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COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract: In this paper, we prove common fixed point theorems for four mappings by using pointwise R-weakly commuting and reciprocally continuous mappings satisfying contractive condition in intuitionistic fuzzy metric spaces.

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Key Words: intuitionistic fuzzy metric spaces, reciprocally continuous mappings, pointwise R-weakly commuting mappings

1. Introduction

Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park [7] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et al. [2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michálek [5]. In 2006, Türkoğlu et al. [9]

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proved Jungck's common fixed point theorem ([4]) in the setting of intuitionistic fuzzy metric spaces for commuting mappings. For more detail, one can refer to papers ([1], [6], [10], [11]).

In this paper, we prove a common fixed point theorem for four mappings by using pointwise R-weakly commuting and reciprocally continuous mappings satisfying contractive condition in intuitionistic fuzzy metric spaces.

2. Preliminaries

Schweizer and Sklar [8] defined the following notions:

Definition 2.1. A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is *continuous t-norm* if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a*1=a$ for all $a\in[0,1]$;
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.2. A binary operation $\diamond: [0,1]\times[0,1]\to[0,1]$ is *continuous t-conorm* if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a\diamond 0=a$ for all $a\in[0,1]$;
(iv) $a\diamond b\le c\diamond d$ whenever $a\le c$ and $b\le d$ for all$a,b,c,d\in[0,1]$.

Alaca et al. [2] defined the notion of intuitionistic fuzzy metric space as follows:

Definition 2.3. A 5-tuple $(X,M,N,*,\diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M,N are fuzzy sets on $X^2\times[0,\infty)$ satisfying

(i) $M(x,y,t)+N(x,y,t)\le 1$ for all $x,y\in X$ and $t>0$;
(ii) $M(x,y,0)=0$ for all $x,y\in X$;
(iii) $M(x,y,t)=1$ for all $x,y\in X$ and $t>0$ if and only if $x=y$;
(iv) $M(x,y,t)=M(y,x,t)$ for all $x,y\in X$ and $t>0$;
(v) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$ for all $x,y,z\in X$ and $s,t>0$;
(vi) for all $x,y\in X$, $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous;
(vii) $\lim_{t\to\infty}M(x,y,t)=1$ for all $x,y\in X$ and $t>0$;
(viii) $N(x,y,0)=1$ for all $x,y\in X$;
(ix) $N(x,y,t)=0$ for all $x,y\in X$ and $t>0$ if and only if $x=y$;
(x) $N(x,y,t)=N(y,x,t)$ for all $x,y\in X$ and $t>0$;

(xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(xii) for all $x, y \in X$, $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is right continuous;
(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.
Then (M, N) is called an *intuitionistic fuzzy metric* on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.4. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated as $x \diamond y = 1 - ((1 - x) * (1 - y))$ for all $x, y \in X$.

Remark 2.5. In an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.

Definition 2.6. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be
(i) *convergent* to a point $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n, x, t) = 0$$

for all $t > 0$,
(ii) *Cauchy sequence* if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$$

for all $t > 0$ and $p > 0$.

Definition 2.7. An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent.

Türkoğlu et al. [10] defined the following notions:

Definition 2.8. Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then a pair (A, S) is said to be *commuting* if

$$M(ASx, SAx, t) = 1 \quad \text{and} \quad N(ASx, SAx, t) = 0$$

for all $x \in X$ and $t > 0$.

Definition 2.9. Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then a pair (A, S) is said to be *weakly commuting* if

$$M(ASx, SAx, t) \geq M(Ax, Sx, t)$$

and

$$N(ASx, SAx, t) \leq N(Ax, Sx, t)$$

for all $x \in X$and $t > 0$.

Definition 2.10. Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then a pair (A, S) is said to be *compatible* if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$ for some $u \in X$.

Definition 2.11. ([9]) Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then a pair (A, S) is said to be *pointwise R-weakly commuting* if given $x \in X$, there exist $R > 0$ such that

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$$

and

$$N(ASx, SAx, t) \leq N(Ax, Sx, t/R)$$

for all $t > 0$.

