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MORPHISMS ON CLOSURE SPACES AND MOORE SPACES

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Abstract: We discuss an equivalence of the concepts of complete lattices, Moore classes, and a one-to-one correspondence between Moore classes of subsets of a set X and closure operators on X. Also, we establish a correspondence between closure operators on sets and complete lattices. We describe morphisms among partially ordered sets, lattices, Moore classes, closure operators and complete lattices and discuss certain inter-relationships between these objects.

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1. Introduction and Preliminaries

A partially ordered set (poset) is a pair $(X, \leq)$, where X is a non empty set and

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$\leq$ is a partial order (a reflexive, transitive and antisymmetric binary relation) on X. For any subset A of X and $x \in X$, x is called a lower bound (upper bound) of A if $x \leq a$ ($a \leq x$ respectively) for all $a \in A$. A poset $(X, \leq)$ is called a lattice if every nonempty finite subset of X has greatest lower bound (or glb or infimum) and least upper bound (or lub or supremum) in X. If $(X, \leq)$ is a lattice and, for any $a, b \in X$, if we define $a \wedge b =$ infimum $\{a, b\}$ and $a \vee b =$ supremum $\{a, b\}$, then $\wedge$ and $\vee$ are binary operations on X which are commutative, associative and idempotent and satisfy the absorption laws $a \wedge (a \vee b) = a = a \vee (a \wedge b)$. Conversely, any algebraic system $(X, \wedge, \vee)$ satisfying the above properties becomes a lattice in which the partial order is defined by $a \leq b \Longleftrightarrow a = a \wedge b \Longleftrightarrow a \vee b = b$. A lattice $(X, \wedge, \vee)$ is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in X$ (equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in X$). A lattice $(X, \wedge, \vee)$ is called a bounded lattice if it has the smallest element 0 and largest element 1; that is, there are elements 0 and 1 in X, such that $0 \leq x \leq 1$ for all $x \in X$.

A partially ordered set in which every subset has infimum and supremum is called a complete lattice. If $(L, \leq)$ is a complete lattice and $X \subseteq L$, we write inf$X$ or $\wedge X$ or $\bigwedge_{x \in X} x$ for the infimum of X and sup X or $\vee X$ or $\bigvee_{x \in X} x$ for the supremum of X. If $X = \{x_1, x_2, \cdots, x_n\}$ is a finite subset, then we write $\bigwedge_{i=1}^{n} x_i$ or $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for inf$X$ and $\bigvee_{i=1}^{n} x_i$ or $x_1 \vee x_2 \vee \cdots \vee x_n$ for sup$X$. Any complete lattice has the smallest element and the greatest element which are denoted by 0 and 1 respectively.Logically, the infimum and supremum of the empty set are 1 and 0 respectively. An element $a \neq 0$ in a complete lattice L is called compact if, for any $A \subseteq L$, $a \leq supA \Longrightarrow a \leq supF$ for some finite $F \subseteq A$. A complete lattice in which every element is the supremum of a set of compact elements is called an algebraic lattice. For elementary properties of posets and lattices we refer to [1] and [2].

2. Moore Classes and Closure Operators

In this section, we introduce the notion of a Moore class and discuss certain important elementary properties of these. To begin with, we have the following.

Definition 2.1. Let X be any non-empty set. A non-empty class $\mathscr{M}$ of subsets of X is called a Moore class on X if $\mathscr{M}$ is closed under arbitrary intersections, in the sense that, if $\{M_\alpha\}_{\alpha \in \Delta}$ is a subclass of $\mathscr{M}$, then $\bigcap_{\alpha \in \Delta} M_\alpha \in$

$\mathscr{M}$.

Example 2.2. Let X be a topological space and $\mathscr{C}$ be the class of all closed subsets of X [3]. Then $\mathscr{C}$ is a Moore class on the set X.

Example 2.3. Let G be a group and $\mathscr{S}$ be the class of all subgroups of G. Then $\mathscr{S}$ is a Moore class on the set G.

