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SUFFICIENT CONDITIONS FOR MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract: The object of the present paper is to consider some sufficient conditions for meromorphic close-to-convex functions in the punctured unit disc.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \cdots \tag{1}$$

which are analytic in the punctured unit disk $\mathfrak{D} = \{z \in C : 0 < |z| < 1\}$.

A function $f \in \Sigma$ is said to be meromorphic starlike of order α if it satisfies the following inequality

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$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathfrak{D}), \tag{2}$$

for some $\alpha(0 \le \alpha < 1)$ .We denote by $\Sigma^*(\alpha)$ the class of all meromorphic starlike functions of order α.

Similarly,a function $f \in \Sigma$ is said to be meromorphic convex of order α if it satisfies the following inequality

$$-Re\left\{(1 + \frac{zf''(z)}{f'(z)})\right\} > \alpha \quad (z \in \mathfrak{D}), \tag{3}$$

for some $\alpha(0 \le \alpha < 1)$ .We denote by $\mathcal{MK}(\alpha)$ the class of all meromorphic convex functions of order α.

Let $\mathcal{MC}(\alpha)$ be the subclass of Σ consisting of functions which satisfy the following inequality

$$-Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha, z \in \mathfrak{D} \tag{4}$$

for some $\alpha(0 \le \alpha < 1)$and for some meromorphic starlike function $g(z) \in \Sigma$. We denote by $\mathcal{MC}(\alpha)$ the class of all meromorphic close-to-convex functions of order α. Also we write $\mathcal{MC}(0)$ simply by $\mathcal{MC}$.

Especially,taking $g(z) = 1/z$,we can get

$$-Re\{z^2 f'(z)\} > \alpha \quad (z \in \mathfrak{D}). \tag{5}$$

This class was introduced by Ganigi and Uralegaddi in [1]. What's more, Cho and Owa investigated this class in [2]. Uralegaddi and Desai also discussed some new criteria for meromorphic close-to-convex functions in [3].

In this paper,we deal with the question of another condition different from [1] , [2] and [3].This is the extension of their works.

2. The Main Results

In order to establish our main results,we require the following lemma:

Lemma 2.1. (see [4]) *Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in the unit disc* $U = \{z \in C : |z| < 1\}$ *and suppose that there exists a point* $z_0 \in U$ *such that*

$$Re\{p(z)\} > 0 \ for |z| < |z_0| \tag{6}$$

and

$$Re\{p(z_0)\} = 0. \tag{7}$$

Then we have

$$z_0 p'(z_0) \leq -\frac{1}{2}(1 + |p(z_0)|^2). \tag{8}$$

Theorem 2.1. *Let $f(z) \in \Sigma$, and suppose that there exists a meromorphic starlike function $g(z)$ such that*

$$Re\left\{\frac{zf'(z)}{g(z)}\Big(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\Big)\right\} > \frac{1}{2}(1 + |\frac{zf'(z)}{g(z)}|^2), z \in \mathfrak{D} \tag{9}$$

then $f(z) \in \mathcal{MC}$.

Proof. Let

$$p(z) = -\frac{zf'(z)}{g(z)}, \tag{10}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (6) and (7) of Lemma 2.1.

Now using (10),it follows that

$$\frac{z_0 f'(z_0)}{g(z_0)}\Big(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\Big) = z_0 p'(z_0). \tag{11}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.1, therefore in view of (8), we obtain

$$Re\left\{\frac{z_0 f'(z_0)}{g(z_0)}\Big(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\Big)\right\} \leq -\frac{1}{2}(1 + |p(z_0)|^2)$$
$$= -\frac{1}{2}\Big(1 + |\frac{z_0 f'(z_0)}{g(z_0)}|^2\Big). \tag{12}$$

This is a contradiction with (9) and therefore the proof of the Theorem 2.1 is completed.

Theorem 2.2. *Let $f(z) \in \Sigma$, and suppose that there exists a meromorphic starlike function $g(z)$ such that*

$$Re\left\{\frac{zf'(z)}{g(z)}\Big(-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)}\Big)\right\} > -\frac{1}{4}(1 + |\frac{zf'(z)}{g(z)}|^2), z \in \mathfrak{D} \tag{13}$$

then $f(z) \in \mathcal{MC}(1/2)$.

Proof. Let

$$p(z) = 2(-\frac{zf'(z)}{g(z)} - \frac{1}{2}), \tag{14}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (6) and (7) of Lemma 2.1.

Now using (15), it follows that

$$\frac{z_0f'(z_0)}{g(z_0)}\Big(-1-\frac{z_0f''(z_0)}{f'(z_0)}+\frac{z_0g'(z_0)}{g(z_0)}\Big)=\frac{1}{2}z_0p'(z_0). \tag{15}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions of Lemma 2.1, therefore in view of (8), we obtain

$$\begin{aligned} Re\Big\{\frac{z_0f'(z_0)}{g(z_0)}\Big(-1-\frac{z_0f''(z_0)}{f'(z_0)}+\frac{z_0g'(z_0)}{g(z_0)}\Big)\Big\} &\le -\frac{1}{4}(1+|p(z_0)|^2) \\ &= -\frac{1}{4}(1+|\frac{z_0f'(z_0)}{g(z_0)}|^2). \end{aligned} \tag{16}$$

This is a contradiction with (14) and therefore the proof of the Theorem 2.2 is completed.

Theorem 2.3. *Let $f(z) \in \Sigma, 0 \le \alpha < 1$ and suppose that there exists a starlike function $g(z)$ such that*

$$Re\Big\{\frac{zf'(z)}{g(z)}\Big(-1-\frac{zf''(z)}{f'(z)}+\frac{zg'(z)}{g(z)}\Big)\Big\} > -\frac{1}{2}(1-\alpha), z \in U \tag{17}$$

then $f(z) \in \mathcal{MC}(\alpha)$.

Proof. Let

$$-\frac{zf'(z)}{g(z)}=(1-\alpha)p(z)+\alpha, \tag{18}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (6) and (7) of Lemma 2.1.

Making use of (20), it follows that

$$-\frac{z_0f'(z_0)}{g(z_0)}\Big(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)}\Big)=(1-\alpha)z_0p'(z_0). \tag{19}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.1, therefore in view of (8), we obtain

$$\begin{aligned} Re\Big\{\frac{z_0f'(z_0)}{g(z_0)}\Big(-1-\frac{z_0f''(z_0)}{f'(z_0)}+\frac{z_0g'(z_0)}{g(z_0)}\Big)\Big\} &\le -\frac{1}{2}(1-\alpha)(1+|p(z_0)|^2) \\ &\le -\frac{1}{2}(1-\alpha). \end{aligned} \tag{20}$$

This is a contradiction with (19) and therefore proof of the Theorem 2.3 is completed.

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