COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract: In this paper, we prove common fixed point theorems for four mappings by using pointwise $R$-weakly commuting and reciprocally continuous mappings satisfying contractive condition in intuitionistic fuzzy metric spaces.

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1. Introduction

Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park [7] defined the notion of intuitionistic fuzzy metric space with the help of continuous $t$-norms and continuous $t$-conorms. Recently, in 2006, Alaca et al. [2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous $t$-norm and continuous $t$-conorms as a generalization of fuzzy metric space due to Kramosil and Michálek [5]. In 2006, Türkoğlu et al. [9]...
proved Jungck’s common fixed point theorem ([4]) in the setting of intuitionistic fuzzy metric spaces for commuting mappings. For more detail, one can refer to papers ([1], [6], [10], [11]).

In this paper, we prove a common fixed point theorem for four mappings by using pointwise $R$-weakly commuting and reciprocally continuous mappings satisfying contractive condition in intuitionistic fuzzy metric spaces.

2. Preliminaries

Schweizer and Sklar [8] defined the following notions:

**Definition 2.1.** A binary operation $\ast : [0, 1] \times [0, 1] \to [0, 1]$ is continuous $t$-norm if $\ast$ satisfies the following conditions:

(i) $\ast$ is commutative and associative;
(ii) $\ast$ is continuous;
(iii) $a \ast 1 = a$ for all $a \in [0, 1]$;
(iv) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

**Definition 2.2.** A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is continuous $t$-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al. [2] defined the notion of intuitionistic fuzzy metric space as follows:

**Definition 2.3.** A 5-tuple $(X, M, N, \ast, \diamond)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm and $M, N$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;
(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(v) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) for all $x, y \in X$, $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;
(vii) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(xi) \( N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s) \) for all \( x, y, z \in X \) and \( s, t > 0 \);
(xii) for all \( x, y \in X \), \( N(x, y, \cdot) : [0, \infty) \to [0, 1] \) is right continuous;
(xiii) \( \lim_{t \to \infty} N(x, y, t) = 0 \) for all \( x, y \in X \).

Then \((M, N)\) is called an \textit{intuitionistic fuzzy metric} on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and the degree of non-nearness between \( x \) and \( y \) with respect to \( t \), respectively.

\textbf{Remark 2.4.} Every fuzzy metric space \((X, M, \ast)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1 - M, \ast, \diamond)\) such that \( t \)-norm \( \ast \) and \( t \)-conorm \( \diamond \) are associated as \( x \diamond y = 1 - ((1 - x) \ast (1 - y)) \) for all \( x, y \in X \).

\textbf{Remark 2.5.} In an intuitionistic fuzzy metric space \((X, M, N, \ast, \diamond)\), \( M(x, y, \cdot) \) is non-decreasing and \( N(x, y, \cdot) \) is non-increasing for all \( x, y \in X \).

\textbf{Definition 2.6.} Let \((X, M, N, \ast, \diamond)\) be an intuitionistic fuzzy metric space. Then a sequence \( \{x_n\} \) in \( X \) is said to be

(i) \textit{convergent} to a point \( x \in X \) if
\[
\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_n, x, t) = 0
\]
for all \( t > 0 \),
(ii) \textit{Cauchy sequence} if
\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0
\]
for all \( t > 0 \) and \( p > 0 \).

\textbf{Definition 2.7.} An intuitionistic fuzzy metric space \((X, M, N, \ast, \diamond)\) is said to be \textit{complete} if and only if every Cauchy sequence in \( X \) is convergent.

Türkoğlu et al. \[10\] defined the following notions:

\textbf{Definition 2.8.} Let \( A \) and \( S \) be self-mappings of an intuitionistic fuzzy metric space \((X, M, N, \ast, \diamond)\). Then a pair \((A, S)\) is said to be \textit{commuting} if
\[
M(ASx, SAx, t) = 1 \quad \text{and} \quad N(ASx, SAx, t) = 0
\]
for all \( x \in X \) and \( t > 0 \).

\textbf{Definition 2.9.} Let \( A \) and \( S \) be self-mappings of an intuitionistic fuzzy metric space \((X, M, N, \ast, \diamond)\). Then a pair \((A, S)\) is said to be \textit{weakly commuting} if
\[
M(ASx, SAx, t) \geq M(Ax, Sx, t)
\]
and
\[
N(ASx, SAx, t) \leq N(Ax, Sx, t)
\]
for all \( x \in X \) and \( t > 0 \).
Definition 2.10. Let $A$ and $S$ be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then a pair $(A, S)$ is said to be compatible if
\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ASx_n, SAx_n, t) = 0
\]
for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = u$ for some $u \in X$.