Example 2.4. Let A be any universal algebra [4],where there is atleast one fundamental nullary operation, and let $\mathscr{S}$ be the class of all subalgebras of A. If a_0 is an element of A corresponding to a fundamental nullary operation on A, then a_0 belongs to every subalgebra of A. From this it follows that the intersection of any class of subalgebras of A is non-empty and hence a subalgebra of A. Thus $\mathscr{S}$ is a Moore class on A.

Example 2.5. For any non-empty set X, the whole power set $\mathscr{P}(X)$ is a Moore class on X and is called the discrete Moore class on X.

Example 2.6. Let X be a non-empty set and $A \subseteq X$. Then the class $\{A, X\}$ is a Moore class on X.

Since the intersection of the empty class of subsets of a set X is the whole set X, it follows that any Moore class necessarily contains X. The following is a straight forward verification.

Theorem 2.7. *Let $\mathscr{M}$ be a Moore class on a set X. For any subset A of X, define*

$$\overline{A} = \cap\{M \in \mathscr{M} \mid A \subseteq M\}.$$

Then the following hold for any subsets A and B of X

(1). $A \subseteq \overline{A}$

(2). $\overline{\overline{A}} = \overline{A}$

(3). $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$

(4). $\overline{A}$ *is the smallest member in* $\mathscr{M}$ *containing* A.

Corollary 2.8. *If $\mathscr{M}$ is a Moore class on X, then* $\mathscr{M} = \{A \subseteq X \mid \overline{A} = A\}$.

Note that in 2.2 above, $\overline{A}$ is the topological closure of A, while in 2.3, $\overline{A}$ is the subgroups generated by A.

Definition 2.9. Let X be a non-empty set and $\mathscr{P}(X)$ the set of all subsets of X. A mapping $c : \mathscr{P}(X) \to \mathscr{P}(X)$ is called a closure operator on X if it satisfies the following for any A and B in $\mathscr{P}(X)$

(1). c is extensive ; that is, $A \subseteq c(A)$

(2). c is idempotent ; that is, $c(c(A)) = c(A)$

(3). c is inclusion preserving ; that is, $A \subseteq B \Rightarrow c(A) \subseteq c(B)$.

The following is an immediate consequence of Theorem 2.7.

Theorem 2.10. *Let $\mathscr{M}$ be a Moore class on a set X and, for any $A \subseteq X$, define*

$$c_{\mathscr{M}}(A) = \cap \{M \in \mathscr{M} \mid A \subseteq M\}.$$

Then $c_{\mathscr{M}}$ is a closure operator on X.

The following is routine verification.

Theorem 2.11. *Let c be a closure operator on a set X and*

$$\mathscr{M}_c = \{A \subseteq X \mid c(A) = A\} = \{c(A) \mid A \subseteq X\}.$$

Then $\mathscr{M}_c$ is a Moore class on X. Also, $c \mapsto \mathscr{M}_c$ is a one-to-one correspondence between the closure operators on X and Moore classes on X.

Definition 2.12. A closure operator c on a set X is called a topological closure operator if $c(\phi) = \phi$ and $c(A \cup B) = c(A) \cup c(B)$ for any subsets A and B of X.

Theorem 2.13. *Let c be a closure operator on a set X and $\mathscr{M}_c$ be the Moore class on X corresponding to c, as given in Theorem 2.11. Then c is a topological closure operator on X if and only if $\mathscr{M}_c$ is closed under finite unions, in the sense that, for any finite subclass $\{M_i\}_{i \in I}$ of $\mathscr{M}_c$, $\bigcup_{i \in I} M_i \in \mathscr{M}_c$.*

