International Journal of Pure and Applied Mathematics
Volume 91 No. 3 2014, 329-347
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i3.6
ijpam.eu

NEW APPROACH TO COVERING ROUGH SETS VIA RELATIONS

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Abstract: The present paper is devoted to discussion of extended covering rough sets from the relations point of view. In fact, we give new definitions to the lower and the upper approximations using a binary general relation. Moreover, our approach represent new generalizations of Pawlak approaches. New closure operators derived from relation are investigated too.

AMS Subject Classification: 54A05, 54B05, 54C05
Key Words: topology, closure operators, rough sets and covering rough sets by relations

1. Introduction

Rough set theory was proposed by Pawlak (see [12]) to deal with the vagueness and granularity in information systems that are characterized by insufficient, inconsistent, and incomplete data. Its successful applications draw attentions from researchers in areas such as artificial intelligence, computational intelligence, data mining and machine learning. The classical rough set model is

Received: November 8, 2013

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based on an equivalence relation on a set, but in some situations, equivalence relations are not suitable for coping with granularity. Instead the classical rough set method is extended to general relation-based rough sets and covering-based rough sets (see [2], [9]-[11], and [14]-[23]). Covering-based rough sets are a natural extension of classical rough sets by relaxing the partitions arising from equivalence relations to coverings that have many applications on real-life problems (see [7] and [8]). In this paper, we introduce new definition for the approximations based on general binary relation. In addition, we introduce a new closure operator based on Galton definition (see [6]), and give some topological properties of them. New closure operators derived from relation using relation coverings concepts are introduced and its properties are studied. Moreover, we give a comparison between our approach and Galton approaches.

2. Basic Concepts

In this section, we introduce the main ideas be behind rough sets, topology and coverings. Moreover, the fundamental concepts, which used through this paper, are investigated.

Definition 2.1. (see [4]) Let U be any set, and R be any binary relation on U. Then, *after set* (resp. *fore set*) of the element $x \in U$ is the class $xR = \{y \in U \mid xRy\}(resp.Rx = \{y \in U \mid yRx\})$.

Definition 2.2. (see [12]) Let U be a finite set, the universe of discourse, and R be an equivalence relation on U, called an indiscernibility relation. The pair $A = (U,R)$ is called Pawlak approximation space. The relation R generates a partition$(U/R = \{[x]R \mid x \in U\})$ on U, where $[x]R$ is the equivalence class with respect to R containing x.

Definition 2.3. (see [12]) For any, $X \subseteq U$ the upper approximation $\overline{Apr}(X)$ and the lower approximation $\underline{Apr}(X)$of a subset $X$are defined as :

$$\overline{Apr}(X) = \cap\{Y \subseteq U/R \mid Y \cap X \neq \phi)\}$$

and

$$\underline{Apr}(X) = \cup\{Y \subseteq U/R \mid Y \subseteq X)\}.$$

Let ϕ be the empty set and X^c is the complement of X in U, then we can get the following properties of the Pawlak's rough sets:

(i) $\underline{Apr}(X) \subseteq X \subseteq \overline{Apr}(X)$.

(ii) $\underline{Apr}(U) = \overline{Apr}(U) = U$.

(iii) $\underline{Apr}(\phi) = \overline{Apr}(\phi) = \phi$.

(iv)$If X \subseteq Y, then \underline{Apr}(X) \subseteq \underline{Apr}(Y) and \overline{Apr}(X) \subseteq \overline{Apr}(Y)$.

(v) $\underline{Apr}(\underline{Apr}(X)) = \underline{Apr}(\overline{Apr}(X)) = \underline{Apr}(X)$.

(vi) $\overline{Apr}(\overline{Apr}(X)) = \overline{Apr}(\underline{Apr}(X)) = \overline{Apr}(X)$.

(vii) $\underline{Apr}(X \cap Y) = \underline{Apr}(X) \cap \underline{Apr}(Y) and \overline{Apr}(X \cup Y) = \overline{Apr}(X) \cup \overline{Apr}(Y)$.

(viii)$\overline{Apr}(X \cap Y) \subseteq \overline{Apr}(X) \cap \overline{Apr}(Y) and \underline{Apr}(X \cup Y) \supseteq \underline{Apr}(X) \cup \underline{Apr}(Y)$.