Clearly, every pair of weakly commuting mappings is pointwise R-weakly commuting with $R = 1$.

Definition 2.12. ([6]) Two self-mappings A and S of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be *reciprocally continuous* if

$$ASu_n \to Az \quad \text{and} \quad SAu_n \to Sz,$$

whenever $\{u_n\}$ is a sequence such that $Au_n \to z$ and $Su_n \to z$ for some $z \in X$ as $n \to \infty$.

If A and S are both continuous, then they are obviously reciprocally continuous, but converse is not true.

Lemma 2.13. ([2], [9]) *Let $\{u_n\}$ is a sequence in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. If there exists a constant $k \in (0, 1)$ such that*

$$M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)$$

and

$$N(u_n, u_{n+1}, kt) \leq N(u_{n-1}, u_n, t)$$

for $n = 1, 2, 3, \ldots$, then $\{u_n\}$ is a Cauchy sequence in X.

Lemma 2.14. ([2], [9]) *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If there exists a constant $k \in (0, 1)$ such that*

$$M(x, y, kt) \geq M(x, y, t) \quad \text{and} \quad N(x, y, kt) \leq N(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

3. Main Results

For our main theorem, we need the following lemma.

Lemma 3.1. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t)$ for all $t \in [0,1]$. Further, let A, B, S and T be four self-mappings of X satisfying*

$(C1)$ $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

$(C2)$ there exists a constant $k \in (0,1)$ such that

$$\begin{aligned}&[1 + aM(Sx, Ty, kt)] * M(Ax, By, kt)\\ &\geq a[M(Ax, Sx, kt) * M(By, Ty, kt) * M(By, Sx, kt)]\\ &\quad + M(Ty, Sx, t) * M(Ax, Sx, t) * M(By, Ty, t)\\ &\quad\quad * M(By, Sx, \alpha t) * M(Ax, Ty, (2-\alpha)t)\end{aligned}$$

and

$$\begin{aligned}&[1 + aN(Sx, Ty, kt)] \diamond N(Ax, By, kt)\\ &\leq a[N(Ax, Sx, kt) \diamond N(By, Ty, kt) \diamond N(By, Sx, kt)]\\ &\quad + N(Ty, Sx, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t)\\ &\quad\quad \diamond N(By, Sx, \alpha t) \diamond N(Ax, Ty, (2-\alpha)t)\end{aligned}$$

for all $x, y \in X$, $a \geq 0$, $\alpha \in (0,2)$ and $t > 0$.

If the pairs (A, S) and (B, T) are pointwise R-weakly commuting, then one continuity of the mappings in compatible pair (A, S) or (B, T) implies their reciprocal continuity.

Proof. First, assume that A and S are compatible and S is continuous. We show that A and S are reciprocally continuous. Let $\{u_n\}$ be a sequence such that $Au_n \to z$ and $Su_n \to z$ for some $z \in X$ as $n \to \infty$. Since S is continuous, we have $SAu_n \to Sz$ and $SSu_n \to Sz$ as $n \to \infty$ and since (A, S) is compatible, we have

$$\lim_{n\to\infty} M(ASu_n, SAu_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ASu_n, SAu_n, t) = 0,$$

which implies that

$$\lim_{n\to\infty} M(ASu_n, Sz, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ASu_n, Sz, t) = 0$$

for all $t > 0$, that is $ASu_n \to Sz$ as $n \to \infty$. By $(C1)$, for each n, there exists $v_n \in X$ such that $ASu_n = Tv_n$. Thus, we have $SSu_n \to Sz$, $SAu_n \to Sz$, $ASu_n \to Sz$ and $Tv_n \to Sz$ as $n \to \infty$, whenever $ASu_n = Tv_n$.