Proof. Suppose that c is a topological closure operator on X. First, let us observe that the empty set ϕ belongs to $\mathscr{M}_c$, since $c(\phi) = \phi$. Let $\{M_i\}_{i \in I}$ be a finite subclass of $\mathscr{M}_c$. If I is empty, then

$$\bigcup_{i \in I} M_i = \phi \in \mathscr{M}_c$$

Therefore, we can assume that I is non-empty, say $I = \{1, 2, \ldots, n\}$, where n is a positive integer. Then

$$\begin{aligned} \bigcup_{i \in I} M_i &= M_1 \cup M_2 \cup \ldots \cup M_n \\ &= c(M_1) \cup c(M_2) \cup \ldots \cup c(M_n) \text{ , since} M_i \in \mathscr{M}_c \end{aligned}$$

$$= c(M_1 \cup M_2 \cup \ldots \cup M_n)$$

and therefore $\bigcup_{i \in I} M_i \in \mathscr{M}_c$. Thus $\mathscr{M}_c$ is closed under finite unions.

Conversely, suppose that $\mathscr{M}_c$ is closed under finite unions. In particular, $\phi \in \mathscr{M}_c$, since the union of the empty class of sets is empty. Therefore $c(\phi) = \phi$. Also, for any subsets A and B of X, we have

$$c(c(A)) = c(A) \text{ and } c(c(B)) = c(B)$$

and hence $c(A)$ and $c(B)$ are members of $\mathscr{M}_c$. Since $\mathscr{M}_c$ is closed under finite unions, it follow that

$$c(A) \cup c(B) \in \mathscr{M}_c$$

and therefore $c(c(A) \cup c(B)) = c(A) \cup c(B)$.

$$\begin{aligned} \text{Now, } c(A) \cup c(B) &\subseteq c(A \cup B) \\ &\subseteq c(c(A) \cup c(B)) \\ &= c(A) \cup c(B) \end{aligned}$$

and hence $c(A \cup B) = c(A) \cup c(B)$. Thus c is a topological closure operator. □

The following can be easily verified.

Theorem 2.14. *Let c be a closure operator on a set X. Then c is topological if and only if there is a unique topology on X with respect to which $c(A)$ is the closure of A, for all subsets A of X.*

3. Complete Lattices

In this section, we establish a correspondence between Moore classes on sets and complete lattices. Let us first recall that two posets $(X, \leq)$ and $(Y, \leq)$ are said to be isomorphic if there is a bijection $f : X \to Y$ such that $a \leq b$ in $X \Leftrightarrow f(a) \leq f(b)$ in Y. If $\mathscr{M}$ is a Moore class on a set X, then $\mathscr{M}$ becomes a complete lattice under the set inclusion ordering in which, for any $\{M_i\}_{i \in I}$ in $\mathscr{M}$,

$$\inf\{M_i\}_{i \in I} = \bigcap_{i \in I} M_i \text{ and } \sup\{M_i\}_{i \in I} = \overline{\bigcup_{i \in I} M_i}$$

The converse of this is proved in the following.

Theorem 3.1. *Let $(L, \leq)$ be a complete lattice. Then $(L, \leq)$ is isomorphic to the Moore class $\mathscr{M}_c$ corresponding to a closure operator c on a suitable set.*

Proof. Consider the set L and define,

$$c: \mathscr{P}(L) \longrightarrow \mathscr{P}(L) \text{ by } c(A) = \{x \in L \mid x \leq supA\}$$

for any subset A of L. Then, clearly $A \subseteq c(A)$ and

$$A \subseteq B \implies supA \leq supB \implies c(A) \subseteq c(B)$$

for any A and B in $\mathscr{P}(L)$. Also, for any $A \subseteq L$,

$$supA \in c(A) \text{ and } x \leq supA \text{ for all } x \in c(A)$$

and hence $supA = sup\ c(A)$, so that $c(c(A)) = c(A)$. Thus c is a closure operator on the set L. Now consider the Moore class $\mathscr{M}_c$ defined by

$$\mathscr{M}_c = \{A \subseteq X \mid c(A) = A\}.$$

For any $a \in L$, let $c(a)$ denote $c(\{a\})$. Define

$$f : L \longrightarrow \mathscr{M}_c \text{ by } f(a) = c(a) \text{ for any} a \in L.$$

Clearly, $a \leq b \iff f(a) \subseteq f(b)$ for all $a, b \in L$. This implies that f is an injection also. Further, for any $A \in \mathscr{M}_c$, we have $A = c(A) = f(supA)$. Therefore f is a surjection too. Thus $f : L \longrightarrow \mathscr{M}_c$ is an isomorphism. □

4. Morphisms

In this section, we describe morphisms among partially ordered sets, lattices, Moore classes, closure operators and complete lattices and discuss certain interrelationships between these. First we recall the following.

