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ON THE RANKS OF POINTS OF TANGENT DEVELOPABLES (SCROLLS AND SEGRE-VERONESE VARIETIES)

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Abstract: We compute the X-rank of points of the tangent developable of embeddings of the Hirzebruch surfaces.

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1. Introduction

For any integral variety $X \subset \mathbb{P}^r$ defined over an algebraically closed field $\mathbb{K}$ let $\tau(X) \subseteq \mathbb{P}^r$ denote the tangent developable of X, i.e. the closure in $\mathbb{P}^r$ of all Zariski tangent spaces $T_PX \subset \mathbb{P}^r$, $P \in X_{reg}$. For each $P \in \mathbb{P}^r$ the X-rank $r_X(P)$ of P is the minimal cardinality of a set $A \subset X$ such that $P \in \langle A\rangle$, where $\langle\ \rangle$ denote the linear span ([7]).

For all integers $a > 0$ and $b > 0$ let $\nu_{a,b} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^N$, $N = ab + a + b$, denote the Segre-Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, b)|$.

Theorem 1. *Fix positive integers a, b and assume that either char($\mathbb{K}$) = 0 or char($\mathbb{K}$) $\neq 2$ or char(K) = 2, $a \neq 2$, $b \neq 2$ and $(a, b) \neq (1, 1)$. Fix $P \in \tau(\nu_{a,b}(\mathbb{P}^1 \times \mathbb{P}^1)) \setminus \nu_{a,b}(\mathbb{P}^1 \times \mathbb{P}^1)$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the degree two zero-dimensional scheme such that $P \in \langle\nu_{a,b}(Z)\rangle$. Set $\rho := r_{\nu_{a,b}(\mathbb{P}^1\times\mathbb{P}^1)}(P)$.*

(i) If Z is contained in some $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 0)|$, then $\rho = b$.

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(ii) If Z is contained in some $E \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$, then $\rho = a$.

(iii) In all other cases we have $\rho = a + b$.

If $\mathrm{char}(\mathbb{K}) = 2$, then it is easy to check that in cases (i), (ii), (iii) the rank of P is 3 in case (i) if $b = 2$, in case (ii) if $a = 2$ and in case (iii) if $(a, b) = (1, 1)$.

Theorem 2. *Fix integers $e > 0$, $a > 0$ and $b \ge ae$. Assume either char($\mathbb{K}$) $= 0$ or char($\mathbb{K}$) $> b$. Let $X_{e,a,b} \subset \mathbb{P}^r$ be the image of F_e by the morphism ϕ induced by the complete linear system $|\mathcal{O}_{F_e}(ah + bf)|$. Fix $P \in \tau(X_{e,a,b}) \setminus X_{e,a,b}$; if $b = ea$ assume the existence of $v \subset F_e \setminus h$ such that $\deg(v) = 2$ and $P \in \langle\phi(v)\rangle$. There is a degree 2 connected scheme $v \subset F_e$ such that $P \in \langle\phi(v)\rangle$. Set $O := v_{red}$. Let F be the fiber of the ruling of F_e containing O. Set $\rho := r_{X_{e,a,b}}(P)$.*

(i) If $v \subset F$, then $\rho = a$,.

(ii) If $v \nsubseteq F$ and $O \notin h$, then $\rho = b$.

(iii) If $v \nsubseteq F$, $O \in h$ and $v \nsubseteq h$, then $\rho = b - ea + a$.

(iv) Assume $v \subset h$. Then $\rho = b - ea$.

Is concision true for Segre-Veronese embeddings? More precisely we may ask it in a weak form (part (a) of Question 1) and in a strong form (part (b) of Question 1).

Question 1. Fix integers $n \ge 2$ and $d_i \ge 1$, $r_i \ge m_i \ge 0$, $1 \le i \le n$. Fix linear subspaces $M_i \subseteq \mathbb{P}^{r_i}$. Let $u : \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n} \to \mathbb{P}^N$, $N := -1 + \prod_{i=1}^{n} \binom{m_i+d_i}{m_i}$, be the Segre-Veronese embedding of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}\mathbb{P}^{r_n}$ of multidegree $(d_1, \dots, d_n)$. Set $X := u(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}\mathbb{P}^{r_n})$ and $Y := u(M_1 \times \cdots \times M_n)$. Fix $P \in \langle Y\rangle$?

(a) Is $r_X(P) = r_Y(P)$?

(b) If (a) is true for P is every set $S \subset X$ evincing the X-rank of P contained in Y ?

If part (a) of Question 1 is true, then we may use Theorem 1 for Segre-Veronese embeddings.

