International Journal of Pure and Applied Mathematics
Volume 91 No. 2 2014, 219-229
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i2.8

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THE GEOMETRY OF VORTEX FILAMENTS FOR MHD IN MINKOWSKI 3-SPACE

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Abstract: In this article, we study geometrical constraints on the magnetic vortex filaments and Beltrami magnetic fields. Later, we give the relation between abnormalities and hydrodynamics for non-null curves. In last section, we derive geometrical constraints on MHD for null curves.

1. Introduction

The geometry of vortex filaments has interesting applications in plasma physics [1], [2]. Recently, Schief denoted the effects of curvature and torsion of lines on the plasma physical phenomena [3], [4]. Andrade investigated the effects of curvature and torsion on vortex filaments in magnetohydrodynamic dynamos (MHD) with the aid of the Gauss-Mainardi-Codazzi equations [5], [6]. Bjorgum studied Beltrami magnetic fields and flows [7].

The paper is organized as follows: In Section 1, we give some definitions. In Section 2, we research geometrical constraints on the magnetic vortex filaments and Beltrami magnetic fields for non-null curves. In Section 3, we study the relation between abnormalities and hydrodynamics for non-null curves. In last section, we derive geometrical constraints on MHD for null curves.

Received: October 28, 2013

Let $\gamma(s)$ a unit speed curve in Minkowski 3-space , s being the arclength parameter. Consider the Frenet frame $\{t=\gamma', n, w\}$ attached to the curve $\gamma = \gamma(s)$ such that is t is the unit tangent vector field, n is the principal normal vector field and w is the binormal vector field. The Frenet -Serret formulas is given by

$$t' = \varepsilon_2 \kappa n, \quad n' = -\varepsilon_1 \kappa t - \varepsilon_3 \tau w, \;\; w' = -\varepsilon_2 \tau n \tag{1}$$

where $\langle t,t\rangle = \varepsilon_1, \langle n,n\rangle = \varepsilon_2, \langle w,w\rangle = \varepsilon_3$. ∇ is the semi Riemannian connection on M and $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of γ, respectively The vector product $\times$ given by [8]

$$w \times t = \varepsilon_3 n, \qquad t \times n = \varepsilon_2 w, \;\; n \times w = \varepsilon_1 t \tag{2}$$

There are three types of vector fields in Minkowski 3-space: spacelike, time-like, and null. Let X be a three dimensional vector field in in Minkowski 3-space $\mathbb{R}^3_1$. If $\langle X, X\rangle = 0$ and $X \neq 0$, X is called null vector. A null curve γ :$[a,b] \to \mathbb{R}^3_1$ has a causal structure if all its tangent vectors are null. There exists a local frame $X = (t=\gamma', n, w)$, called Cartan equations satisfying $\langle t,t\rangle = 0$, $\langle n,n\rangle = 0$, $\langle t,w\rangle = 0, \langle w,w\rangle = 1$, $\langle t,n\rangle = 1$. The vector product is defined : $t \times w = -t$, $t \times n = -w$, $w \times n = -n$ [9], [10].

Intrinsic derivatives of the Null Frenet type orthonormal triad are given as

$$t' = \kappa w, \;\; n' = \tau w, \;\; w' = -\tau t - \kappa n \tag{3}$$

where κ and τ are the curvature and torsion functions of γ, respectively. The null localized induced equation(LIE) for null curves is given as [9], [10]

$$\frac{\delta t}{\delta w} = \frac{\delta^2 t}{\delta s^2}$$

2. Geometrical Constraints on MHD Dynamos for Non-Null Curves

$\frac{\delta}{\delta s}, \frac{\delta}{\delta n}$ and $\frac{\delta}{\delta w}$ show directional derivatives in the tangential, principal normal and binormal directions in $\mathbb{E}^3_1$. $\{t, n, w\}$ denote the directional derivatives of the orthonormal triad in the n-and w-directions in $\mathbb{R}^3_1$. The gradient operator is

$$grad = t\frac{\delta}{\delta s} + n\frac{\delta}{\delta n} + w\frac{\delta}{\delta w} \tag{4}$$

and $\psi_{ns} = \left\langle n, \frac{\delta t}{\delta n}\right\rangle$, $\psi_{ws} = \left\langle w, \frac{\delta t}{\delta w}\right\rangle$ are the quantities which first introduced by Bjorgum in $\mathbb{R}^3$ [6]. Using (1) and (2), we compute