Now we claim that $Bv_n \to Sz$ as $n \to \infty$. By $(C2)$, take $\alpha = 1$,

$$\begin{aligned}&[1 + aM(SSu_n, Tv_n, kt)] * M(ASu_n, Bv_n, kt)\\ &\geq a[M(ASu_n, SSu_n, kt) * M(Bv_n, Tv_n, kt) * M(Bv_n, SSu_n, kt)]\\ &\quad + M(Tv_n, SSu_n, t) * M(ASu_n, SSu_n, t) * M(Bv_n, Tv_n, t)\\ &\quad * M(Bv_n, SSu_n, t) * M(ASu_n, Tv_n, t)\end{aligned}$$

and

$$\begin{aligned}&[1 + aN(SSu_n, Tv_n, kt)] \diamond N(ASu_n, Bv_n, kt)\\ &\leq a[N(ASu_n, SSu_n, kt) \diamond N(Bv_n, Tv_n, kt) \diamond N(Bv_n, SSu_n, kt)]\\ &\quad + N(Tv_n, SSu_n, t) \diamond N(ASu_n, SSu_n, t) \diamond N(Bv_n, Tv_n, t)\\ &\quad \diamond N(Bv_n, SSu_n, t) \diamond N(ASu_n, Tv_n, t).\end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned}&[1 + aM(Sz, Sz, kt)] * M(Sz, Bv_n, kt)\\ &\geq a[M(Sz, Sz, kt) * M(Bv_n, Sz, kt) * M(Bv_n, Sz, kt)]\\ &\quad + M(Sz, Sz, t) * M(Sz, Sz, t) * M(Bv_n, Sz, t)\\ &\quad * M(Bv_n, Sz, t) * M(Sz, Sz, t)\end{aligned}$$

and

$$\begin{aligned}&[1 + aN(Sz, Sz, kt)] \diamond N(Sz, Bv_n, kt)\\ &\leq a[N(Sz, Sz, kt) \diamond N(Bv_n, Sz, kt) \diamond N(Bv_n, Sz, kt)]\\ &\quad + N(Sz, Sz, t) \diamond N(Sz, Sz, t) \diamond N(Bv_n, Sz, t)\\ &\quad \diamond N(Bv_n, Sz, t) \diamond N(Sz, Sz, t),\end{aligned}$$

which implies that

$$M(Sz, Bv_n, kt) \geq M(Bv_n, Sz, t)$$

and

$$N(Sz, Bv_n, kt) \geq N(Bv_n, Sz, t).$$

By Lemma 2.14, we have $Bv_n \to Sz$ as $n \to \infty$. Again by $(C2)$, take $\alpha = 1$,

$$\begin{aligned}&[1 + aM(Sz, Tv_n, kt)] * M(Az, Bv_n, kt)\\ &\geq a[M(Az, Sz, kt) * M(Bv_n, Tv_n, kt) * M(Bv_n, Sz, kt)]\\ &\quad + M(Tv_n, Sz, t) * M(Az, Sz, t) * M(Bv_n, Tv_n, t)\\ &\quad * M(Bv_n, Sz, t) * M(Az, Tv_n, t)\end{aligned}$$

and

$$\begin{aligned}&[1+aN(Sz,Tv_n,kt)]\diamond N(Az,Bv_n,kt)\\&\le a[N(Az,Sz,kt)\diamond N(Bv_n,Tv_n,kt)\diamond N(Bv_n,Sz,kt)]\\&\quad+N(Tv_n,Sz,t)\diamond N(Az,Sz,t)\diamond N(Bv_n,Tv_n,t)\\&\quad\diamond N(Bv_n,Sz,t)\diamond N(Az,Tv_n,t).\end{aligned}$$