2. The Proofs

For any smooth surface X, every effective divisor $D \subset X$ and every zero-dimensional scheme $Z \subset X$ let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_X : \mathcal{I}_D$ as its ideal sheaf.

Lemma 1. *Fix integers $a \geq 0$ and $b \geq 0$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme such that* $\deg(Z) \leq a + b + 1$. *We have $h^1(\mathcal{I}_Z(a,b)) > 0$ if and only if either there is $D \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)|$ such that* $\deg(Z \cap D) \geq b + 2$ *or there is $E \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)|$ such that* $\deg(E \cap Z) \geq a + 2$.

Proof. Since the " if " part is obvious by the cohomology of line bundles on D and E, it is sufficient to prove the " only if " part. We use induction on the integer $a + b$, the case $(a, b) = (0, 0)$ being obvious, since $h^1(\mathcal{I}_P) = 0$ for all $P \in \mathbb{P}^1 \times \mathbb{P}^1$. Similarly, we conclude if either $a = 0$ or $b = 0$. Hence we may assume $a > 0$ and $b > 0$. Let α be the maximum of all integers $\deg(Z \cap D)$ for all $D \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)|$ and β the maximum of all integers $\deg(E \cap Z)$ for all $E \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)|$. Assume $\alpha \leq b+1$ and $\beta \leq a+1$. Take $D \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)|$ such that $\deg(Z \cap D) = \alpha$. We have $\deg(\mathrm{Res}_D(Z)) = \deg(Z \cap D) \leq a + b - 1$. Since $\deg(Z \cap D) \leq b + 1$, we have $h^1(D, \mathcal{I}_{Z \cap D}(a, b)) = 0$. Hence the Castelnuovo's sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(a-1, b)) > 0$. By the inductive assumption either there is $T \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)|$ such that $\deg(\mathrm{Res}_D(Z) \cap T) \geq b+2$ or there is $E \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)|$ such that $\deg(E \cap \mathrm{Res}_D(Z)) \geq a + 1$. The first case is impossible, because $Z \supseteq \mathrm{Res}_D(Z)$ and so $\deg(\mathrm{Res}_D(Z) \cap T) \leq \deg(Z \cap T) \leq \alpha$. Therefore the latter case occurs, i.e. $\beta = a+1$ and $D \cap Z \cap E = \emptyset$. Exchanging the role of the rulings we conclude by induction on $a + b$, unless $\alpha = b + 1$. The scheme $Z \cap (D \cup E)$ has degree $a + b + 2$ and hence $\deg(Z) \geq a + b + 2$, a contradiction. □

Lemma 2. *Fix a rational normal curve $C \subset \mathbb{P}^r$ and $P \in \tau(C) \setminus C$. If $r = 2$ and char$(\mathbb{K}) = 2$, then either $r_C(P) = 3$ or $r_C(P) = 3$ and the latter occurs if and only if P is the strange point of C. In all other cases we have $r_C(P) = r$.*

Proof. Since C is a smooth variety and $P \in \tau(C) \setminus C$, there is a degree two zero-dimensional connected scheme $Z \subset C$ such that $P \in \langle Z \rangle$. Since $P \notin C$, we have $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Take $A \subset C$ evincing the C-rank of P. Since Z is not reduced, we have $Z \neq A$. Hence $h^1(\mathcal{I}_{A \cup Z}(1)) > 0$ ([5], Lemma 1). Since $C \cong \mathbb{P}^1$, the cohomology of line bundles on $\mathbb{P}^1$ gives $\deg(Z \cup A) \geq r + 2$. Therefore $r_C(P) = \sharp(A) \geq r$. For degree reasons P may be a strange point of C only if char$(K) = 2 = r$. If P is not a strange point of C, then $r_C(P) \leq r$ ([1]). If char$(\mathbb{K}) = 2 = r$ and P is the strange point of C, then $r_C(P) = 3$ ([1]) □

Proof of Theorem 1. The scheme Z exists, because $\nu_{a,b}(\mathbb{P}^1 \times \mathbb{P}^1)$ is a smooth variety. In all cases the proof below easily check the uniqueness of Z; anyway, the proof works for any Z without using its uniqueness. Set $\{O\} := Z_{red}$. In the set-up of part (i) Z is contained in a the linear span of the degree b

rational normal curve $\nu_{a,b}(D)$. Since $r_{\nu_{a,b}(D)}(P)) \le b$ (Lemma 2) we get the inequality $\rho \le b$ in the set-up of part (i). In the same way the curve $r_{a,b}(E)$ gives the inequality $\rho \le a$ in the set-up of (ii). Take the set-up of (iii). Since $\dim(|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|) = 3 \ge 2 = \deg(Z)$, there is $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ containing Z. Since Z is neither as in case (i) nor in case (ii), C is irreducible and hence $P \in \langle \nu_{a,b}(C)\rangle$ with $\nu_{a,b}(C)$ rational normal curve of degree $a+b$. Hence $\rho \le a+b$ by Lemma 2.

