ON THE DIOPHANTINE EQUATION $143^x + 145^y = z^2$

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Abstract: In this paper, we show that $(1,0,12)$ is a unique non-negative integer solution $(x,y,z)$ for the Diophantine equation $143^x + 145^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.

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1. Introduction

In 2012, Chotchaisthit [1] showed that, for any positive prime $p$, the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where $x$, $y$, $z$ are non-negative integers. In the same year, Sroysang [6] showed that $(1,0,2)$ is a unique non-negative integer solution $(x,y,z)$ for the Diophantine equation $3^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. In 2013, Sroysang [8] showed that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. For similar equations, we refer to [4, 5, 7, 9, 10, 11, 12, 13, 14]. Recently, Rabago [3] showed that $(1,0,3)$, $(1,1,5)$, $(2,1,9)$ and $(3,1,23)$ are only four solutions $(x,y,z)$ for the Diophantine equation $8^x + 17^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.

In this paper, we show that $(1,0,12)$ is a unique non-negative integer solution $(x,y,z)$ for the Diophantine equation $143^x + 145^y = z^2$ where $x$, $y$ and $z$
are non-negative integers.

2. Preliminaries

**Proposition 2.1.** [2] (Catalan’s conjecture) \((3, 2, 2, 3)\) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers such that \(\min\{a, b, x, y\} > 1\).

**Lemma 2.2.** \((1, 12)\) is a unique solution \((x, z)\) for the Diophantine equation \(143^x + 1 = z^2\) where \(x\) and \(z\) are non-negative integers.

**Proof.** Let \(x\) and \(z\) be non-negative integers such that \(143^x + 1 = z^2\). If \(x = 0\), then \(z^2 = 2\) which is impossible. Then \(x \geq 1\). It follows that \(z^2 = 143^y + 1 \geq 143^1 + 1 = 144\). Then \(z \geq 12\). Now, we consider on the equation \(z^2 - 143^x = 1\). By Proposition 2.1, we have \(x = 1\). We obtain that \(z^2 = 144\) and then \(z = 12\).

**Lemma 2.3.** The Diophantine equation \(1 + 145^y = z^2\) has no non-negative integer solution where \(y\) and \(z\) are non-negative integers.

**Proof.** Suppose that there are non-negative integers \(y\) and \(z\) such that \(1 + 145^y = z^2\). If \(y = 0\), then \(z^2 = 2\) which is impossible. Then \(y \geq 1\). It follows that \(z^2 = 1 + 145^y \geq 1 + 145^1 = 146\). Then \(z \geq 13\). Now, we consider on the equation \(z^2 - 145^y = 1\). By Proposition 2.1, we have \(y = 1\). We obtain that \(z^2 = 146\). This is a contradiction.

3. Main Results

**Theorem 3.1.** \((1, 0, 12)\) is a unique non-negative integer solution \((x, y, z)\) for the Diophantine equation \(143^x + 145^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

**Proof.** Let \(x, y\) and \(z\) be non-negative integers such that \(143^x + 145^y = z^2\). By Lemma 2.3, it follows that \(x \geq 1\). Note that \(z\) is even. This implies that \(z^2 \equiv 0 \pmod{4}\). Since \(145^y \equiv 1 \pmod{4}\), we obtain that \(143^x \equiv 3 \pmod{4}\). Then \(x\) is odd. Now, we will divide the number \(y\) into two cases.

Case \(y = 0\). By Lemma 2.2, we have \(x = 1\) and \(z = 12\).
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Case $y \geq 1$. Note that $145^y \equiv 0 \pmod{5}$. Since $143^y \equiv 2 \pmod{5}$ or $143^y \equiv 3 \pmod{5}$, it follows that $z^2 \equiv 2 \pmod{5}$ or $z^2 \equiv 3 \pmod{5}$. This implies that $z$ is odd. This is a contradiction.

Hence, $(1, 0, 12)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $143^x + 145^y = z^2$ where $x, y$ and $z$ are non-negative integers.

**Corollary 3.2.** The Diophantine equation $143^x + 145^y = w^4$ has no non-negative integer solution where $x, y$ and $w$ are non-negative integers.

**Proof.** Suppose that there are non-negative integers $x, y$ and $w$ such that $143^x + 145^y = w^4$. Let $z = w^2$. This implies that $143^x + 145^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $w^2 = z = 12$. This is a contradiction.

**Corollary 3.3.** $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $143^x + 145^y = 9u^4$ where $x, y$ and $u$ are non-negative integers.

**Proof.** Let $x, y$ and $u$ be non-negative integers such that $143^x + 145^y = 9u^4$. Let $z = 3u^2$. This implies that $143^x + 145^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $3u^2 = z = 12$. It follows that $u = 2$.

**Corollary 3.4.** $(1, 0, 3)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $143^x + 145^y = 16v^2$ where $x, y$ and $v$ are non-negative integers.

**Proof.** Let $x, y$ and $v$ be non-negative integers such that $143^x + 145^y = 16v^2$. Let $z = 4v$. This implies that $143^x + 145^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $4v = z = 12$. It follows that $v = 3$.

**References**