$$divt = \left\langle t\frac{\delta}{\delta s} + n\frac{\delta}{\delta n} + w\frac{\delta}{\delta w}, t\right\rangle = \psi_{ns} + \psi_{ws},$$

$$\mathrm{div}n = \left\langle t\frac{\delta}{\delta s} + n\frac{\delta}{\delta n} + w\frac{\delta}{\delta w}, n \right\rangle = -\kappa + \left\langle w, \frac{\delta n}{\delta w} \right\rangle$$

$$\mathrm{div}w = \left\langle t\frac{\delta}{\delta s} + n\frac{\delta}{\delta n} + w\frac{\delta}{\delta w}, w \right\rangle = \left\langle n, \frac{\delta w}{\delta n} \right\rangle$$

$$curlt = \left\langle t \times \frac{\delta}{\delta s} + n \times \frac{\delta}{\delta n} + w \times \frac{\delta}{\delta w}, t \right\rangle = \kappa w + \varepsilon_3 \Omega_s t \tag{5}$$

where

$$\Omega_s = \langle curlt, t \rangle = \varepsilon_2 [\left\langle w, \frac{\delta t}{\delta n} \right\rangle - \left\langle n, \frac{\delta t}{\delta w} \right\rangle] \tag{6}$$

is called the non-null abnormality of t-field

$$curln = \left\langle t \times \frac{\delta}{\delta s} + n \times \frac{\delta}{\delta n} + w \times \frac{\delta}{\delta w}, n \right\rangle = -\varepsilon_1 \varepsilon_3 \mathrm{div}wt + \varepsilon_2 \Omega_n n + \varepsilon_1 \varepsilon_2 \psi_{ns} w$$

where

$$\Omega_n = \langle curln, n \rangle = \varepsilon_2 (\tau + \varepsilon_1 \varepsilon_3 \left\langle t, \frac{\delta n}{\delta w} \right\rangle) \tag{7}$$

is the non-null abnormality of the n-field.

$$\mathrm{curl}w = \left\langle t \times \frac{\delta}{\delta s} + n \times \frac{\delta}{\delta n} + w \times \frac{\delta}{\delta w}, w \right\rangle = \varepsilon_3 \Omega_w w - \varepsilon_1 \varepsilon_3 \psi_{ws} n + \varepsilon_1 \varepsilon_2 (\kappa + divn) t$$

where

$$\Omega_w = \langle \mathrm{curl}w, w \rangle = \varepsilon_3 (\tau - \varepsilon_1 \varepsilon_2 \left\langle t, \frac{\delta w}{\delta n} \right\rangle) \tag{8}$$

is the non-null abnormality of the w-field.

$$\Omega_s + \Omega_n - \varepsilon_1 \Omega_w = 2\varepsilon_2 \tau$$

Theorem. *For non-null curves, using the magnetic helicity equation, the components of the magnetic field of the vortex filaments under geodesic motions* $\Omega_s = 0$ *are*

$$\frac{B_w}{B_s} = -\frac{2\varepsilon_3 \tau}{\kappa}$$

Proof. We consider the magnetic field of vortex filament as $B = B_s t + B_w w$. Expansion of LHS of the magnetic helicity is given for non-null curves as following:

$$B_s \nabla \times t + \nabla B_s \times t + B_w \nabla \times w + \nabla B_w \times w = \lambda (B_s t + B_w w)$$

$$\nabla \times B = [\varepsilon_1\varepsilon_2(\kappa + \mathrm{div}n)B_w + \varepsilon_3\Omega_s B_s]t + [\varepsilon_3\frac{\delta B_s}{\delta w} - \varepsilon_1\varepsilon_3\psi_{ws}B_w - \varepsilon_3\frac{\delta B_w}{\delta s}]n$$
$$+[\varepsilon_3\Omega_w B_w + \kappa B_s]w = \lambda(B_s t + B_w w)$$

$$\varepsilon_1\varepsilon_2(\kappa + \mathrm{div}n)B_w + \varepsilon_3\Omega_s B_s = \lambda B_s \tag{9}$$

$$\varepsilon_3\frac{\delta B_w}{\delta s} - \varepsilon_3\frac{\delta B_s}{\delta w} = -\varepsilon_1\varepsilon_3\psi_{ws}B_w \tag{10}$$

$$\varepsilon_3\Omega_w B_w + \kappa B_s = \lambda B_w \tag{11}$$

From (9),(11)

$$\frac{B_s}{B_w} = \frac{\varepsilon_3\Omega_w - \lambda}{\kappa} \tag{12}$$

$$\frac{B_w}{B_s} = \frac{\Omega_s - \varepsilon_3\lambda}{\kappa + \mathrm{div}n} \tag{13}$$