Definition 2.4. (see [22]) Let U be a domain of discourse, C a family of subsets of U. If none subsets in C is empty, and $\cup C = U$, then C is called a covering of U.

It is clear that a partition of U is certainly a covering of U, so the concept of a covering is an extension of the concept of a partition.

Definition 2.5. A topological space (see [4]) is the pair (U, τ) consisting of a set U and family τ of subsets of U satisfying the following conditions:

(T1) $\phi \in \tau$ and $U \in \tau$.
(T2) τ is closed under finite intersection.
(T3) τ is closed under arbitrary union.

The pair (U, τ) is called *space*, the elements of U are called *points* of the space, the subsets of U belonging to τ are called *open* sets in the space and the complement of the subsets of U belonging to τ are called *closed* sets in the space; the family τ of open subsets of U is also called a *topology* on U.

Definition 2.6. (see [6]) For an operator $cl\colon P(U) \longrightarrow P(U)$, if it satisfies the following rules, then we call it a closure operator on U. For $X, Y \subseteq U$:

(i) $cl(\phi) = \phi$.

(ii) $X \subseteq cl(X)$.

(iii)$cl(X \cup Y) = cl(X) \cup cl(Y)$.

Definition 2.7. (see [6]) For an operator $int\colon P(U) \longrightarrow P(U)$, if it satisfies the following four rules, then we call it an interior operator on U. For $X, Y \subseteq U$,

(i) $int(U) = U$.

(ii) $int(X) \subseteq X$.

(iii) $int(X \cap Y) = int(X) \cap int(Y)$.

A topological space can be defined as a closure space in which the closure operator is idempotent i.e. if $cl(cl(X)) = cl(X)$.

3. Covering Rough Sets from the Relations Point of View

The present section is devoted to introduce new approach to covering rough sets based on binary relations. Moreover, we introduce new generalizations to Pawlak approximation space. In fact, we introduce new general concepts to the approximations which extend the domain of the applications of rough set theory. The properties of the operators are investigated. In addition, we introduce some comparisons between our approach and the others approach.

Definition 3.1. Let the non-empty set U be finite, called *the universe*, and R be a general binary relation on U. Then, the pair $\boldsymbol{G} = (U, R)$ is called *a generalized approximation space*, in briefly $\boldsymbol{GAS}$.

Definition 3.2. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$ and R be a serial relation. Then we can define a covering for U by using the concept of after set and the fore set as the following:

(i) Right Covering: $\boldsymbol{C}_r = \{xR \mid \forall x \in U\}$, such that $U = \cup xR, \forall x \in U$.

(ii) Left Covering: $\boldsymbol{C}_l = \{Rx \mid \forall x \in U\}$, such that $U = \cup Rx, \forall x \in U$.

Definition 3.3. Let $\boldsymbol{G} = (U, R)$be a $\boldsymbol{GAS}$, for each element $x \in U$, we can define two neighborhoods of it as following:

(i) R-right neighborhood: $N_r(x) = \cap yR, \forall x \in yR$ such that $yR \in \boldsymbol{C}_r$.
(ii) R-left neighborhood: $N_l(x) = \cap Ry, \forall x \in Ry$ such that $Ry \in \boldsymbol{C}_l$.

Lemma 3.1. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, then for each $i = r, l$:*

(i) $x \in N_i(x), \forall x \in U$.
(ii) $N_i(x) \neq \phi, \forall x \in U$.

Proof. (i) Straightforward from Definition 3.3

(ii) Since, from Definition3.3, $xR \in \boldsymbol{C}_r$ and $Rx \in \boldsymbol{C}_l$. Then $\forall x \in U$, there is at least $y \in U$ such that $x \in yR$ and $x \in Ry$. Thus, $N_i(x) \neq \phi, \forall x \in U$. □

Definition 3.4. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, we say $X \subseteq U$ is *right-elementary set* (resp. *left-elementary set*), if it is a finite union of R-right neighborhoods (resp. R-left neighborhoods) of its elements, the class of all right-elementary (resp. left-elementary) sets of U is given by the class:

$\boldsymbol{E}_r = \{X \mid X = \cup N_r(x), \forall x \in U\}$ (resp. $\boldsymbol{E}_l = \{X \mid X = \cup N_l(x), \forall x \in U\}$).