Definition 2.11. ([9]) Let $A$ and $S$ be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then a pair $(A, S)$ is said to be pointwise $R$-weakly commuting if given $x \in X$, there exist $R > 0$ such that
\[
M(ASx, SAx, t) \geq M(Ax, Sx, t/R)
\]
and
\[
N(ASx, SAx, t) \leq N(Ax, Sx, t/R)
\]
for all $t > 0$.

Clearly, every pair of weakly commuting mappings is pointwise $R$-weakly commuting with $R = 1$.

Definition 2.12. ([6]) Two self-mappings $A$ and $S$ of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be reciprocally continuous if
\[
ASu_n \to Az \quad \text{and} \quad SAu_n \to Sz,
\]
whenever $\{u_n\}$ is a sequence such that $Au_n \to z$ and $Su_n \to z$ for some $z \in X$ as $n \to \infty$.

If $A$ and $S$ are both continuous, then they are obviously reciprocally continuous, but converse is not true.

Lemma 2.13. ([2], [9]) Let $\{u_n\}$ is a sequence in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. If there exists a constant $k \in (0, 1)$ such that
\[
M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)
\]
and
\[
N(u_n, u_{n+1}, kt) \leq N(u_{n-1}, u_n, t)
\]
for $n = 1, 2, 3, \ldots$, then $\{u_n\}$ is a Cauchy sequence in $X$.

Lemma 2.14. ([2], [9]) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If there exists a constant $k \in (0, 1)$ such that
\[
M(x, y, kt) \geq M(x, y, t) \quad \text{and} \quad N(x, y, kt) \leq N(x, y, t)
\]
for all $x, y \in X$ and $t > 0$, then $x = y$. 
3. Main Results

For our main theorem, we need the following lemma.

**Lemma 3.1.** Let \((X, M, N, \ast, \diamond)\) be a complete intuitionistic fuzzy metric space with \(t \ast t \geq t\) and \((1 - t) \diamond (1 - t) \leq (1 - t)\) for all \(t \in [0, 1]\). Further, let \(A, B, S\) and \(T\) be four self-mappings of \(X\) satisfying

(C1) \(A(X) \subset T(X)\) and \(B(X) \subset S(X)\),

(C2) there exists a constant \(k \in (0, 1)\) such that

\[
[1 + aM(Sx, Ty, kt)] \ast M(Ax, By, kt) \\
\geq a[M(Ax, Sx, kt) \ast M(By, Ty, kt) \ast M(By, Sx, kt)] \\
+ M(Ty, Sx, t) \ast M(Ax, Sx, t) \ast M(By, Ty, t) \\
\ast M(By, Sx, \alpha t) \ast M(Ax, Ty, (2 - \alpha)t)
\]

and

\[
[1 + aN(Sx, Ty, kt)] \diamond N(Ax, By, kt) \\
\leq a[N(Ax, Sx, kt) \diamond N(By, Ty, kt) \diamond N(By, Sx, kt)] \\
+ N(Ty, Sx, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \\
\diamond N(By, Sx, \alpha t) \diamond N(Ax, Ty, (2 - \alpha)t)
\]

for all \(x, y \in X, a \geq 0, \alpha \in (0, 2)\) and \(t > 0\).

If the pairs \((A, S)\) and \((B, T)\) are pointwise \(R\)-weakly commuting, then one continuity of the mappings in compatible pair \((A, S)\) or \((B, T)\) implies their reciprocal continuity.

**Proof.** First, assume that \(A\) and \(S\) are compatible and \(S\) is continuous. We show that \(A\) and \(S\) are reciprocally continuous. Let \(\{u_n\}\) be a sequence such that \(Au_n \to z\) and \(Su_n \to z\) for some \(z \in X\) as \(n \to \infty\). Since \(S\) is continuous, we have \(SAu_n \to Sz\) and \(SSu_n \to Sz\) as \(n \to \infty\) and since \((A, S)\) is compatible, we have

\[
\lim_{n \to \infty} M(ASu_n, SAu_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ASu_n, SAu_n, t) = 0,
\]

which implies that

\[
\lim_{n \to \infty} M(ASu_n, Sz, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ASu_n, Sz, t) = 0
\]