Definition 4.1. Let $(X_1, \leq)$ and $(X_2, \leq)$ be partially ordered sets. A mapping $f : X_1 \longrightarrow X_2$ is called an order-preserving mapping or an order homomorphism, if

$$a \leq b \text{ in } X_1 \implies f(a) \leq f(b) \text{ in } X_2$$

for any a and $b \in X_1$. A bijection $f : X_1 \longrightarrow X_2$ is called an order isomorphism if

$$a \leq b \text{ in } X_1 \iff f(a) \leq f(b) \text{ in } X_2$$

for any a and b in X_1.

Definition 4.2. Let $(X_1, \wedge, \vee)$ and $(X_2, \wedge, \vee)$ be lattices and $f : X_1 \longrightarrow X_2$ be a mapping. Then

(1). f is called a meet homomorphism if $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in X_1$.

(2). f is called a join homomorphism if $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in X_1$.

(3). f is called a lattice homomorphism if f is both a meet and join homomorphism.

(4). f is called a meet (or join or lattice) isomorphism if f is a bijection and meet (respectively join or lattice) homomorphism.

Remark 4.3. For any $f : X_1 \to X_2$, it is clear that (3) implies (1) or (2) and that either of (1) or (2) implies that f is an order homomorphism. One can construct examples of lattices to establish that the converse of each of these implications is not true.

Definition 4.4. A pair $(X, \mathscr{M})$ is called a Moore space if X is a non-empty set and $\mathscr{M}$ is a Moore class on X.

Definition 4.5. A pair (X, c) is called a closure space if X is a non-empty set and c is a closure operator on X.

Definition 4.6. Let $(X_1, \mathscr{M}_1)$ and $(X_2, \mathscr{M}_2)$ be Moore spaces. A mapping $f : X_1 \longrightarrow X_2$ is called a homomorphism of Moore spaces if $f^{-1}(A) \in \mathscr{M}_1$ for all $A \in \mathscr{M}_2$.

Definition 4.7. Let (X_1, c_1) and (X_2, c_2) be closure spaces and $f : X_1 \longrightarrow X_2$ a mapping. Then f is said to be a homomorphism of closure spaces if

$$f(c_1(A)) \subseteq c_2(f(A)) \text{ for all } A \subseteq X_1.$$

The following result describes the inter relationships between homomorphisms of Moore spaces and those of closure spaces. First, recall from 2.10 that a Moore space $(X, \mathscr{M})$ induces a closure operator $c_{\mathscr{M}}$ on X where $c_{\mathscr{M}}$ is defined by

$$c_{\mathscr{M}}(A) = \cap \{M \in \mathscr{M} \mid A \subseteq M\}, \text{for any } A \subseteq X.$$

Theorem 4.8. *Let $(X_1, \mathscr{M}_1)$ and $(X_2, \mathscr{M}_2)$ be Moore spaces and $(X_1, c_{\mathscr{M}_1})$ and $(X_2, c_{\mathscr{M}_2})$ be the corresponding closure spaces. Then any mapping $f : X_1 \longrightarrow X_2$ is a homomorphism of Moore spaces if and only if it is a homomorphism of closure spaces.*

Proof. For simplicity, Let us write c_1 and c_2 for $c_{\mathscr{M}_1}$ and $c_{\mathscr{M}_2}$ respectively. Let $f : X_1 \longrightarrow X_2$ be a mapping. First suppose that f is a homomorphism of Moore spaces; that is,

$$f^{-1}\ (M)\ \in \mathscr{M}_1 \quad \text{for all} \quad M \in\ \mathscr{M}_2.$$