Taking $n\to\infty$,

$$\begin{aligned}&[1+aM(Sz,Sz,kt)]*M(Az,Sz,kt)\\&\ge a[M(Az,Sz,kt)*M(Sz,Sz,kt)*M(Sz,Sz,kt)]\\&\quad+M(Sz,Sz,t)*M(Az,Sz,t)*M(Sz,Sz,t)\\&\quad*M(Sz,Sz,t)*M(Az,Tv_n,t)\end{aligned}$$

and

$$\begin{aligned}&[1+aN(Sz,Sz,kt)]\diamond N(Az,Sz,kt)\\&\le a[N(Az,Sz,kt)\diamond N(Sz,Sz,kt)\diamond N(Sz,Sz,kt)]\\&\quad+N(Sz,Sz,t)\diamond N(Az,Sz,t)\diamond N(Sz,Sz,t)\\&\quad\diamond N(Sz,Sz,t)\diamond N(Az,Sz,t),\end{aligned}$$

which implies that

$$M(Az,Sz,kt)\ge M(Az,Sz,t)\quad\text{and}\quad N(Az,Sz,kt)\ge N(Az,Sz,t).$$

By Lemma 2.14, $Az=Sz$. Therefore, $SAu_n\to Sz$ and $ASu_n\to Sz=Az$ as $n\to\infty$. Hence, A and S are reciprocally continuous on X.

Similarly, if the pair (B,T) is compatible and T is continuous, then the proof is similar. This completes the proof. □

Now, we prove main theorem.

Theorem 3.2. *Let $(X,M,N,*,\diamond)$ be a complete intuitionistic fuzzy metric space with $t*t\ge t$ and $(1-t)\diamond(1-t)\le(1-t)$ for all $t\in[0,1]$. Further, let A,B,S and T be four self-mappings of X satisfying $(C1)$ and $(C2)$. If the pairs (A,S) and (B,T) are pointwise R-weakly commuting and one of the mappings in compatible pair (A,S) or (B,T) is continuous, then A,B,S and T have a unique common fixed point.*

Proof. By $(C1)$ since $A(X)\subset T(X)$, for any point $x_0\in X$, there exists a point $x_1\in X$ such that $Ax_0=Tx_1$. Since $B(X)\subset S(X)$, for this point $x_1\in X$, we can choose a point x_2 in X such that $Bx_1=Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that for $n=0,1,2,\dots$

$$y_{2n}=Ax_{2n}=Tx_{2n+1}\quad\text{and}\quad y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}.$$

By $(C2)$, for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0,1)$, we have

$$\begin{aligned}
&[1 + aM(y_{2n}, y_{2n+1}, kt) * M(y_{2n+1}, y_{2n+2}, kt)\\
&\geq a[M(y_{2n+2}, y_{2n+1}, kt) * M(y_{2n+1}, y_{2n}, kt) * M(y_{2n+1}, y_{2n+1}, kt)]\\
&\quad + M(y_{2n}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t)\\
&\qquad * M(y_{2n+1}, y_{2n+1}, (1-q)t) * M(y_{2n+2}, y_{2n}, (1+q)t)\\
&\geq a[M(y_{2n+2}, y_{2n+1}, kt) * M(y_{2n+1}, y_{2n}, kt)]\\
&\quad + M(y_{2n}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, qt)
\end{aligned}$$

and

$$\begin{aligned}
&[1 + aN(y_{2n}, y_{2n+1}, kt)] \diamond N(y_{2n+1}, y_{2n+2}, kt)\\
&\leq a[N(y_{2n+2}, y_{2n+1}, kt) \diamond N(y_{2n+1}, y_{2n}, kt) \diamond N(y_{2n+1}, y_{2n+1}, kt)]\\
&\quad + N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+2}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n}, t)\\
&\qquad \diamond N(y_{2n+1}, y_{2n+1}, (1-q)t) \diamond N(y_{2n+2}, y_{2n}, (1+q)t)\\
&\leq a[M(y_{2n+2}, y_{2n+1}, kt) \diamond M(y_{2n+1}, y_{2n}, kt)]\\
&\quad + M(y_{2n}, y_{2n+1}, t) \diamond M(y_{2n+2}, y_{2n+1}, t) \diamond M(y_{2n}, y_{2n+1}, qt).
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)&\\
* M(y_{2n}, y_{2n+1}, qt)&
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n+2}, t)&\\
\diamond N(y_{2n}, y_{2n+1}, qt).&
\end{aligned} \tag{3.2}$$

