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OPEN RANK (OR WIDERANK) FOR REDUCIBLE PROJECTIVE SETS: A CLASSIFICATION OF AN EXTREMAL CASE

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Abstract: Let $X \subset \mathbb{P}^r$, $r \ge 2$, be a reduced projective set such that $\langle X \rangle = \mathbb{P}^r$, where $\langle\ \rangle$ denote the linear span of X. For any $P \in \mathbb{P}^r$ the X-widerank $w_X(P)$ of P is the minimal integer $t > 0$ such that for each closed set $B \subset X$ containing no irreducible component of X there is $S \subset X \setminus B$ with $P \in \langle S \rangle$. Here we classify all (r, X) such that $w_X(P) \ge r+1$ for a general P (r is odd and X is the union of $(r+1)/2$ linearly independent lines). We give conditions on X which imply that $w_X(P) \le r$ (or $w_X(P) \le r+1-\dim(X)$) for every $P \in \mathbb{P}^r \setminus X$.

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1. The Statements

For each set $B \subseteq \mathbb{P}^r$ let $\langle B \rangle$ denote its linear span.

Let $X \subset \mathbb{P}^r$, $r \ge 2$, be a reduced projective set such that $\langle X \rangle = \mathbb{P}^r$. For each $P \in \mathbb{P}^r$ the X-rank $r_X(P)$ of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. Now assume that no component of X is a single point. The X-widerank $w_X(P)$ of P is the minimal integer $t > 0$ such that for each closed set $B \subset X$ containing no irreducible component of X there is $S \subset X \setminus B$ with

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$P \in \langle B \rangle$ and $\sharp(S) \leq t$. We have $w_X(P) \leq r+1$ for all P and all X and there are known several pairs (X, P) with $w_X(P) = r+1$ (e.g., if X is a rational normal curve, then take all $P \in X$). A different problem is the generic X-widerank, i.e. the X-widerank of a general $P \in \mathbb{P}^r$. In this note we prove the following results.

Theorem 1. *There is a non-empty open subset U of $\mathbb{P}^r$ such that $w_X(P) = r+1$ for all $P \in U$ if and only if r is odd and X is a disjoint union of $(r+1)/2$ lines spanning $\mathbb{P}^r$.*

All X's as in Theorem 1 are projectively equivalent. For these X's the set of all $P \in \mathbb{P}^r$ with $w_X(P) = r+1$ is the set of all P not contained in one of the $(r+1)/2$ codimension 2 linear subspaces of $\mathbb{P}^r$ spanned by $(r-1)/2$ lines of X.

Proposition 1. *Assume the existence of an irreducible component Y of X such that $\deg(Y) > 1$. Then $w_X(P) \leq r+1-\dim(Y)$ for a general $P \in \mathbb{P}^r$.*

If X is a Veronese embedding of $\mathbb{P}^m$, then the definition of $w_X(P)$ is related to the open rank $or(P)$ defined in [2]: when $or(P)$ is defined, then $or(P) = w_X(P)$, but $or(P)$ is defined only for points associated to forms depending on all $m+1$ homogeneous variables.

We work over an algebraically closed field $\mathbb{K}$ with arbitrary characteristic.

2. The Proofs

To prove Proposition 1 we need the following classical definition. Fix any integral variety $Y \subset \mathbb{P}^r$ with positive dimension and $P \in \mathbb{P}^r$. P is said to be a *strange point* of Y if every tangent space to Y at a smooth point of Y contains P. Let $\Sigma(Y)$ denote the set of all strange points of Y. In characteristic zero $\Sigma(Y) \neq \emptyset$ if and only if Y is a cone and in this case $\Sigma(Y)$ is the vertex of Y. In positive characteristic $\Sigma(Y)$ is a linear space with dimension at most $\dim(Y)$ (unless Y is a linear space).

Lemma 1. *Assume that X has an irreducible component Y which is a linear space of dimension $m \geq 2$. Then $r_X(P) \leq r+1-m$ for all $P \in \mathbb{P}^r$ and $w_X(P) \leq r+2-m$ for all $P \in \mathbb{P}^r \setminus Y$.*

Proof. Fix $P \in \mathbb{P}^r$. If $P \in Y$, then $r_X(P) = 1$. Now assume $P \notin Y$ and take a closed subset $B \subset X$ containing no irreducible components of X. Since $\mathbb{P}^r \setminus Y \neq \emptyset$, we have $r > m$. Since $\langle Y \rangle = Y$ and X spans $\mathbb{P}^r$, the set $X \setminus (Y \cup B)$ spans a linear space M such that $\langle M \cup Y \rangle = \mathbb{P}^r$. Hence there are $S \subset X \setminus (Y \cup B)$ and $O \in Y$ such that $\sharp(S) = r-m$ and $P \in \langle S \cup \{O\} \rangle$. Since