for all \(t > 0\), that is \(ASu_n \to Sz\) as \(n \to \infty\). By (C1), for each \(n\), there exists \(v_n \in X\) such that \(ASu_n = Tv_n\). Thus, we have \(SSu_n \to Sz, SAu_n \to Sz, ASu_n \to Sz\) and \(Tv_n \to Sz\) as \(n \to \infty\), whenever \(ASu_n = Tv_n\).
Now we claim that $Bv_n \to Sz$ as $n \to \infty$. By (C2), take $\alpha = 1$,

$$[1 + aM(SSu_n, Tv_n, kt)] \ast M(ASu_n, Bv_n, kt)$$

$$\geq a[M(ASu_n, SSu_n, kt) \ast M(Bv_n, Tv_n, kt) \ast M(Bv_n, SSu_n, kt)]$$

$$+ M(Tv_n, SSu_n, t) \ast M(ASu_n, SSu_n, t) \ast M(Bv_n, Tv_n, t)$$

$$\ast M(Bv_n, SSu_n, t) \ast M(ASu_n, Tv_n, t)$$

and

$$[1 + aN(SSu_n, Tv_n, kt)] \circ N(ASu_n, Bv_n, kt)$$

$$\leq a[N(ASu_n, SSu_n, kt) \circ N(Bv_n, Tv_n, kt) \circ N(Bv_n, SSu_n, kt)]$$

$$+ N(Tv_n, SSu_n, t) \circ N(ASu_n, SSu_n, t) \circ N(Bv_n, Tv_n, t)$$

$$\circ N(Bv_n, SSu_n, t) \circ N(ASu_n, Tv_n, t).$$

Taking $n \to \infty$,

$$[1 + aM(Sz, Sz, kt)] \ast M(Sz, Bv_n, kt)$$

$$\geq a[M(Sz, Sz, kt) \ast M(Bv_n, Sz, kt) \ast M(Bv_n, Sz, kt)]$$

$$+ M(Sz, Sz, t) \ast M(Sz, Sz, t) \ast M(Bv_n, Sz, t)$$

$$\ast M(Bv_n, Sz, t) \ast M(Sz, Sz, t)$$

and

$$[1 + aN(Sz, Sz, kt)] \circ N(Sz, Bv_n, kt)$$

$$\leq a[N(Sz, Sz, kt) \circ N(Bv_n, Sz, kt) \circ N(Bv_n, Sz, kt)]$$

$$+ N(Sz, Sz, t) \circ N(Sz, Sz, t) \circ N(Bv_n, Sz, t)$$

$$\circ N(Bv_n, Sz, t) \circ N(Sz, Sz, t),$$

which implies that

$$M(Sz, Bv_n, kt) \geq M(Bv_n, Sz, t)$$

and

$$N(Sz, Bv_n, kt) \geq N(Bv_n, Sz, t).$$

By Lemma 2.14, we have $Bv_n \to Sz$ as $n \to \infty$. Again by (C2), take $\alpha = 1$,

$$[1 + aM(Sz, Tv_n, kt)] \ast M(Az, Bv_n, kt)$$

$$\geq a[M(Az, Sz, kt) \ast M(Bv_n, Tv_n, kt) \ast M(Bv_n, Sz, kt)]$$

$$+ M(Tv_n, Sz, t) \ast M(Az, Sz, t) \ast M(Bv_n, Tv_n, t)$$

$$\ast M(Bv_n, Sz, t) \ast M(Az, Tv_n, t)$$
and

\[ [1 + aN(Sz, Tv_n, kt)] \odot N(Az, Bv_n, kt) \leq a[N(Az, Sz, kt) \odot N(Bv_n, Tv_n, kt) \odot N(Bv_n, Sz, kt)] \]

\[ + N(Tv_n, Sz, t) \odot N(Az, Sz, t) \odot N(Bv_n, Tv_n, t) \]

\[ \odot N(Bv_n, Sz, t) \odot N(Az, Tv_n, t). \]

Taking \( n \to \infty \),

\[ [1 + aM(Sz, Sz, kt)] \ast M(Az, Sz, kt) \geq a[M(Az, Sz, kt) \ast M(Sz, Sz, kt) \ast M(Sz, Sz, kt)] \]

\[ + M(Sz, Sz, t) \ast M(Az, Sz, t) \ast M(Sz, Sz, t) \]

\[ \ast M(Sz, Sz, t) \ast M(Az, Tv_n, t) \]

and

\[ [1 + aN(Sz, Sz, kt)] \odot N(Az, Sz, kt) \leq a[N(Az, Sz, kt) \odot N(Sz, Sz, kt) \odot N(Sz, Sz, kt)] \]

\[ + N(Sz, Sz, t) \odot N(Az, Sz, t) \odot N(Sz, Sz, t) \]

\[ \odot N(Sz, Sz, t) \odot N(Az, Sz, t), \]

which implies that

\[ M(Az, Sz, kt) \geq M(Az, Sz, t) \quad \text{and} \quad N(Az, Sz, kt) \geq N(Az, Sz, t). \]

