ON THE DIOPHANTINE EQUATION $323x + 325y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $323x + 325y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 18)$.

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1. Introduction

In 2011, Suvarnamani, Singta and Chotchaisthit [13] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where $x, y$ and $z$ are non-negative integers. In 2012, Chotchaisthit [1] proved that the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where $x, y, z$ are non-negative integers and $p$ is a positive prime. In 2013, Sroysang [11, 12] proved that the two Diophantine equation $47^x + 49^y = z^2$ and $89^x + 91^y = z^2$ have no non-negative integer solution where $x, y$ and $z$ are non-negative integers. Moreover, Sroysang [4, 5, 6, 7, 8, 9, 10] solve similar equations. Recently, Rabago [3] solved the Diophantine equation $8^x + 17^y = z^2$ where $x, y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(1, 0, 3)$, $(1, 1, 5)$, $(2, 1, 9)$ and $(3, 1, 23)$. In this paper, we solve the Diophantine equation $323^x + 325^y = z^2$ where $x, y$ and $z$ are non-negative integers.
2. Preliminaries

Proposition 2.1. [2] (Catalan’s conjecture) $(3,2,2,3)$ is a unique solution $(a,b,x,y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2.2. $(1,18)$ is a unique solution $(x,z)$ for the Diophantine equation $323^x + 1 = z^2$ where $x$ and $z$ are non-negative integers.

Proof. Let $x$ and $z$ be non-negative integers such that $323^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. This implies that $x \geq 1$. Then $z^2 = 323^y + 1 \geq 323^1 + 1 = 324$. Thus, $z \geq 18$. Now, we consider on the equation $z^2 - 323^x = 1$. By Proposition 2.1, we have $x = 1$. It follows that $z^2 = 324$. Hence, $z = 18$.

Lemma 2.3. The Diophantine equation $1 + 325^y = z^2$ has no non-negative integer solution where $y$ and $z$ are non-negative integers.

Proof. Suppose that there are non-negative integers $y$ and $z$ such that $1 + 325^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. This implies that $y \geq 1$. Then $z^2 = 1 + 325^y \geq 1 + 325^1 = 326$. Thus, $z \geq 19$. Now, we consider on the equation $z^2 - 325^y = 1$. By Proposition 2.1, we have $y = 1$. It follows that $z^2 = 326$. This is a contradiction.

3. Main Results

Theorem 3.1. $(1,0,18)$ is a unique non-negative integer solution $(x,y,z)$ for the Diophantine equation $323^x + 325^y = z^2$ where $x, y$ and $z$ are non-negative integers.

Proof. Let $x, y$ and $z$ be non-negative integers such that $323^x + 325^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Note that $z$ is even. We obtain that $z^2 \equiv 0 \pmod{4}$. Moreover, $z^2 \equiv 0 \pmod{5}$ or $z^2 \equiv 1 \pmod{5}$ or $z^2 \equiv 4 \pmod{5}$. Since $325^y \equiv 1 \pmod{4}$, we have $323^x \equiv 3 \pmod{4}$. This implies that $x$ is odd. Now, we will divide the number $y$ into two cases.

Case $y = 0$. By Lemma 2.2, it follows that $x = 1$ and $z = 18$.

Case $y \geq 1$. Note that $323^y \equiv 2 \pmod{5}$ or $323^y \equiv 3 \pmod{5}$. Since $325^y \equiv 0 \pmod{5}$, we obtain that $z^2 \equiv 2 \pmod{5}$ or $z^2 \equiv 3 \pmod{5}$. This is a contradiction.

Hence, the solution $(x,y,z)$ is $(1,0,18)$.
Corollary 3.2. The Diophantine equation $323^x + 325^y = w^4$ has no non-negative integer solution where $x, y$ and $w$ are non-negative integers.

Proof. Suppose that there are non-negative integers $x, y$ and $w$ such that $323^x + 325^y = w^4$. Let $z = w^2$. Thus, $323^x + 325^y = z^2$. By Theorem 3.1, it follows that $(x, y, z) = (1, 0, 18)$. Hence, $w^2 = z = 18$. This is a contradiction.

Corollary 3.3. $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + 325^y = 9u^2$ where $x, y$ and $u$ are non-negative integers.

Proof. Let $x, y$ and $u$ be non-negative integers such that $323^x + 325^y = 4u^4$. Let $z = 3u$. Then $323^x + 325^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 18)$. Then $3u = z = 18$. Thus, $u = 6$.

Corollary 3.4. $(1, 0, 3)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + 325^y = 4v^4$ where $x, y$ and $v$ are non-negative integers.

Proof. Let $x, y$ and $v$ be non-negative integers such that $323^x + 325^y = 4v^4$. Let $z = 2v^2$. Then $323^x + 325^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 18)$. Then $2v^2 = z = 18$. Thus, $v = 3$.

References