$B \cap Y$ is a proper closed subset of Y there is a line $L \subset Y$ such that $O \in L$ and $L \cap B$ is finite. Take $E \subset L \setminus L \cap B$ such that $\sharp(E) = 2$. Since $\langle E \rangle = L$, we have $O \in \langle E \rangle$. Hence $P \in \langle S \cup E \rangle$. Since $S \cup E \subset X \setminus B$, we get $w_X(P) \le r + 2 - m$. Since $S \cup \{O\} \subset X$, we get $r_X(P) \le r + 1 - m$. □

Lemma 2. *Let $Y \subsetneq \mathbb{P}^n$ be an integral and non-degenerate variety. Then $w_Y(P) \le n + 1 - \dim(Y)$ for all $P \in \mathbb{P}^r \setminus (Y \cup \Sigma(Y))$.*

Proof. Fix $P \in \mathbb{P}^r \setminus (Y \cup \Sigma(Y))$ and a closed subset $B \subsetneq Y$. Since $P \notin \Sigma(Y)$, a characteristic free version of Bertini's theorem gives that for a general linear subspace $M \subset \mathbb{P}^n$ with $P \in M$ and $\dim(M) = n - \dim(Y)$ the scheme $M \cap Y$ is a reduced set spanning M ([1], proof of Theorem 1 up to line 12 of page 6). Hence there is $S \subset M \cap Y$ such that $\sharp(S) = n + 1 - \dim(Y)$ and $P \in \langle S \rangle$. Since $P \notin Y$, $\dim(B) \le \dim(Y) - 1$ and M is a general linear space of dimension $n - \dim(Y)$ containing P, we have $M \cap B = \emptyset$. Hence $S \subset Y \setminus B$. □

Proposition 2. *Let $X \subset \mathbb{P}^r$ be equidimensional and non-degenerate with dimension $m > 0$. Assume that X is connected in codimension $m - 1$, i.e. it is connected if $m = 1$, $X \setminus A$ is connected for all closed sets $A \subset X$ with $\dim(A) \le m - 2$ if $m \ge 2$. Let Σ be the union of the strange sets of the irreducible components of X. Then $w_X(P) \le r + 1 - m$ for all $P \in \mathbb{P}^r \setminus (X \cup \Sigma)$.*

Proof. Apply $m - 1$ times [1], proof of Theorem 1 up to line 12 of page 6, and the proof of Lemma 2. In the case $m = 1$ and $\mathrm{char}(\mathbb{K}) = 0$, then this is the proof of [2], Lemma 11. □

Proof of Proposition 1. Fix $P \in \mathbb{P}^r \setminus (Y \cup \Sigma(Y))$ and take a closed subset $B \subset X$ containing no irreducible component of X. Set $e := \dim(\langle Y \rangle)$. Since $\deg(Y) > 1$, we have $e > \dim(Y)$. If $e = r$, then we may apply Lemma 2, because $w_Y(P)$ is defined and $w_X(P) \le w_Y(P)$. Hence we may assume $r > e$. Since X spans $\mathbb{P}^r$ there is $S \subset X \setminus (Y \cup B)$ such that $\sharp(S) = r - e$ and $\langle Y \rangle \cup \mathbb{P}^r$. Hence there is $O \in \langle Y \rangle$ such that $P \in \langle \{O\} \cup S \rangle$. If $O \notin Y \cup \Sigma(Y)$, then there is $E \subset Y \setminus Y \cap B)$ such that $\sharp(E) \le e - \dim(Y)$ and $O \in \langle E \rangle$. In this case the set $E \cup S \subset X \setminus B$ shows that $w_X(P) \le r + 1 - \dim(Y)$. For a general $P \in \mathbb{P}^r$ we may find S and O with $O \notin Y \cup \Sigma(Y)$. Hence $w_X(P) \le r + 1 - \dim(Y)$ for a general $P \in \mathbb{P}^r$. □