Since the t-norm and the t-conorm are continuous and $M(x, y, \cdot)$ is left continuous and $N(x, y, \cdot)$ is right continuous. Letting $q \to 1$ in (3.1) and (3.2), we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$$

and

$$N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n+2}, t).$$

Similarly, we also have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t)$$

and

$$N(y_{2n+2}, y_{2n+3}, kt) \leq N(y_{2n+1}, y_{2n+2}, t) \diamond N(y_{2n+2}, y_{2n+3}, t).$$

In general, for $m = 1, 2, 3, \ldots$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t) * M(y_{m+1}, y_{m+2}, t)$$

and

$$N(y_{m+1}, y_{m+2}, kt) \leq N(y_m, y_{m+1}, t) \diamond N(y_{m+1}, y_{m+3}, t).$$

Consequently, it follows that for $m = 1, 2, 3, \ldots$ and $p = 1, 2, 3, \ldots$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t) * M(y_{m+1}, y_{m+2}, t/k^p)$$

and

$$N(y_{m+1}, y_{m+2}, kt) \leq N(y_m, y_{m+1}, t) \diamond N(y_{m+1}, y_{m+3}, t/k^p).$$

As $p \to \infty$,

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t)$$

and

$$N(y_{m+1}, y_{m+2}, kt) \leq N(y_m, y_{m+1}, t).$$

Hence by Lemma 2.13, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to $z \in X$. Its subsequences $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$ and $\{Sx_{2n+2}\}$ also converges to z.

Now, suppose that (A, S) is a compatible pair and S is continuous. Then by Lemma 3.1, A and S are reciprocally continuous, thus $SAx_n \to Sz$ and $ASx_n \to Az$ as $n \to \infty$. As (A, S) is a compatible pair, we have

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0,$$

that is,

$$M(Az, Sz, t) = 1 \quad \text{and} \quad N(Az, Sz, t) = 0.$$

Hence $Az = Sz$. Since $A(X) \subset T(X)$, there exists a point $p \in X$ such that $Az = Tp = Sz$. By $(C2)$, take $\alpha = 1$,

$$\begin{aligned}
&[1 + aM(Sz, Tp, kt)] * M(Az, Bp, kt) \\
&\geq a[M(Az, Sz, kt) * M(Bp, Tp, kt) * M(Bp, Sz, kt)] \\
&\quad + M(Tp, Sz, t) * M(Az, Sz, t) * M(Bp, Tp, t) \\
&\quad\quad * M(Bp, Sz, t) * M(Az, Tp, t)
\end{aligned}$$

and

$$\begin{aligned}
&[1 + aN(Sz, Tp, kt)] \diamond N(Az, Bp, kt) \\
&\leq a[N(Az, Sz, kt) \diamond N(Bp, Tp, kt) \diamond N(Bp, Sz, kt)] \\
&\quad + N(Tp, Sz, t) \diamond N(Az, Sz, t) \diamond N(Bp, Tp, t) \\
&\quad\quad \diamond N(Bp, Sz, t) \diamond N(Az, Tp, t),
\end{aligned}$$

which implies that

$$M(Az,Bp,kt) \leq M(Az,Bp,t) \quad \text{and} \quad N(Az,Bp,kt) \geq N(Az,Bp,t)$$

for all $t > 0$. By Lemma 2.14, we have $Az = Bp$. Thus, $Az = Bp = Sz = Tp$. Since A and S are pointwise R-weakly commuting mappings, there exists $R > 0$ such that

$$M(ASz, SAz, t) \geq M(Az, Sz, t/R) = 1$$

and

$$N(ASz, SAz, t) \leq N(Az, Sz, t/R) = 0.$$

Therefore, $ASz = SAz$ and $AAz = ASz = SAz = SSz$.