Fix A evincing $r_X(P)$ and assume $\sharp(A) \le a+b-1$. We have $h^1(\mathcal{I}_Z(a,b)) > 0$ ([5, Lemma 1]). Set $\{O\} := Z_{red}$. Set $W := Z \cup A$. We have $h^1(\mathcal{I}_A(a,b)) = 0$ and $h^1(\mathcal{I}_W(a,b)) > 0$. Since $\deg(W) \le a+b+1$, either there is $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ such that $\deg(D\cap W) \ge b+2$ or there is some $E \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ such that $\deg(E\cap W) \ge a+2$ (Lemma 1). Assume for instance the existence of D. We need to prove that we are in case (i). Assume that $Z \nsubseteq D$. Since A evinces ρ, $\nu_{a,b}(A)$ is linearly independent. Hence $\sharp(A\cap D) \le b+1$. Since $Z \nsubseteq D$, we get $O \in D$, $O \notin A$ and $\sharp(A\cap D) = d+1$. We have $\deg(\mathrm{Res}_D(W)) = \deg(W) - b - 2$. Assume for the moment $h^1(\mathcal{I}_{\mathrm{Res}_D(W)}(a-1,b)) = 0$. By [6, Lemma 5.1] we get $\mathrm{Res}_D(Z) = A \setminus A\cap D$. Since $\mathrm{Res}_D(Z) = \{O\} \nsubseteq A$, we get a contradiction. Now assume $h^1(\mathcal{I}_{\mathrm{Res}_D(W)}(a-1,b)) > 0$. Since $\deg(W) \le (a-1)+b$, either there is $D' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ such that $\deg(D' \cap \mathrm{Res}_D(W)) \ge b+2$ or there is some $E' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ such that $\deg(E' \cap \mathrm{Res}_D(W)) \ge a+1$ (Lemma 1). First assume the existence of D'. Since $\mathrm{Res}_D(W) = \{O\} \cup (A \setminus A\cap D)$, we have $D' \ne D$. Hence $\sharp(A\cap D') \ge b+2$. Hence $\nu_{a,b}(A)$ is linearly dependent, a contradiction. Now assume the existence of E'. Since $\deg(\{O\}\cap E') \le 1$ we get $\sharp(A\cap E') \ge a$. Since $\deg(D\cap E') = 1$, we get $\sharp(A) \ge b+2+a-1 = a+b+1$, a contradiction. □

Lemma 3. *Fix integers $e > 0$, $a > 0$ and $b \ge ae$. Fix a degree 2 connected scheme $v \subset F_e$ and set $O := v_{red}$. If $O \in h$, then assume $b > ae$. Let $A \subset F_e$ be any finite set such that $h^1(\mathcal{I}_{v\cup A}(ah+bf)) > h^1(\mathcal{I}_{\{O\}\cup A}(ah+bf))$ and with minimal cardinality, ρ, among all such finite sets. Let F be the fiber of the ruling of F_e containing O.*

(i) If $v \subset F$, then $\rho = a$, $O \notin A$ and $A \subset F$.

(ii) If $v \nsubseteq F$ and $O \notin h$, then $\rho = b$.

(iii) If $v \nsubseteq F$, $O \in h$ and $v \nsubseteq h$, then $\rho = b - ea + a$.

(iv) If $v \subset h$, then $\rho = b - ea$.

Proof. It is easy to check that the minimality condition for the set A implies $O \notin A$, $h^1(\mathcal{I}_{A\cup\{O\}}(ah+bf)) = 0$, and $h^1(\mathcal{I}_{v\cup A}(ah+bf)) = 1$. Let $\pi : F_e \to \mathbb{P}^1$ denote the ruling of F_e.

