International Journal of Pure and Applied Mathematics
Volume 91 No. 2 2014, 191-196
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i2.5
ijpam.eu

NOTES ON λ-COMMUTING OPERATORS

Krutan Rasimi[1 §], Alit Ibraimi[2], Luigj Gjoka[3]

[1,2]Faculty of Natural Sciences and Mathematics
State University of Tetova
Ilinden St., 1200, Tetovo, MACEDONIA

[3]Faculty of Mathematical Engineering and Physical Engineering
Polytechnic University of Tirana
"Dëshmorët e Kombit" Blvd., Square "Mother Teresa" 4
1001, Tirana, ALBANIA

Abstract: The aim of the paper is to give some results relating to λ-commuting operators where pairs of different classes of operators are included. Under considerations we take pairs of λ-commuting operators that include normal, hyponormal, quasihyponormal and isometric operators. We focus on algebraic relations between different operators. In fact, we have worked on operators products and we have presented some results.

AMS Subject Classification: 47B20, 47A30
Key Words: Hilbert space, hyponormal, quasihyponormal, isometric operators

1. Introduction

Throughout this article, we denote with H the complex infinite separable Hilbert space and with $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators acting on H. We say that an operator T from $\mathcal{B}(H)$ is a normal operator if is commutative with his adjoint operator T^*, that is, if $T^*T = TT^*$. For the operator T from $\mathcal{B}(H)$ we say that is hyponormal operator if $T^*T \geq TT^*$. This condition is equivalent to $||T^*x|| \leq ||Tx||$, $x \in H$. Operator T is a

Received: September 2, 2013

§Correspondence author

quasihyponormal operator if $T^{*2}T^2 \geq (T^*T)^2$ holds. This is equivalent to $||T^*Tx|| \leq ||TTx||$, $x \in H$. An operator $T \in \mathcal{B}(H)$ is said to be paranormal if $||Tx||^2 \leq ||T^2x|| \cdot ||x||$ holds for all $x \in H$. An operator $T \in \mathcal{B}(H)$ is said to be isometric if $T^*T = I$. Finally, operators $A, B \in \mathcal{B}(H)$ are said to be λ-commutative if $AB = \lambda BA$, non trivially provided $AB \neq 0$.

2. Main Results

Recall that if A is a hyponormal operator and B a normal operator then operators AB and BA are hyponormal operators that provides$AB = BA$. Proof is straightforward using Fuglede-Putnam theorem.

In [2], [4], and [5], authors have used definition of λ-commutative operators and have determined, for various classes of operators, restrictions on $\lambda \in \mathbb{C}$. For example, in [4] authors have proved what values can take λ in specific case. They have proved the following result.

Theorem 2.1. (see [4]) *Let $A, B \in \mathcal{B}(H)$ and $AB = \lambda BA \neq 0$. Then:*

1. *If A^* and B are hyponormal, then $|\lambda| \leq 1$;*
2. *If A and B^* are hyponormal, then $|\lambda| \geq 1$.*

Moreover, in papers cited above, authors give some relation between values of $\lambda \in \mathbb{C}$ and spectrum of specific operator. Our intention is to study the product of operators for given values of λ.

Theorem 2.2. *Let be $A, B \in \mathcal{B}(H)$ bounded linear operator such that $AB = \lambda BA \neq O$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. If A^* and B are hyponormal then A^*B and BA^* are hyponormal operators.*

Proof. Let $(A^*B)^*(A^*B) = B^*AA^*B \geq B^*A^*AB$ because A^* is a hyponormal operator. In other hand we have $(A^*B)(A^*B)^* = A^*BB^*A \leq A^*B^*BA = \overline{\lambda}^{-1}\lambda^{-1}B^*A^*AB = |\lambda|^{-2}B^*A^*AB \leq B^*A^*AB$ since B is hyponormal and $|\lambda| \geq 1$.

Similarly we can prove the following result.