Magnetic vortex lines have relations $\frac{\mathrm{div}n}{\kappa} \leq 1$. Thus (13) yields

$$\frac{B_w}{B_s} = \frac{\Omega_s - \varepsilon_3\lambda}{\kappa} \tag{14}$$

For $\Omega_n = 0$,

$$\Omega_s - \varepsilon_1\Omega_w = 2\varepsilon_2\tau \tag{15}$$

From (12),(14) and hypothesis theorem $\Omega_s = 0$, it is obtained

$$\frac{-2\varepsilon_1\varepsilon_2\varepsilon_3\tau - \lambda}{\kappa} = -\varepsilon_3\frac{\kappa}{\lambda} \tag{16}$$

With aid (16) ,

$$\lambda^2 - 2\tau\lambda + \varepsilon_3\kappa^2 = 0 \tag{17}$$

(15) gives

$$\lambda = \tau \pm \tau\sqrt{1 - \frac{\varepsilon_3\kappa^2}{\tau^2}}$$

(14) yields

$$\frac{B_w}{B_s} = -\frac{2\varepsilon_3\tau}{\kappa} \tag{18}$$

From (18), the ratio of the components of the magnetic field is constant. Using (10), we express

$$\frac{\delta B_w}{\delta s} = \frac{\delta B_s}{\delta w} \tag{19}$$

Maxwell equation in $\mathbb{E}_1^3$ is given

$$\langle \nabla, B \rangle = \varepsilon_1 \frac{\delta B_s}{\delta s} + \varepsilon_3 \frac{\delta B_w}{\delta w} = 0 \tag{20}$$

(19) and (20) give

$$B_w = \varepsilon_3 \frac{\delta \varnothing}{\delta s}, \qquad B_s = \varepsilon_1 \frac{\delta \varnothing}{\delta w}$$

and

$$\varepsilon_1 \frac{\delta^2 \varnothing}{\delta s^2} + \varepsilon_3 \frac{\delta^2 \varnothing}{\delta w^2} = 0 \tag{21}$$

(21) is a Minkowski Laplacian-like equation

2.1. Beltrami Flows and Fields for Non-Null Curves in Minkowski 3-Space

Beltrami magnetic fields and flows have an important role in magnetodynamic [11], plasma physics, and other contexts [12]. For instance, Etnyre and Ghrist denoted geometric properties of Beltrami fields [13]. They also stated connections between the field of contact topology and the study of Beltrami fields in hydrodynamics on Riemannian manifolds in three dimension [13,14]. Dombre and Urish mentioned the role of ABC fields on the Euclidean 3-torus with aid Beltrami fields [15]. For this reason, it is useful to study Beltrami magnetic dynamos as an application of the Gauss-Weingarten equations for non-null curves.

Beltrami magnetic flow are defined

$$\nabla \times v = mv \tag{22}$$

where $v{=}v_t t$ is non-null flow velocity in $\mathbb{R}_1^3$. Together (5) and (2) give

$$v_t(\kappa w + \varepsilon_3 \Omega_s t) - \varepsilon_2 w \frac{\delta v_t}{\delta n} + \varepsilon_3 n \frac{\delta v_t}{\delta w} = m v_t t \tag{23}$$

(23) yield

$$\Omega_s = \varepsilon_3 m$$

$$\frac{\delta v_t}{\delta n} = \varepsilon_2 \kappa v_t$$

$$\frac{\delta v_t}{\delta w} = 0$$

As similar, non-null Beltrami dynamos are given

$$\nabla \times B = mB \tag{24}$$

where $B = B(s)t$ Beltrami field.

The expansion of the left-hand side (LHS) of (24) yields

$$B(\kappa w + \varepsilon_3 \Omega_s t) - \varepsilon_2 w \frac{\delta B}{\delta n} + \varepsilon_3 n \frac{\delta B}{\delta w} = mBt \tag{25}$$

(25) yield three equations

$$\kappa B - \varepsilon_2 \frac{\delta B}{\delta n} = 0 \tag{26}$$

$$\Omega_s = \varepsilon_3 m = \text{constant} \tag{27}$$

(26) shows, hydrodynamical motion is not geodesic.

$$\frac{\delta B}{\delta w} = 0$$

From (26),

$$B(w) = e^{\varepsilon_2 \int (\kappa)(n) dn} \tag{28}$$

Equation (28) denotes that the non-null Beltrami field B depends on just n-direction.