By Lemma 2.14, \( Az = Sz \). Therefore, \( SAu_n \to Sz \) and \( ASu_n \to Sz = Az \) as \( n \to \infty \). Hence, \( A \) and \( S \) are reciprocally continuous on \( X \).

Similarly, if the pair \((B,T)\) is compatible and \( T \) is continuous, then the proof is similar. This completes the proof. \( \square \)

Now, we prove main theorem.

**Theorem 3.2.** Let \((X, M, N, *, \odot)\) be a complete intuitionistic fuzzy metric space with \( t \ast t \geq t \) and \( (1 - t) \odot (1 - t) \leq (1 - t) \) for all \( t \in [0,1] \). Further, let \( A, B, S \) and \( T \) be four self-mappings of \( X \) satisfying \( (C1) \) and \( (C2) \). If the pairs \((A,S)\) and \((B,T)\) are pointwise R-weakly commuting and one of the mappings in compatible pair \((A,S)\) or \((B,T)\) is continuous, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** By \((C1)\) since \( A(X) \subset T(X) \), for any point \( x_0 \in X \), there exists a point \( x_1 \in X \) such that \( Ax_0 = Tx_1 \). Since \( B(X) \subset S(X) \), for this point \( x_1 \in X \), we can choose a point \( x_2 \) in \( X \) such that \( Bx_1 = Sx_2 \) and so on. Inductively, we can define a sequence \( \{y_n\} \) in \( X \) such that for \( n = 0, 1, 2, \ldots \)

\[ y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \]
By (C2), for all \( t > 0 \) and \( \alpha = 1 - q \) with \( q \in (0, 1) \), we have

\[
\begin{align*}
[1 + aM(y_{2n}, y_{2n+1}, kt) &\ast M(y_{2n+1}, y_{2n+2}, kt) \\
&\geq a[M(y_{2n+2}, y_{2n+1}, kt) \ast M(y_{2n+1}, y_{2n}, kt) \\
&\ast M(y_{2n+1}, y_{2n+1}, (1 - q)t) \ast M(y_{2n+2}, y_{2n}, (1 + q)t) \\
&\geq a[M(y_{2n+2}, y_{2n+1}, kt) \ast M(y_{2n+1}, y_{2n}, kt) \\
&\ast M(y_{2n+1}, y_{2n+1}, t) \ast M(y_{2n+2}, y_{2n+1}, t) \ast M(y_{2n}, y_{2n+1}, qt)
\end{align*}
\]

and

\[
\begin{align*}
[1 + aN(y_{2n}, y_{2n+1}, qt) &\circ N(y_{2n+1}, y_{2n+2}, qt) \\
&\leq a[N(y_{2n+2}, y_{2n+1}, qt) \circ N(y_{2n+1}, y_{2n}, qt) \circ N(y_{2n+1}, y_{2n+1}, qt) \\
&\circ N(y_{2n+1}, y_{2n+1}, (1 - q)t) \circ N(y_{2n+2}, y_{2n}, (1 + q)t) \\
&\leq a[M(y_{2n+2}, y_{2n+1}, qt) \circ M(y_{2n+1}, y_{2n}, qt) \\
&\circ M(y_{2n+1}, y_{2n+1}, t) \circ M(y_{2n+2}, y_{2n+1}, t) \circ M(y_{2n}, y_{2n+1}, qt)
\end{align*}
\]

Thus it follows that

\[
M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t) \\
\ast M(y_{2n}, y_{2n+1}, qt)
\] (3.1)

and

\[
N(y_{2n+1}, y_{2n+2}, qt) \leq N(y_{2n}, y_{2n+1}, t) \circ N(y_{2n+1}, y_{2n+2}, t) \\
\circ N(y_{2n}, y_{2n+1}, qt)
\] (3.2)

Since the \( t \)-norm and the \( t \)-conorm are continuous and \( M(x, y, \cdot) \) is left continuous and \( N(x, y, \cdot) \) is right continuous. Letting \( q \to 1 \) in (3.1) and (3.2), we have

\[
M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t)
\]

and

\[
N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \circ N(y_{2n+1}, y_{2n+2}, t)
\]