Corollary 2.3. *Let $A, B \in \mathcal{B}(H)$ be bounded linear operator such that $AB = \lambda BA \neq O$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. If A and B^* are hyponormal then AB^* and B^*A are hyponormal operators.*

Proof. We have

$$||(AB^*)^*x|| = ||BA^*x|| \leq ||B^*A^*x|| \quad \text{(since } B^* \text{ is hyponormal)}$$

$$
\begin{aligned}
&=||\overline{\lambda}A^*B^*x||\\
&=|\overline{\lambda}|||A^*B^*x||\\
&\leq||AB^*x|| \quad \text{(since } |\overline{\lambda}| = |\lambda| \leq 1 \text{ and } A \text{ is hyponormal).}
\end{aligned}
$$

In [5], the authors proved the following result.

Theorem 2.4. (see [5]) *Let A be hyponormal operator and B normal such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}$. Then following statements are equivalent:*

1. *AB is hyponormal;*
2. *$\sigma(AB) \neq \{0\}$;*
3. *$|\lambda| = 1$.*

Remarks 2.5. *If $|\lambda| = 1$, then BA is a hyponormal operator.*

We state the following theorem.

Theorem 2.6. *Let A be quasihyponormal operator and B normal such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}$. If $|\lambda| = 1$, then AB is quasihyponormal operator.*

Proof. First of all let us show that from $AB = \lambda BA \neq 0$, we have $B^*A^* = \overline{\lambda}A^*B^*$, $A^*B = \lambda^{-1}BA^*$ and $A^*B^* = \overline{\lambda}^{-1}B^*A^*$. Meanwhile, because B is normal, then λB is also a normal operator. From Fuglede-Putnam theorem, condition $AB = \lambda BA$ imply that $AB^* = \overline{\lambda}B^*A$. Now we can calculate

$$
\begin{aligned}
||(AB)^*(AB)x|| &= ||B^*A^*ABx|| = ||BA^*ABx|| = ||\lambda A^*BABx||\\
&= ||\lambda\lambda^{-1}A^*ABBx|| \leq ||AABBx|| = |\lambda|||ABABx|| = ||(AB)^2x||.
\end{aligned}
$$

In similar way we prove that BA is a quasihyponormal operator.

Note that in above results we use commutativity and λ-commutativity of an operator with a normal operator. This fact is very important because enables us to use Fuglede-Putnam Theorem.

We know (see [1]) that products of two hyponormal operators A and B, that is, AB and BA are hyponormal whenever A commute with B^*. In similar way we can say that if A^* and B are hyponormal such that $AB = BA$, then A^*B and BA^* are hyponormal. This is a trivial case of Theorem 2.2, for $\lambda = 1$.

Let A be a quasihyponormal and B a hyponormal operator. Then the conclusion of Theorem 2.6 is not true. In such and other cases, the pairs of operators that not include normal operators, we need additional condition like double commutativity. That is, along $AB = BA$ we need to take $AB^* = B^*A$. See, for example, [3].

For those and similar cases, we introduce the following definition.

Definition 2.7. We say that operators A and B are (λ, μ)-*commuting operators*, if they satisfy $AB = \lambda BA$ and $AB^* = \mu B^*A$.

Theorem 2.8. *Let A and B be (λ, μ)-commuting operators such that A is a quasihyponormal, B a hyponormal and $|\mu| \leq |\lambda| \leq 1$. Then operator AB is a quasihyponormal operator.*

Proof. First of all, let us re-write relations: $AB = \lambda BA$, $B^*A^* = \overline{\lambda}A^*B^*$, $BA = \lambda^{-1}AB$, $A^*B^* = \overline{\lambda}^{-1}B^*A^*$, $AB^* = \mu B^*A$, $BA^* = \overline{\mu}A^*B$, $B^*A = \mu^{-1}AB^*$, $A^*B = \overline{\mu}^{-1}BA^*$. Let us calculate now.

$$\begin{aligned}
||(AB)^*(AB)x|| =& ||B^*A^*ABx|| \\
\leq& ||BA^*ABx|| \quad \text{(because } B \text{ is hyponormal)} \\
=& ||\overline{\mu}A^*BABx|| \\
=& ||\overline{\mu}\lambda^{-1}A^*ABBx|| \\
=& \left|\frac{\overline{\mu}}{\lambda}\right| ||A^*ABBx|| \\
=& \frac{|\overline{\mu}|}{|\lambda|}||A^*ABBx|| \\
\leq& ||AABBx|| \quad \text{(because } A \text{ is quasihyponormal and } |\overline{\mu}|/|\lambda| \leq 1) \\
=& ||\lambda ABABx|| = |\lambda|||ABABx|| \leq ||(AB)^2x||.
\end{aligned}$$