Now, for any subset A of X_1 and $M \in \mathscr{M}_2$, we have

$$\begin{aligned} f(A) \subseteq M &\Longrightarrow A \subseteq f^{-1}(f(A))\ \subseteq\ f^{-1}(M)\ \in \mathscr{M}_1 \\ &\Longrightarrow c_1(A)\ \subseteq\ f^{-1}(M) \\ &\Longrightarrow f(c_1(A))\ \subseteq\ f(f^{-1}(M))\ \subseteq\ M. \end{aligned}$$

From this it follows that$f(c_1(A))\ \subseteq\ \cap\ \{M \in \mathscr{M}_2\ \mid\ f(A) \subseteq M\ \}\ =\ c_2(f(A))$. Thus $f : X_1 \longrightarrow X_2$ is a homomorphism of closure spaces.

Conversely, suppose that f is a homomorphism of closure spaces; that is,

$$f(c_1(A)) \subseteq c_2(f(A)) \ \text{ for all } \ A \subseteq\ X.$$

For any $M \in \mathscr{M}_2$, we have

$$\begin{aligned} f(c_1(f^{-1}(M)))\ &\subseteq\ c_2\ (f(f^{-1}(M))) \\ &\subseteq c_2(M), \ \text{ Since } f(f^{-1}(M))\ \subseteq M \\ &= M, \ \text{ since } M \in \mathscr{M}_2. \end{aligned}$$

So that $c_1(f^{-1}(M))\ \subseteq f^{-1}(M)$ and therefore $f^{-1}(M) = c_1(f^{-1}(M)) \in \mathscr{M}_1$. Therefore f is a homomorphism of Moore spaces. □

Definition 4.9. Let $(L_1, \leq)$ and $(L_2, \leq)$ be complete lattices. A function $f : L_1 \longrightarrow L_2$ is called complete join homomorphism if

$$f(\sup A)\ =\ \sup f(A) \text{ for all } A \subseteq L_1;$$

f is called a complete meet homomorphism if

$$f(\inf A)\ =\ \inf f(A), \text{ for all } A\ \subseteq\ L_1.$$

Clearly every complete join or meet homomorphism is an order homomorphism. Also, any complete join (meet) homomorphism is a join (meet respectively) homomorphism. The converse of these are false. In fact a lattice homomorphism need not be a complete meet (or join) homomorphism. For, consider the following .

Example 4.10. Let $\mathbb{R}$ be the topological space of real numbers with respect to the usual topology. Let $\mathscr{O}(\mathbb{R})$ be the set of all open subsets of $\mathbb{R}$. Then $\mathscr{O}(\mathbb{R})$ together with the inclusion ordering is a complete lattice in which, for any $\{A_\alpha\}_{\alpha\in\Delta} \subseteq \mathscr{O}(\mathbb{R})$,

$$sup\{A_\alpha\}_{\alpha\in\Delta} = \bigcup_{\alpha\in\Delta} A_\alpha \text{ and } \inf\{A_\alpha\}_{\alpha\in\Delta} = \text{ The interior of } \bigcap_{\alpha\in\Delta} A_\alpha.$$

Let $\mathscr{P}(\mathbb{R})$ be the complete lattice of all subsets of $\mathbb{R}$ in which the supremums and infimums are simply the set unions and set intersections. Let $i : \mathscr{O}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ be the inclusion map (that is, $i(A) = A$ for all $A \in \mathscr{O}(\mathbb{R})$). Then clearly i is a lattice homomorphism, since $\mathscr{O}(\mathbb{R})$ is closed under arbitrary unions and finite intersections. Also, for any $\{A_\alpha\}_{\alpha\in\Delta} \subseteq \mathscr{O}(X)$,

$$i\,(\sup_{\mathscr{O}(\mathbb{R})}\{A_\alpha\}_{\alpha\in\Delta}) = i\,(\bigcup_{\alpha\in\Delta} A_\alpha) = \bigcup_{\alpha\in\Delta} i\,(A_\alpha) = \sup_{\mathscr{P}(\mathbb{R})}\{i(A_\alpha)\}_{\alpha\in\Delta}$$