Proof of Theorem 1. We first check the " if " part. Take $r = 2k - 1$, $k \ge 2$, and write $X = L_1 \sqcup \cdots \sqcup L_k$ with each L_i a line and $\langle X \rangle = \mathbb{P}^r$. Any two X's as above are projectively equivalent. Let U be the set of all $P \in \mathbb{P}^r$ not contained in one of the k codimension 2 linear subspaces of $\mathbb{P}^r$ spanned by $k - 1$

lines of X. Fix any $P \in U$. Linear algebra teach us that $r_X(P) = k$, that there is a unique set $S \subset X$ with $\sharp(S) = k$ and $P \in \langle S\rangle$ and that $\sharp(S \cap L_i) = 1$ for all i. Set $P_i := S \cap L_i$. Moreover, for each $i \in \{1, \dots, k\}$ P_i is the only $O \in L_i$ such that $P \in \langle\{O\} \cup (X \setminus L_i)$. Take any finite $B \subset X$ with $B \supseteq S$ and any $A \subset X \setminus B$ with $\sharp(A) < 2k$. There is $i \in \{1, \dots, k\}$ such that $\sharp(A \cap L_i) \leq 1$. Take $F \supseteq A$ such that $F \subset X \setminus S$, $\sharp(F \cap L_i) = 1$ and $\sharp(F \cap L_j) \geq 2$. Write $\{O\} := F \cap L_i$. Since $F \subset \{O\} \cup (X \setminus L_i)$ and $O \neq P_i$, we have $P \notin \langle F\rangle$. Hence $w_X(P) \geq 2k$. Since the inequality $w_X(P) \leq r + 1$ is obvious for any X and P, we get the " if " part.

Now assume the existence of a non-empty open subset U of $\mathbb{P}^r$ such that $w_X(P) \geq r + 1$ for all $P \in U$. By Proposition 1 and Lemma 1 we may assume that each irreducible component of X is a line. Taking a smaller U we may assume that no $Q \in U$ is contained in a proper linear subspace of $\mathbb{P}^r$ spanned by some of the lines in X. First assume $r = 2$. Since X spans $\mathbb{P}^2$ we have $\deg(X) \geq 2$. In this case we immediately see that $w_X(P) = 2$ for all $P \in \mathbb{P}^2$. Now assume $r > 2$ and that Theorem 1 is true in $\mathbb{P}^{r-1}$. We also use induction on the integer $\deg(X)$. Fix an irreducible component L of X and call W the union of the other irreducible components of X. Let $\ell_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-2}$ be the linear projection from L.

(i) First assume $L \cap \langle W\rangle = \emptyset$. Our assumption on U gives the existence of a unique $O_P \in \langle W\rangle$ such that $P \in \langle L \cup \{O_P\}\rangle$. Take $S \subset W \setminus W \cap B$ evincing $w_W(O_P)$. Take any $E \subset L \setminus L \cap B$ with $\sharp(E) = 2$. Since $E \cup S \subset X \setminus B$, we get $\sharp(E \cup S) \geq w_X(P) = r + 1$. Hence $\sharp(S) \geq r - 1$. Varying P in U the point O_P varies in a non-empty open subset of $\langle W\rangle$. Hence $r_W(O) \geq r - 1$ for a a general $O \in \langle W\rangle$. The inductive assumption gives that $r = 2k - 1$ is odd and that W is the union of $k - 1$ lines spanning $\langle W\rangle$. Hence X is as in the statement of Theorem 1.

(ii) Now assume that $L \cap \langle W\rangle$ is a unique point, Q. Hence $\dim(\langle W\rangle) = r - 1$.