Theorem 2.9. *Let A and B be (λ, μ)-commuting operators such that A is a quasihyponormal, B is an isometric operator, and $|\mu| \geq |\lambda| \geq 1$. Then operator AB is a quasihyponormal operator.*

Proof. We have

$$\begin{aligned}
||(AB)^*(AB)x|| =& ||B^*A^*ABx|| \\
=& ||\lambda B^*A^*BAx|| \quad \text{(because } AB = \lambda BA) \\
=& ||\lambda\overline{\mu}^{-1}B^*BA^*Ax|| \quad \text{(because } A^*B = \overline{\mu}^{-1}BA^*) \\
=& |\lambda/\overline{\mu}|\,||B^*BA^*Ax|| = (|\lambda|/|\overline{\mu}|)||B^*BA^*Ax|| \\
\leq& ||A^*Ax|| \quad \text{(because } B \text{ is isometric and } |\lambda|/|\overline{\mu}| \leq 1) \\
\leq& ||AAx|| \quad \text{(because } A \text{ is quasihyponormal)} \\
=& ||B^2(A^2x)|| \quad \text{(because } B \text{ is isometric)}
\end{aligned}$$

$$
\begin{aligned}
&= ||BBAAx|| \\
&= ||\lambda^{-3}ABABx|| \\
&= \left|1/\lambda^3\right| \, ||ABABx|| = (1/|\lambda|^3)||ABABx|| \\
&\leq ||(AB)^2x|| \quad \text{(because } |\lambda| \geq 1\text{).}
\end{aligned}
$$

Theorem 2.10. *Let A and B be (λ, μ)-commuting operators such that A is a quasihyponormal, B is an isometric operator. Then the operator BA is a quasihyponormal operator whenever $|\lambda| \geq 1$, $|\mu| \geq 1$.*

Proof. We have

$$
\begin{aligned}
||(BA)^*(BA)x|| &= ||A^*B^*BAx|| \\
&= ||\overline{\lambda}^{-1}B^*A^*BAx|| \quad \text{(because } A^*B^* = \overline{\lambda}^{-1}B^*A^*\text{)} \\
&= ||\overline{\lambda}^{-1}\overline{\mu}^{-1}B^*BA^*Ax| \quad |\text{(because } A^*B = \overline{\mu}^{-1}BA^*\text{)} \\
&= (1/|\overline{\lambda}\overline{\mu}|)||B^*BA^*Ax|| = (1/|\overline{\lambda}||\overline{\mu}|)||B^*BA^*Ax|| \\
&\leq ||A^*Ax|| \quad \text{(because } B \text{ is isometric and } 1/|\overline{\lambda}||\overline{\mu}| \leq 1\text{)} \\
&\leq ||AAx|| \quad \text{(because } A \text{ is quasihyponormal)} \\
&= ||B^2(A^2x)|| \quad \text{(because } B\text{is isometric)} \\
&= ||BBAAx|| \\
&= ||\lambda^{-1}BABAx|| \\
&= |1/\lambda| \, ||BABAx|| = (1/|\lambda|)||BABAx|| \\
&\leq ||(BA)^2x|| \quad \text{(because } |\lambda| \geq 1\text{).}
\end{aligned}
$$

References

[1] J.B. Conway, W. Szymanski, Linear combinations of hyponormal operators, *Roucky Mountain Journal of Mathematics*, **18**, No. 3 (1988), 695-705.

[2] J.A. Brooke, P. Busch, D.B. Pearson, Commutativity up to a factor of bounded operators in complex Hilbert space, *ArXiv:*math/0007049v3 [math.FA] (2001).

[3] T. Veluchamy, A. Devika, Some properties of quasi-*paranormal operators, *Journal of Modern Mathematics and Statistics*, **1**, No-s: 1-4 (2007), 35- 38.

[4] Cm. Chõ, J.I. Lee, T. Yamazaki, On the operator equation $AB = zBA$, *Scientiae Mathematicae Japonicae Online*, **e-2009**, 49-55.

[5] L. Zhang, T. Ohwada, M. Chõ, On λ-commutating operators, *International Mathematical Forum*, **6**, No. 34 (2011), 1685-1690.