(a) In this step we prove that in each case (i), (ii), (iii) and (iv) the integer ρ is is at most the one claimed in the statement of the lemma. Take v as in case (i). For each $E \subset F \setminus \{O\}$ with $\sharp(E) = a$ we have $h^1(F, \mathcal{I}_{E\cup v}(ah+bf)) = 1$ and $h^1(F, \mathcal{I}_{\{O\}\cup E}(ah+bf)) = 0$. Since $b \ge ea$, we have $h^1(\mathcal{O}_{F_e}(ah+(b-1)f)) = 0$. Hence $h^1(\mathcal{I}_{E\cup v}(ah+bf)) = 1$ and $h^1(\mathcal{I}_{\{O\}\cup E}(ah+bf)) = 0$. Therefore $\rho \le a$. Take v as in case (ii). Since $O \notin h$, the linear system $|\mathcal{O}_{F_e}(h+ef)|$ induces an embedding at O and it injective outside h. Take a general $C \in |\mathcal{I}_v(h+ef)|$. Since $v \nsubseteq F$, a dimensional count gives the irreducibility of C. Hence $C \cong \mathbb{P}^1$. Since $\deg(\mathcal{O}_C(ah+bf)) = b$, we have $h^1(C, \mathcal{I}_{E\cup v}(ah+bf)) = 1$ and $h^1(C, \mathcal{I}_{v\cup E}(ah+bf)) = 0$ for all $E \subset C \setminus \{O\}$ with $\sharp(E) = b$. Since $h^1(\mathcal{O}_{F_e}((a-1)h+(b-e)f)) = 0$, we get $\rho \le b$. In case (iv) we use the curve $h \cong \mathbb{P}^1$. In case (iii) we use the curve $h\cup F$ with $E \subset h\cup F\setminus\{O\}$, $\sharp(E\cap h) = b-ea$ and $\sharp(E\cap F) = a$; indeed, we have $h^0(h\cup F, \mathcal{O}_{h\cup F}(ah+bf)) = b-ea+a$.

(b) Take the set-up of (ii) and take any $E \subset F_e$ with $\sharp(E) \le b-1$ and $h^1(\mathcal{I}_{\{O\}\cup E}(ah+bf)) = 0$. We need to prove that $h^1(\mathcal{I}_{v\cup E}(ah+bf)) = 0$. Assume $h^1(\mathcal{I}_{v\cup E}(ah+bf)) > 0$. Set $B_0 := \{O\}\cup E$. Take $D_1 \in |\mathcal{O}_{F_e}(f)|$ such that $a_1 := \sharp(B_0\cap D_1)$ is maximal and set $B_1 := B_0 \setminus D_1$. For all $i = 2, \dots, b$ define recursively D_i, a_i and B_i in the following way Take $D_i \in |\mathcal{O}_{F_e}(f)|$ such that $a_i := \sharp(B_{i-1}\cap D_i)$ is maximal with the only restriction that $D_i \ne D_j$ for $j < i$ if $B_{i-1} = \emptyset$. Set $B_i := B_{i-1} \setminus B_{i-1}\cap B_{i-1}\cap D_i$. The sequence $a_1, \dots, a_b$ is non-increasing. Since $\sharp(B) \le b$, we have $B_b = \emptyset$ and $a_1 + \cdots + a_b = \sharp(B)$. Set $T := ah + D_1 + \cdots + D_b$. We have $B \subset T$, but $v \nsubseteq T$, because F appears only once in the sequence $D_1, \dots, D_b$ and $O \notin h$. Hence $\mathrm{Res}_T(A\cup v) = \{O\}$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_T(v\cup A)}) = 0$. Since $h^1(\mathcal{I}_B(ah+bf)) = 0$, the Castelnuovo's sequence (e.g. [2], eq. (1)) gives a contradiction.

(c) Take the set-up of (iv). Hence $b > ae$. Take any $E \subset F_e$ such that $h^1(\mathcal{I}_{\{O\}\cup E}(ah+bf)) = 0$, $\sharp(E) \le b-ea-1$, $O \notin E$, $h^1(\mathcal{I}_{v\cup E}(ah+bf)) > 0$ and $h^1(\mathcal{I}_{v\cup E'}(ah+bf)) = 0$ for all $E' \subsetneq E$. Take $B_0 := \{O\}\cup E$ and make the construction a_i, D_i, B_i as in step (b). Let x be the minimal integer such that $a_{x+1} = 0$. In the set-up of (iv) we have $x \le b-eb$. Set $T' := h\cup D_1 \cup \cdots \cup D_{x-1}$. Since $v\cup E \subset T' \cup D_x$, we have $\mathrm{Res}_{T'}(v\cup E) \subset D_x$. Since $v \subset h \subset T'$, we have $\mathrm{Res}_{T'}(v\cup E) = D_x \cap (E \setminus E\cap D_x \cap h)$. First assume $x \ge 2$ and $1 \le \sharp(E\setminus E\cap D_x\cap h) \le a$. Since $\mathrm{Res}_{T'}(v\cup E) \ne \emptyset$, minimality of A gives $h^1(\mathcal{I}_{(v\cup E)\cap T'}(ah+bf)) = 0$. The Castelnuovo' sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_{T'}(v\cup E)}((a-1)h+(b-x)f)) > 0$. Since $\sharp(\mathrm{Res}_{T'}(v\cup E)) \le a$ and $\mathrm{Res}_{T'}(v\cup E)\cap h) = \emptyset$, we get a contradiction. Now assume $x \ge 2$ and $\sharp(E\setminus E\cap D_x\cap h) \ge a+1$. Since $h^1(\mathcal{I}_E(ah+bf)) = 0$, we get $a_x = a+1$ and that $D_x \cap E \cap h = \emptyset$. Since the sequence a_i is non-decreasing, we get