(ii1) First assume $Q \notin W$. In this case ℓ_L is defined at each point of W. The algebraic set $\ell_L(W)$ is a union of finitely many lines and it spans $\mathbb{P}^{r-2}$. For a general P the point $\ell_L(P)$ is a general point of $\mathbb{P}^{r-2}$. Assume for the moment $w_{\ell_L(W)}(\ell_L(P)) \leq r - 2$ and take $S \subset \ell_L(W) \setminus \ell_L(B \cap W)$ such that $\ell_L(P) \in \langle S\rangle$. Since $L \cap W = \emptyset$, $\ell_L|W$ is a morphism and so there is $S' \subset W$ such that $\sharp(S') = \sharp(S)$ and $\ell_L(S') = S$. Since $S \cap \ell_L(B \setminus L \cap B) = \emptyset$ and $W \cap L = \emptyset$, we have $S' \subset W \setminus W \cap B$. Take any $E \subset L \setminus L \cap B$ such that $\sharp(E) = 2$. Since $\ell_L(P) \in \langle S\rangle$ and $\langle E\rangle = L$, we have $P \in \langle E \cup S'\rangle$. Since $S' \cup E \subset X \setminus B$, we get $w_X(P) \leq r$, a contradiction. Hence $w_{\ell_L(W)}(\ell_L(P)) \geq r - 1$. By the inductive assumption $r - 2 = 2k - 3$ is odd and $\ell_L(W)$ is a disjoint union of $k - 1$ lines spanning $\mathbb{P}^{r-2}$. Of course, ℓ_L must send at least two components of W onto the same line of $\ell_L(W)$, because $\langle W\rangle$ has dimension $r-1$. Take an ordering $L_1,\dots,L_{k-1}$ of the lines of $\ell_L(W)$ and call E_i the union of the components of W whose image is L_i. Since $L_i\cap L_j=\emptyset$ for all $i\neq j$, we have $E_i\cap E_j=\emptyset$ for all $i\neq j$, each $\langle E_i\rangle$ is a plane containing Q and $\langle E_i\rangle\cap\langle E_j\rangle=\{Q\}$ for all $i\neq j$. Since $P\in\langle X\rangle$, there is $o_P\in L$, $o_W\in\langle W\rangle$ and $O_i\in\langle E_i\rangle$, $1\le i\le k-1$, such that $P\in\langle\{o_P,o_W\}\rangle$ and $o_W\in\langle\{O_1,\dots,O_{k-1}\rangle$. Since $P\notin\langle W\rangle$, we have $o_P\neq Q$. Since $L\cap\langle W\rangle\neq\emptyset$, the k-ple (o_P,o_W) is not unique. Indeed, for suitable $o_W\in\langle W\rangle$ we may take as o_P any point of $L\setminus\{Q\}$. In particular we may take o_W so that $o_P\notin B$. We fix any such a choice of o_W. Either E_i is a line or E_i is a non-degenerate plane curve. In both cases we have $w_{E_i}(O)=2$ for all $O\in\langle E_i\rangle$. Take $F_i\subset E_i\setminus E_i\cap B$ such that $\sharp(F_i)=2$ and $O_i\subset\langle E_i\rangle$. Set $F:=\{o_P\}\cup F_1\cup\dots\cup F_{k-1}$. We have $\sharp(F)=r$, $F\subset X\setminus B$ and $P\in\langle F\rangle$.

(ii2). Now assume $Q\in W$. Hence X contains two lines L,R such that $L\cup R$ is a reducible conic. The case $r=2$ gives $w_{L\cup R}(O)=2$ for all $O\in\langle L\cup R\rangle$. As in the proof of Lemma 1 we get $w_X(O)\le r$ for all $O\in\mathbb{P}^r$.

(iii) Now assume $L\subset\langle W\rangle$, i.e. $\langle W\rangle=\mathbb{P}^r$. Hence $r_W(O)$ and $w_W(O)$ are defined for every $O\in\mathbb{P}^r$. We have $w_X(O)\le w_W(O)$. By the inductive assumption on the integer $\deg(X)$ we get that $r=2k-1$ is odd and that W is a disjoint union of k lines. Hence $\deg(X)=k+1$. Taking another irreducible component L' of X instead of L and using steps (i), (ii1) and (iv) we get that any k of the components of X span $\mathbb{P}^r$. Let U be the complement in $\mathbb{P}^{2k-1}$ of the codimension 2 linear subspaces of $\mathbb{P}^r$ spanned by $k-2$ of the lines of X. Fix any $P\in U$ and any finite set $B\subset X$. First assume $r=3$. Let $H\subset\mathbb{P}^3$ be a general plane through P. The set $X\cap H$ is formed by 3 points spanning H. For general H we have $B\cap H=\emptyset$. Hence $w_X(P)\le 3$, a contradiction. Now assume $r\ge 5$. Fix a line $L\subset X$ and set $W:=X\setminus L$. Since any k of the components of X span $\mathbb{P}^r$, $\ell_L(W)$ is a disjoint union of k lines, any $k-1$ of them spanning $\mathbb{P}^{r-2}$. Hence $r_{\ell_L(W)}(O)\le r-2$ for a general $O\in\mathbb{P}^{r-2}$ by the inductive assumption. Since $P\in U$, we have $P\notin L$. Hence $\ell_L(P)$ is well-defined. Set $B':=\ell_L(B\setminus B\cap L)$. For general P the point $\ell_L(P)$ is a general point of $\mathbb{P}^{r-2}$. Hence there is $S\subset\ell_L(W)\setminus B'$ such that $\sharp(S)\le r-2$ and $\ell_L(P)\in\langle S\rangle$. Since $\ell_L|W$ is an embedding, there is a unique $S'\subset W$ such that $\ell_L(S')=S$. Since $S\cap B'=\emptyset$, we have $S'\cap B=\emptyset$. Take any $E\subset L\setminus L\cap B$ such that $\sharp(E)=2$. Since $\ell_L(P)\in\langle S\rangle$, we have $P\in\langle E\cup S'\rangle$. Hence $w_X(P)\le r$, a contradiction. □

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