Similarly, we also have

\[
M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) \ast M(y_{2n+2}, y_{2n+3}, t)
\]

and

\[
N(y_{2n+2}, y_{2n+3}, kt) \leq N(y_{2n+1}, y_{2n+2}, t) \circ N(y_{2n+2}, y_{2n+3}, t).
\]
In general, for \( m = 1, 2, 3, \ldots \)

\[
M(y_{m+1}, y_{m+2}, kt) \geq M(y_{m}, y_{m+1}, t) \ast M(y_{m+1}, y_{m+2}, t)
\]

and

\[
N(y_{m+1}, y_{m+2}, kt) \leq N(y_{m}, y_{m+1}, t) \diamond N(y_{m+1}, y_{m+3}, t).
\]

Consequently, it follows that for \( m = 1, 2, 3, \ldots \) and \( p = 1, 2, 3, \ldots \)

\[
M(y_{m+1}, y_{m+2}, kt) \geq M(y_{m}, y_{m+1}, t) \ast M(y_{m+1}, y_{m+2}, t/k^p)
\]

and

\[
N(y_{m+1}, y_{m+2}, kt) \leq N(y_{m}, y_{m+1}, t) \diamond N(y_{m+1}, y_{m+3}, t/k^p).
\]

As \( p \to \infty \),

\[
M(y_{m+1}, y_{m+2}, kt) \geq M(y_{m}, y_{m+1}, t)
\]

and

\[
N(y_{m+1}, y_{m+2}, kt) \leq N(y_{m}, y_{m+1}, t).
\]

Hence by Lemma 2.13, \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, \( \{y_n\} \) converges to \( z \in X \). Its subsequences \( \{Ax_{2n}\} \), \( \{Tx_{2n+1}\} \), \( \{Bx_{2n+1}\} \) and \( \{Sx_{2n+2}\} \) also converges to \( z \).

Now, suppose that \((A, S)\) is a compatible pair and \( S \) is continuous. Then by Lemma 3.1, \( A \) and \( S \) are reciprocally continuous, thus \( SAx_n \to Sz \) and \( ASx_n \to Az \) as \( n \to \infty \). As \((A, S)\) is a compatible pair, we have

\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ASx_n, SAx_n, t) = 0,
\]

that is,

\[
M(Az, Sz, t) = 1 \quad \text{and} \quad N(Az, Sz, t) = 0.
\]

Hence \( Az = Sz \). Since \( A(X) \subset T(X) \), there exists a point \( p \in X \) such that \( Az = Tp = Sz \). By \((C2)\), take \( \alpha = 1 \),

\[
[1 + aM(Sz, Tp, kt)] \ast M(Az, Bp, kt) \\
\geq a[M(Az, Sz, kt) \ast M(Bp, Tp, kt) \ast M(Bp, Sz, kt)] \\
+ M(Tp, Sz, t) \ast M(Az, Sz, t) \ast M(Bp, Tp, t) \\
\ast M(Bp, Sz, t) \ast M(Az, Tp, t)
\]

and

\[
[1 + aN(Sz, Tp, kt)] \diamond N(Az, Bp, kt) \\
\leq a[N(Az, Sz, kt) \diamond N(Bp, Tp, kt) \diamond N(Bp, Sz, kt)] \\
+ N(Tp, Sz, t) \diamond N(Az, Sz, t) \diamond N(Bp, Tp, t) \\
\diamond N(Bp, Sz, t) \diamond N(Az, Tp, t),
\]
which implies that

\[ M(Az, Bp, kt) \leq M(Az, Bp, t) \quad \text{and} \quad N(Az, Bp, kt) \geq N(Az, Bp, t) \]

for all \( t > 0 \). By Lemma 2.14, we have \( Az = Bp \). Thus, \( Az = Bp = Sz = Tp \). Since \( A \) and \( S \) are pointwise \( R \)-weakly commuting mappings, there exists \( R > 0 \) such that

\[ M(ASz, SAz, t) \geq M(Az, Sz, t/R) = 1 \]

and

\[ N(ASz, SAz, t) \leq N(Az, Sz, t/R) = 0. \]

Therefore, \( ASz = SAz \) and \( AAz = ASz = SAz = SSz \).