Similarly, B and T are pointwise R-weakly commuting mappings, we have $BBp = BTp = TBp = TTp$. Again by $(C2)$, take $\alpha = 1$,

$$\begin{aligned}&[1 + aM(SAz, Tp, kt)] * M(AAz, Bp, kt)\\ &\geq a[M(AAz, SAz, kt) * M(Bp, Tp, kt) * M(Bp, SAz, kt)]\\ &\quad + M(Tp, SAz, t) * M(AAz, SAz, t) * M(Bp, Tp, t)\\ &\quad\quad * M(Bp, SAz, t) * M(AAz, Tp, t)\end{aligned}$$

and

$$\begin{aligned}&[1 + aN(SAz, Tp, kt)] \diamond N(AAz, Bp, kt)\\ &\leq a[N(AAz, SAz, kt) \diamond N(Bp, Tp, kt) \diamond N(Bp, SAz, kt)]\\ &\quad + N(Tp, SAz, t) \diamond N(AAz, SAz, t) \diamond N(Bp, Tp, t)\\ &\quad\quad \diamond N(Bp, SAz, t) \diamond N(AAz, Tp, t),\end{aligned}$$

which implies that

$$M(AAz, Az, kt) \leq M(AAz, Az, t)$$

and

$$N(AAz, Az, kt) \geq N(AAz, Az, t).$$

Hence by Lemma 2.14, $AAz = Az = SAz$. Hence Az is common fixed point of A and S. Similarly by $(C2)$, $Bp = Az$ is a common fixed point of B and T. Hence, Az is a common fixed point of A, B, S and T.

Finally, suppose that Ap $(\neq Az)$ is another common fixed point of A, B, S and T. Again by $(C2)$, take $\alpha = 1$,

$$\begin{aligned}&[1 + aM(SAz, TAp, kt)] * M(AAz, BAp, kt)\\ &\geq a[M(AAz, SAz, kt) * M(BAp, TAp, kt) * M(BAp, SAz, kt)]\\ &\quad + M(TAp, SAz, t) * M(AAz, SAz, t) * M(BAp, TAp, t)\\ &\quad\quad * M(BAp, SAz, t) * M(AAz, TAp, t)\end{aligned}$$

and

$$\begin{aligned}&[1 + aN(SAz, TAp, kt)] \diamond N(AAz, BAp, kt)\\&\le a[N(AAz, SAz, kt) \diamond N(Bp, TAp, kt) \diamond N(BAp, SAz, kt)]\\&\quad + N(TAp, SAz, t) \diamond N(AAz, SAz, t) \diamond N(BAp, TAp, t)\\&\quad\quad \diamond N(BAp, SAz, t) \diamond N(AAz, TAp, t),\end{aligned}$$

which implies that

$$M(Az, Ap, kt) \ge M(Az, Ap, t) \quad \text{and} \quad N(Az, Ap, kt) \le N(Az, Ap, t).$$

Hence by using Lemma 2.14, $Az = Ap$. Thus uniqueness follows. This completes the proof. □

If S and T are identity mappings in Theorem 3.2, we get following result.

Corollary 3.3. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \ge t$ and $(1 - t) \diamond (1 - t) \le (1 - t)$ for all $t \in [0, 1]$. Further, let A and B be reciprocally continuous mappings on X satisfying*

$(C3)$ there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned}&[1 + aM(x, y, kt)] * M(Ax, By, kt)\\&\ge a[M(Ax, x, kt) * M(By, y, kt) * M(By, x, kt)]\\&\quad + M(y, x, t) * M(Ax, x, t) * M(By, y, t)\\&\quad\quad * M(By, x, \alpha t) * M(Ax, y, (2 - \alpha)t)\end{aligned}$$

and

$$\begin{aligned}&[1 + aN(x, y, kt)] \diamond N(Ax, By, kt)\\&\le a[N(Ax, x, kt) \diamond N(By, y, kt) \diamond N(By, x, kt)]\\&\quad + N(y, x, t) \diamond N(Ax, x, t) \diamond N(By, y, t)\\&\quad\quad \diamond N(By, x, \alpha t) \diamond N(Ax, y, (2 - \alpha)t)\end{aligned}$$

for all $x, y \in X$, $a \ge 0$, $\alpha \in (0, 2)$ and $t > 0$.

Then A and B has a unique common fixed point.

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