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ON THE DIOPHANTINE EQUATION $323^x + 325^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $323^x + 325^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 18)$.

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1. Introduction

In 2011, Suvarnamani, Singta and Chotchaisthit [13] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers. In 2012, Chotchaisthit [1] proved that the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where x, y, z are non-negative integers and p is a positive prime. In 2013, Sroysang [11, 12] proved that the two Diophantine equation $47^x + 49^y = z^2$ and $89^x + 91^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers. Moreover, Sroysang [4, 5, 6, 7, 8, 9, 10] solve similar equations. Recently, Rabago [3] solved the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers. The solutions (x, y, z) are $(1, 0, 3)$, $(1, 1, 5)$, $(2, 1, 9)$ and $(3, 1, 23)$. In this paper, we solve the Diophantine equation $323^x + 325^y = z^2$ where x, y and z are non-negative integers.

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2. Preliminaries

Proposition 2.1. [2] (**Catalan's conjecture**) $(3,2,2,3)$ *is a unique solution* (a,b,x,y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a,b,x *and* y *are integers such that* $\min\{a,b,x,y\} > 1$.

Lemma 2.2. $(1,18)$ *is a unique solution* (x,z) *for the Diophantine equation* $323^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Proof. Let x and z be non-negative integers such that $323^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. This implies that $x \geq 1$. Then $z^2 = 323^y + 1 \geq 323^1 + 1 = 324$. Thus, $z \geq 18$. Now, we consider on the equation $z^2 - 323^x = 1$. By Proposition 2.1, we have $x = 1$. It follows that $z^2 = 324$. Hence, $z = 18$. □

Lemma 2.3. *The Diophantine equation* $1+325^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 325^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. This implies that $y \geq 1$. Then $z^2 = 1 + 325^y \geq 1 + 325^1 = 326$. Thus, $z \geq 19$. Now, we consider on the equation $z^2 - 325^y = 1$. By Proposition 2.1, we have $y = 1$. It follows that $z^2 = 326$. This is a contradiction. □

3. Main Results

Theorem 3.1. $(1,0,18)$ *is a unique non-negative integer solution* (x,y,z) *for the Diophantine equation* $323^x+325^y = z^2$ *where* x,y *and* z *are non-negative integers.*

Proof. Let x,y and z be non-negative integers such that $323^x + 325^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Note that z is even. We obtain that $z^2 \equiv 0 \pmod 4$. Moreover, $z^2 \equiv 0 \pmod 5$ or $z^2 \equiv 1 \pmod 5$ or $z^2 \equiv 4 \pmod 5$. Since $325^y \equiv 1 \pmod 4$, we have $323^x \equiv 3 \pmod 4$. This implies that x is odd. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, it follows that $x = 1$ and $z = 18$.

Case $y \geq 1$. Note that $323^y \equiv 2 \pmod 5$ or $323^y \equiv 3 \pmod 5$. Since $325^y \equiv 0 \pmod 5$, we obtain that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. This is a contradiction.

Hence, the solution (x,y,z) is $(1,0,18)$. □

Corollary 3.2. *The Diophantine equation* $323^x + 325^y = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $323^x + 325^y = w^4$. Let $z = w^2$. Thus, $323^x + 325^y = z^2$. By Theorem 3.1, it follows that $(x, y, z) = (1, 0, 18)$. Hence, $w^2 = z = 18$. This is a contradiction. □

Corollary 3.3. $(1, 0, 2)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $323^x + 325^y = 9u^2$ *where* x, y *and* u *are non-negative integers.*

Proof. Let x, y and u be non-negative integers such that $323^x + 325^y = 4u^4$. Let $z = 3u$. Then $323^x + 325^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 18)$. Then $3u = z = 18$. Thus, $u = 6$. □

Corollary 3.4. $(1, 0, 3)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $323^x + 325^y = 4v^4$ *where* x, y *and* v *are non-negative integers.*

Proof. Let x, y and v be non-negative integers such that $323^x + 325^y = 4v^4$. Let $z = 2v^2$. Then $323^x + 325^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 18)$. Then $2v^2 = z = 18$. Thus, $v = 3$. □

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