partial\widehat{K}} (\pi_{\widehat{K}}\hat{p} - \hat{p})\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}\; d\hat{s} - \int_{\widehat{K}} (\pi_{\widehat{K}}\hat{p} - \hat{p})\widehat{\nabla} \cdot \hat{\mathbf{q}}\; d\hat{\mathbf{x}}.$$

Since $\hat{\mathbf{q}} \cdot \hat{\mathbf{n}} \in P_{k-1}(\hat{e})$ for each edge $\hat{e}$ of $\widehat{K}$ and $\widehat{\nabla} \cdot \hat{\mathbf{q}} \in \Phi_k(\widehat{K})$, so the right-hand side vanishes, using the degrees of freedom (5) and (6) for $\pi_{\widehat{K}}$. This completes the proof. □

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