and therefore i is a complete join homomorphism. However, i is not a complete meet homomorphism, since

$$\begin{aligned} i\,(\inf_{\mathscr{O}(\mathbb{R})}\{(-1, \tfrac{1}{n})\}_{n\in\mathbb{Z}^+}) &= \text{ The interior of } \bigcap_{n\in\mathbb{Z}^+}(-1, \tfrac{1}{n}) \\ &= \text{ The interior of } (-1,0] = (-1,0) \end{aligned}$$

$$\text{and } \inf_{\mathscr{P}(\mathbb{R})}\{i(-1, \tfrac{1}{n})\}_{n\in\mathbb{Z}^+} = \bigcap_{n\in\mathbb{Z}^+}(-1, \tfrac{1}{n}) = (-1,0]$$

and hence i is not a complete meet homomorphism.

By considering the dual lattices of the above, we can ascertain that there are complete meet homomorphisms which are not complete join homomorphisms.

Note 4.11. Let L_1 and L_2 be complete lattices and $f : L_1 \longrightarrow L_2$ be an order homomorphism. Then, for any subset A of L_1, we have

$$f(supA) \geq sup(f(A)) \text{ and } f(\inf A) \leq \inf(f(A)).$$

From this it follows that f is a complete join homomorphism if and only if

$$f(supA) \leq sup(f(A)) \text{ for all } A \subseteq L_1$$

and that f is a complete meet homomorphism if and only if

$$f(\inf A) \geq \inf(f(A)) \text{ for all } A \subseteq L_1.$$

Let us recall, form Sections 2 and 3, that each of Moore classes, complete lattices and closure operators are in one-to-one correspondence with each other. If $(L, \leq)$ is a complete lattice, then we have the closure space (L, c), where c is the closure operator on L defined by $c(A) = \{x \in L \mid x \leq supL\}$ for any $A \subseteq L$. Also, we have the Moore space $(L, \mathscr{M})$ where $\mathscr{M}$ is given by

$$\mathscr{M} = \{A \subseteq L \mid c(A) = A\} = \{A \subseteq A \mid x \leq supA \Longleftrightarrow x \in A \text{ for any } x \in L\}.$$

Now, we prove the following.

Theorem 4.12. *Let $(L_1, \leq)$ and $(L_2, \leq)$ be complete lattices, (L_1, c_1) and (L_2, c_2) be the corresponding closure spaces and $(L_1, \mathscr{M}_1)$ and $(L_2, \mathscr{M}_2)$ be the corresponding Moore spaces respectively. Then the following are equivalent to each other for any mapping $f : L_1 \longrightarrow L_2$.*

(1). f is a complete join homomorphism of $(L_1, \leq)$ into $(L_2, \leq)$.

(2). f is a homomorphism of the closure space (L_1, c_1) into the closure space (L_2, c_2).

(3). f is a homomorphism of the Moore space $(L_1, \mathscr{M}_1)$ into the Moore space $(L_2, \mathscr{M}_2)$

Proof. (2) $\Longleftrightarrow$ (3) is proved in 4.8.
(1) $\Longrightarrow$ (2) : Suppose that f is a complete join homomorphism. Let A be any subset of L_1. Then

$$x \in c_1(A) \Longrightarrow x \leq supA \Longrightarrow f(x) \leq f(supA) = supf(A) \Longrightarrow f(x) \in c_2(f(A))$$

and hence $f(c_1(A)) \subseteq c_2(f(A))$. Thus f is a homomorphism of closure spaces.
(2) $\Longrightarrow$ (1) : Suppose that f is a homomorphism of closure spaces. Let $A \subseteq L_1$ and $x = supA$. Then $x \in c_1(A)$
and hence $f(x) \in f(c_1(A)) \subseteq c_2(f(A))$, which implies that $f(x) \leq sup(f(A))$. Therefore

$$f(supA) \leq sup(f(A)) \text{ for all } A \subseteq L_1.$$

Thus f is a complete join homomorphism. □

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