Similarly, \( B \) and \( T \) are pointwise \( R \)-weakly commuting mappings, we have \( BBp = BTp = TBp = TTp \). Again by \((C2)\), take \( \alpha = 1 \),

\[
[1 + aM(SAz, Tp, kt)] \ast M(AAz, Bp, kt) \\
\geq a[M(AAz, SAz, kt) \ast M(Bp, Tp, kt) \ast M(Bp, SAz, kt)] \\
+ M(Tp, SAz, t) \ast M(AAz, SAz, t) \ast M(Bp, Tp, t) \\
\ast M(Bp, SAz, t) \ast M(AAz, Tp, t)
\]

and

\[
[1 + aN(SAz, Tp, kt)] \odot N(AAz, Bp, kt) \\
\leq a[N(AAz, SAz, kt) \odot N(Bp, Tp, kt) \odot N(Bp, SAz, kt)] \\
+ N(Tp, SAz, t) \odot N(AAz, SAz, t) \odot N(Bp, Tp, t) \\
\odot N(Bp, SAz, t) \odot N(AAz, Tp, t),
\]

which implies that

\[ M(AAz, Az, kt) \leq M(AAz, Az, t) \]

and

\[ N(AAz, Az, kt) \geq N(AAz, Az, t). \]

Hence by Lemma 2.14, \( AAz = Az = SAz \). Hence \( Az \) is common fixed point of \( A \) and \( S \). Similarly by \((C2)\), \( Bp = Az \) is a common fixed point of \( B \) and \( T \). Hence, \( Az \) is a common fixed point of \( A, B, S \) and \( T \).

Finally, suppose that \( Ap \neq Az \) is another common fixed point of \( A, B, S \) and \( T \). Again by \((C2)\), take \( \alpha = 1 \),

\[
[1 + aM(SAz, TAp, kt)] \ast M(AAz, BAp, kt) \\
\geq a[M(AAz, SAz, kt) \ast M(BAp, TAp, kt) \ast M(BAp, SAz, kt)] \\
+ M(TAp, SAz, t) \ast M(AAz, SAz, t) \ast M(BAp, TAp, t) \\
\ast M(BAp, SAz, t) \ast M(AAz, TAp, t)
\]
and
\[
[1 + aN(SAz, TAp, kt)] \odot N(AAz, BAp, kt)
\]
\[
\leq a[N(AAz, SAz, kt) \odot N(Bp, TAp, kt) \odot N(BAp, SAz, kt)]
\]
\[
+ N(TAp, SAz, t) \odot N(AAz, SAz, t) \odot N(BAp, TAp, t)
\]
\[
\odot N(BAp, SAz, t) \odot N(AAz, TAp, t),
\]
which implies that
\[
M(Az, Ap, kt) \geq M(Az, Ap, t) \quad \text{and} \quad N(Az, Ap, kt) \leq N(Az, Ap, t).
\]
Hence by using Lemma 2.14, \(Az = Ap\). Thus uniqueness follows. This completes the proof. \(\square\)

If \(S\) and \(T\) are identity mappings in Theorem 3.2, we get following result.

**Corollary 3.3.** Let \((X, M, N, *, \odot)\) be a complete intuitionistic fuzzy metric space with \(t \ast t \geq t\) and \((1 - t) \ast (1 - t) \leq (1 - t)\) for all \(t \in [0, 1]\). Further, let \(A\) and \(B\) be reciprocally continuous mappings on \(X\) satisfying
\((C3)\) there exists a constant \(k \in (0, 1)\) such that
\[
[1 + aM(x, y, kt)] \ast M(Ax, By, kt)
\]
\[
\geq a[M(Ax, x, kt) \ast M(By, y, kt) \ast M(By, x, kt)]
\]
\[
+ M(y, x, t) \ast M(Ax, x, t) \ast M(By, y, t)
\]
\[
\ast M(By, x, \alpha t) \ast M(Ax, y, (2 - \alpha)t)
\]
and
\[
[1 + aN(x, y, kt)] \odot N(Ax, By, kt)
\]
\[
\leq a[N(Ax, x, kt) \odot N(By, y, kt) \odot N(By, x, kt)]
\]
\[
+ N(y, x, t) \odot N(Ax, x, t) \odot N(By, y, t)
\]
\[
\odot N(By, x, \alpha t) \odot N(Ax, y, (2 - \alpha)t)
\]
for all \(x, y \in X\), \(a \geq 0\), \(\alpha \in (0, 2)\) and \(t > 0\).

Then \(A\) and \(B\) has a unique common fixed point.

**References**