3. The Relation between Abnormalities and this Hydrodynamics for Non-Null Curves

We can express the vortex filament rotation of non-null curves as following:

$$\nabla \times v = \Omega \tag{29}$$

where ∇ is the non-null gradient operator, Ω is defined as rotation or curl the of the velocity field and $v = \kappa w$ is the curvature binormal vector..

Our aim is to study the relation between abnormalities and this hydrodynamics for non-null curves. From the Eqs.(29), we have

$$\langle v, \nabla \times v \rangle = \langle v, \Omega \rangle \,. \tag{30}$$

$$\begin{aligned} \langle v, \nabla \times v \rangle &= \langle \kappa w, \nabla \times (\kappa w) \rangle \qquad (31) \\ &= \langle \kappa w, \kappa(\varepsilon_3 \Omega_w w - \varepsilon_1 \varepsilon_3 \psi_{ws} n + \varepsilon_1 \varepsilon_2 (\kappa + divn) t \rangle = \kappa^2 \Omega_w \end{aligned}$$

Together (30) and (31) give,

$$\langle w, \Omega \rangle = \kappa^2 \Omega_w \tag{32}$$

The non-null vorticity is computed as

$$\begin{aligned} \Omega &= t\times(\frac{\delta\kappa}{\partial s}w+\kappa\frac{\delta w}{\partial s})+\frac{w}{\kappa}\times(\frac{\delta\kappa}{\delta w}w+\kappa\frac{\delta w}{\delta w}) \\ &= -\varepsilon_1\varepsilon_2(\kappa+\mathrm{div}n)t-\varepsilon_3(\varepsilon_1\psi_{ws}+\kappa_s)n-\varepsilon_1\varepsilon_2\kappa\tau w \end{aligned} \tag{33}$$

(33) reduces to

$$\langle w,\Omega\rangle=-\varepsilon_1\varepsilon_2\varepsilon_3\kappa\tau=\kappa\tau \tag{34}$$

A comparison of Eqs.(32) and (34) shows that

$$\Omega_w=\tau \tag{35}$$

4. Geometrical Constraints on the Magnetic Vortex Filaments for Null Curves

We take the null gradient vector $\nabla=\mathrm{grad}=t\frac{\delta}{\delta s}+n\frac{\delta}{\delta n}+w\frac{\delta}{\delta w}.\frac{\delta}{\delta s}$. We define the quantities ψ_{ns} and ψ_{ws} which first introduced by Bjorgum in $\mathbb{R}^3$

$$\psi_{ns}=\left\langle n,\frac{\delta t}{\delta n}\right\rangle,\qquad \psi_{ws}=\left\langle w,\frac{\delta t}{\delta w}\right\rangle \tag{36}$$

We reproduce some formulas derived in this section, see [16].

$$\mathrm{div}t=\langle t,\kappa w\rangle+\left\langle n,\frac{\delta t}{\delta n}\right\rangle+\left\langle w,\frac{\delta t}{\delta w}\right\rangle=\psi_{ns}+\psi_{ws}, \tag{37}$$

$$\mathrm{div}n=\left\langle t\frac{\delta}{\delta s}+n\frac{\delta}{\delta n}+w\frac{\delta}{\delta w},n\right\rangle=\langle t,\tau w\rangle+\left\langle w,\frac{\delta n}{\delta w}\right\rangle=\left\langle w,\frac{\delta n}{\delta w}\right\rangle \tag{38}$$

$$\mathrm{div}w=\left\langle t\frac{\delta}{\delta s}+n\frac{\delta}{\delta n}+w\frac{\delta}{\delta w},w\right\rangle=-\kappa+\left\langle n,\frac{\delta w}{\delta n}\right\rangle \tag{39}$$

$$\mathrm{curl}t=-\kappa t+n\Phi_s$$

$$\mathrm{curl}n=\Phi_n t-(\kappa+\mathrm{div}b)n-\psi_{ns}w$$

$$\mathrm{curl}w=-\psi_{ws}t+n\mathrm{div}n+\Phi_w w$$

where respectively,

$$\Phi_s=\langle\mathrm{curl}t,t\rangle=\left\langle w,\frac{\delta t}{\delta n}\right\rangle-\left\langle n,\frac{\delta t}{\delta w}\right\rangle, \tag{40}$$

$$\Phi_n = \langle \text{curl} n, n \rangle = -\tau + \left\langle t, \frac{\delta n}{\delta w} \right\rangle, \tag{41}$$

$$\Phi_w = \langle \text{curl} w, w \rangle = \kappa + \left\langle t, \frac{\delta w}{\delta n} \right\rangle \tag{42}$$

are called the abnormalities of t- field, n- field and w- field in $\mathbb{R}^3_1$.