Definition 3.5. The subset $X \subseteq U$ is called *right-definable (exact) set* (resp. *left-definable (exact) set*) if $X$and X^c are right-elementary (resp. left-elementary) sets. Otherwise, $X \subseteq U$ is *right-undefinable (rough) set* (resp. *left-undefinable (rough) set*).

Remark 3.1. It is clear that for each $i = r, l$: $\boldsymbol{E}_i$ represent two different topologies on U.

Example 3.1. Let $U = \{a, b, c, d\}$ and R be a binary relation on U, where $R = \{(a, b), (a, d), (b, b), (b, c), (c, d), (d, a)\}$. Then $aR = \{b, d\}, bR = \{b, c\}, cR = \{d\}$ and $dR = \{a\}$. Also,$Ra = \{d\}, Rb = \{a, b\}, Rc = \{b\}$ and $Rd = \{a, c\}$ which implies $N_r(a) = \{a\}, N_r(b) = \{b\}, N_r(c) = \{b, c\}$ and $N_r(d) = \{d\}$. Also,$N_l(a) = \{a\}, N_l(b) = \{b\}, N_l(c) = \{a, c\}$and $N_l(d) = \{d\}$. Thus the class of all right-elementary (resp. left-elementary) sets of U is given by the class: $\boldsymbol{E}_r = \{U, \phi, \{a\}, \{b\}, \{d\}, \{b, c\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$ (resp. $\boldsymbol{E}_l = \{U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$). It is clear that the only right-exact sets are $\{a\}, \{d\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}$. Similarly, left-exact sets are $\{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}$.

The rough set can be defined approximately using two exacts sets, namely the lower and the upper approximations, which are given in the following definition.

Definition 3.6. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$ and $X \subseteq U$. Then:

(i) The right-lower (resp. left-lower) approximation of $X \subseteq U$ is:

$$\underline{R}_r(X) = \{x \in X \mid N_r(x) \subseteq X\} \quad \text{(resp. } \underline{R}_l(X) = \{x \in X \mid N_l(x) \subseteq X\}.$$

(ii) The right-upper (resp. left-upper) approximation of $X \subseteq U$ is:

$$\overline{R}_r(X) = \{x \in U \mid N_r(x) \cap X \neq \phi\} \quad \text{(resp. } \overline{R}_l(X) = \{x \in U \mid N_l(x) \cap X \neq \emptyset\}).$$

Remark 3.2. It is clear that, $X \subseteq U$ is an $i-$exact set if and only if $\underline{R}_i(X) = \overline{R}_i(X) = X, \forall i = r, l$. Otherwise, X is i-rough.

Remark 3.3. By using the above operators, we can divide the universe U into different disjoints regions are: $\forall X \subseteq U$ and $\forall i = r, l$

(i) Boundary region: $\boldsymbol{B}_i(X) = \overline{R}_i(X) - \underline{R}_i(X)$.

(ii) Positive region: $\underline{R}_i(X)$.

(iii) Negative region: $U - \overline{R}_i(X)$.

Definition 3.7. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$. Then *the accuracy of the approximations* of a subset $X \subseteq U$is given by:

$$\mu_i(X) = |\underline{R}_i(X)|/|\overline{R}_i(X)|,$$

such that $|\overline{R}_i(X)| \neq 0, \forall i = r, l$. Here $|.|$ represents the cardinality of elements.

Remark 3.4. From the above definition, it is clear that; for each $X \subseteq U$:

(i) $0 \leq \mu_i(X) \leq 1$.

(ii) If $\mu_i(X) = 1$, then X is $i-$exact set. Otherwise X is i-rough set.

(iii) $\mu_i(\phi) = 0$.

Remark 3.5. If R is an equivalence relation, then $\boldsymbol{C}_i$ represents a partition on U and in this case $\underline{R}_i(X)$ and $\overline{R}_i(X)$represent the lower and upper approximation operations as specified by Pawlak's original definitions, see [12]. So, we can say that our approaches can be considered as a generalization to Pawlak definition.