$a_i = a+1$ for all $i \le x$ and hence $\sharp(B) = x(a+1)$. In this case we may exchange the role of the curves D_i, $1 \le i \le x$, and win if there is at least one $i \in \{1, \dots, x\}$ such that $D_i \cap A \cap h \neq \emptyset$. Since $O \notin E$, the curve D_i containing O has this property. Now assume $x = 1$. Since $O \in B$, we get $D_1 = F$. Since $h^1(\mathcal{I}_{\{O\}\cup E}(ah+bf)) = 0$, we have $\sharp(A) \le a$. Since $b > ae$ and $h^1(\mathcal{O}_{F_e}((a-1)h+(b-1)f)) = 0$, the linear system $|\mathcal{O}_{F_e}(ah+bf)|$ gives an embedding ϕ of the curve $h \cup F$ with $\langle \phi(h+F)\rangle \cong \mathbb{P}^{a+b-ea}$, $\langle \phi(h)\rangle = \mathbb{P}^a$, $\langle \phi(h)\rangle = \mathbb{P}^{b-ea}$ and $\phi(h)$, $\phi(F)$ rational normal curves in their linear span. Since $p_a(h+F) = 0$, we get $h^1(\mathcal{I}_{v\cup F}(ah+bf)) = 0$, a contradiction.

(d) Take the set-up of (iii). Hence $b > ae$. Take any $E \subset F_e$ such that $h^1(\mathcal{I}_{\{O\}\cup E}(ah+bf)) = 0$, $\sharp(E) \le b-ea-1$, $O \notin E$, $h^1(\mathcal{I}_{v\cup E}(ah+bf)) > 0$ and $h^1(\mathcal{I}_{v\cup E'}(ah+bf)) = 0$ for all $E' \subsetneq E$. Take $B_0 := \{O\} \cup E$ and make the construction a_i, D_i, B_i as in steps (b) and (c). Let x be the minimal integer such that $a_{x+1} = 0$. In the set-up of (iii) we have $x \le b-eb-a$. Let $c \in \{1, \dots, x\}$ be the integer such that $D_c = F$. First assume $a_c \le a$. Take $T := h \cup (\bigcup_{i\neq c} D_i)$. We have $\mathrm{Res}_T(v \cup E) = D_c \cap B$, because $\mathrm{Res}_T(v)) = \{O\}$. We conclude as in step (c). Now assume $a_c = a+1$. If $c < x$ and $a_x \le a$, then we conclude as in step (c) using T' instead of T, because $v \subset F \cup h \subset T'$. Hence we may assume $a_i = a+1$ for all i. Let $J := D_1 \cup \cdots \cup D_x$. Since $F \subseteq J$ and $v \nsubseteq F$, we have $\mathrm{Res}_{v\cup J}(v \cup E)) = \{O\}$. Hence $(v\cup E)\cap J \neq v \cup A$. Hence $h^1(\mathcal{I}_{(v\cup E)\cap J}(ah+bf)) = 0$. Since $(a+1)x \le b-ea+a$, we have $b-x \ge ea$. Hence $\mathcal{O}_{F_e}(ah+(b-x)f)$ is spanned. Therefore $h^1(\mathcal{I}_{\{O\}}(ah+(b-x)f)) = 0$. The Castelnuovo's sequence gives $h^1(\mathcal{I}_{v\cup E}(ah+bf)) = 0$, a contradiction. □

Proof of Theorem 2. In the case $b = ae$ we may assume $b > ea$. Use Lemma 3 to get that ρ is at least the value claimed in the statement of Theorem 2. In all cases we get the opposite inequality is true using [1] and the curves introduced in step (a) of the proof of Lemma 3. □

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