The Gauss-Weingarten equations are

$$\frac{\delta}{\delta w} \begin{bmatrix} t \\ n \\ w \end{bmatrix} = \begin{bmatrix} -(\Phi_n + \tau) & 0 & \psi_{ws} \\ 0 & \Phi_n + \tau & \text{div} n \\ -\text{div} n & -\psi_{ws} & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ w \end{bmatrix}, \tag{43}$$

Using $\Phi_n = 0$, the Gauss-Weingarten equations Eq.(43) turns into

$$\begin{aligned} \tfrac{\delta \kappa}{\delta w} &= \psi_{ws}^2 + \kappa^2 + \tfrac{\delta \psi_{\mathbf{ws}}}{\delta s} - \kappa\tau \\ \tfrac{\delta \tau}{\delta w} &= \tau^2 + \tfrac{\delta(\text{div} n)}{\delta s} + \kappa\tau + \text{div} n \psi_{ws} \\ \tfrac{\delta \tau}{\delta s} &= \kappa \text{div} n - 2\tau\psi_{ws} \ . \end{aligned} \tag{44}$$

Theorem. *Using the magnetic helicity equation , the components of the magnetic field of the vortex filaments are obtained as*

$$\frac{B_w}{B_s} = -\frac{\tau}{\kappa}$$

constant for null curves.

Proof. The left expansion the null magnetic helicity is

$$B_s \nabla \times t + \nabla B_s \times t + B_w \nabla \times w + \nabla B_w \times w = \lambda(B_s t + B_w w)$$

$$\begin{aligned} \nabla \times B &= B_s(-\kappa t + n\Phi_s) + (w \times t)\frac{\delta B_s}{\delta w} + B_w(\Phi_w w - t\psi_{ws} + n \text{div} n) - t\frac{\delta B_w}{\delta s} \\ &= (-\kappa B_s + \frac{\delta B_s}{\delta w} - \psi_{ws} B_w - \frac{\delta B_w}{\delta s})t + (B_s \Phi_s + \text{div} n B_w)n + B_w \Phi_w w \\ &= \lambda(B_s t + B_w w) \end{aligned}$$

We obtain PDE equations as following:

$$\begin{aligned} B_s \Phi_s + \text{div} n B_w &= 0 \Rightarrow \frac{B_w}{B_s} = -\frac{\Phi_s}{\text{div} n} \\ B_w \Phi_w &= \lambda B_w \Rightarrow \lambda = \Phi_w \end{aligned}$$

$$\frac{\delta B_w}{\delta s} - \frac{\delta B_s}{\delta w} = \psi_{ws} B_w - (\kappa + \lambda) B_s \tag{45}$$

For $\Phi_n = 0$,

$$\Phi_s + \Phi_w = \kappa + \tau$$

$$\Phi_s = (\kappa + \tau) - \lambda$$

$$\frac{B_w}{B_s} = -\frac{(\kappa + \tau) - \lambda}{\text{div} n} \tag{46}$$

$$\begin{aligned} \Phi &= t \times (\frac{\delta \kappa}{\partial s} w + \kappa \frac{\delta w}{\partial s}) + \frac{w}{\kappa} \times (\frac{\delta \kappa}{\delta w} w + \kappa \frac{\delta w}{\delta w}) \\ &= -\frac{\delta \kappa}{\delta s} t + \kappa^2 w + \psi_{ws} n - \text{div} n t \end{aligned} \tag{47}$$

From Eq.(47), we have

$$\Phi_w = \kappa = \lambda$$

And

$$\Phi_s = \tau$$

$$\frac{B_w}{B_s} = -\frac{\tau}{\text{div} n} \tag{48}$$

Magnetic vortex lines have $\frac{\text{div} n}{\kappa} \leq 1$.Therefore it is obtained

$$\frac{B_w}{B_s} = -\frac{\tau}{\kappa}$$

Using (44) and $\psi_{ws} = 0$ is expressed

$$\frac{\delta B_w}{\delta s} - \frac{\delta B_s}{\delta w} = -2\kappa B_s. \tag{49}$$

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