Lemma 3.2. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, if $x \in N_i(y)$. Then, $N_i(x) \subseteq N_i(y), \forall i = r, l$.*

Proof. Let $x \in N_i(y)$, then x is contained in all after sets (resp. fore sets) that contains also the element y.Now, if $z \in N_i(x)$. Then z is contained in all after sets (resp. fore sets) that contains the element x and thus z is contained in all after sets (resp. fore sets) that contains also the element y. Then $z \in N_i(y)$ which implies $N_i(x) \subseteq N_i(y), \forall i = r, l$. □

The following proposition gives the basic properties of the approximations which can be considered as the comparison between our approach and the others generalizations such as given in [13]-[16] and [18]-[23].

Proposition 3.1. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$ and $X, Y \subseteq U$. Then, ,for each $i = r, l$:*

(i)$\underline{R}_i(U) = \overline{R}_i(U) = U$ *and* $\underline{R}_i(\phi) = \overline{R}_i(\phi) = \phi$.

(ii) $\underline{R}_i(X) \subseteq X \subseteq \overline{R}_i(X)$, *for any* $X \subseteq U$.

(iii) If $X \subseteq Y$, *then* $\underline{R}_i(X) \subseteq \underline{R}_i(Y)$ *and* $\overline{R}_i(X) \subseteq \overline{R}_i(Y)$.

(iv)$\underline{R}_i(X \cup Y) \supseteq \underline{R}_i(X) \cup \underline{R}_i(Y)$ *and* $\overline{R}_i(X \cap Y) \subseteq \overline{R}_i(X) \cap \overline{R}_i(Y)$.

(v) $\underline{R}_i(X \cap Y) = \underline{R}_i(X) \cap \underline{R}_i(Y)$ *and* $\overline{R}_i(X \cup Y) = \overline{R}_i(X) \cup \overline{R}_i(Y)$.

(vi) $\underline{R}_i(\underline{R}_i(X)) = \underline{R}_i(X)$. *(vii)* $\overline{R}_i(\overline{R}_i(X)) = \overline{R}_i(X)$.

Proof. (i) and (ii) Straightforward, from the definition of approximations.

(iii) Suppose that $X \subseteq Y$, and let $x \in \underline{R}_i(X)$. Then $x \in X$ and $N_i(x) \subseteq X$, which means that $x \in Y$ and $N_i(x) \subseteq Y$. Thus, $\underline{R}_i(X) \subseteq \underline{R}_i(Y)$. By the same way, $\overline{R}_i(X) \subseteq \overline{R}_i(Y)$.

(iv) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$.Then $\underline{R}_i(X) \subseteq \underline{R}_i(X \cup Y)$ and $\underline{R}_i(Y) \subseteq \underline{R}_i(X \cup Y)$. Thus $\underline{R}_i(X) \cup \underline{R}_i(Y) \subseteq \underline{R}_i(X \cup Y)$. By the same way, $\underline{R}_i(X \cap Y) \subseteq \underline{R}_i(X) \cap \underline{R}_i(Y)$.

(v) Let $x \in (\underline{R}_i(X) \cap \underline{R}_i(Y))$, then $x \in \underline{R}_i(X)$ and $x \in \underline{R}_i(Y)$. Thus $x \in X$, $N_i(x) \subseteq X$ and $x \in Y, N_i(x) \subseteq Y$ which means that $x \in X \cap Y$, $N_i(x) \subseteq (X \cap Y)$. Then $x \in \underline{R}_i(X \cap Y)$ and this implies $\underline{R}_i(X) \cap \underline{R}_i(Y) \subseteq \underline{R}_i(X \cap Y)$. Now let $x \in \underline{R}_i(X \cap Y)$, then $x \in (X \cap Y)$ and $N_i(x) \subseteq (X \cap Y)$. Thus $x \in X, N_i(x) \subseteq X$ and $x \in Y, N_i(x) \subseteq Y$ which implies $x \in \underline{R}_i(X)$ and $x \in \underline{R}_i(Y)$. Then, $x \in \underline{R}_i(X) \cap \underline{R}_i(Y)$ and thus, $\underline{R}_i(X \cap Y) \subseteq \underline{R}_i(X) \cap \underline{R}_i(Y)$. Hence, $\underline{R}_i(X \cap Y) = \underline{R}_i(X) \cap \underline{R}_i(Y)$. By the same way, $\overline{R}_i(X \cup Y) = \overline{R}_i(X) \cup \overline{R}_i(Y)$.

(vi) First, it is clear that $\underline{R}_i(\underline{R}_i(X)) \subseteq \underline{R}_i(X)$, (By (ii)). Now, we will prove that $\underline{R}_i(X) \subseteq \underline{R}_i(\underline{R}_i(X))$: Let $x \in \underline{R}_i(X)$,then $N_i(x) \subseteq X$, we must prove that$N_i(x) \subseteq \underline{R}_i(X)$. Let $z \in N_i(x)$, then (By lemma 3.2) $N_i(z) \subseteq N_i(x)$, which implies $N_i(z) \subseteq X$. Thus $z \in \underline{R}_i(X)$ and this means that $N_i(x) \subseteq \underline{R}_i(X)$. Then, $\underline{R}_i(X) \subseteq \underline{R}_i(\underline{R}_i(X))$ and hence $\underline{R}_i(\underline{R}_i(X)) = \underline{R}_i(X)$.

(vii) First, it is clear that $\overline{R}_i(X) \subseteq \overline{R}_i(\overline{R}_i(X))$, (By (ii)). Now, we will prove that $\overline{R}_i(\overline{R}_i(X)) \subseteq \overline{R}_i(X)$ as follow: Let $y \in \overline{R}_i(\overline{R}_i(X))$. Then $y \in \{x \in U\colon N_i(x) \cap \overline{R}_i(X) \neq \phi\}$ which means that $N_i(y) \cap \overline{R}_i(X) \neq \phi$. Thus $\exists z$ such that $z \in N_i(y)$ and $z \in \overline{R}_i(X)$ which implies $N_i\ (z) \subseteq N_i(y)$ and $N_i(z) \cap X \neq \phi$.

Then $N_i(y) \cap X \neq \phi$ and this means that $y \in \overline{R}_i(X)$.Thus $\overline{R}_i(\overline{R}_i(X)) \subseteq \overline{R}_i(X)$. Hence, $\overline{R}_i(\overline{R}_i(X)) = \overline{R}_i(X)$. □

Remark 3.6. In the above proposition, the converse of property (iv)is not true in general as the following example illustrates.

Example 3.2. Let $U = \{a, b, c, d, e\}$ and R be a binary relation on U, where $R = \{(a, a), (a, d), (b, a), (b, c), (c, c), (d, e), (e, b), (e, d)\}$.Then $aR = \{a, d\}, bR = \{a, c\}, cR = \{c\}, dR = \{e\}$and $eR = \{b, d\}$. Also, $Ra = \{a, b\}, Rb = \{e\}, Rc = \{b, c\}, Rd = \{a, e\}$ and $Re = \{d\}$, which implies $N_r(a) = \{a\}, N_r(b) = \{b, d\}, N_r(c) = \{c\}, N_r(d) = \{d\}$ and $N_r(e) = \{e\}$. Also, $N_l(a) = \{a\}, N_l(b) = \{b\}, N_l(c) = \{b, c\}, N_l(d) = \{d\}$ and $N_l(e) = \{e\}$. Now, let $X = \{a, b, c\}$ and $Y = \{c, d\}$. Then $X \cup Y = \{a, b, c, d\}, X \cap Y = \{c\}$ and thus $\underline{R}_r(X) = \{a, c\}, \underline{R}_r(Y) = \{c, d\}, \underline{R}_r(X \cup Y) = \{a, b, c, d\}, \overline{R}_r(X) = \{a, b, c\}, \overline{R}_r(Y) = \{b, c, d\}$ and $\overline{R}_r(X \cap Y) = \{c\}$. Clearly, $\underline{R}_r(X \cup Y) \neq \underline{R}_r(X) \cup \underline{R}_r(Y)$ and

$\overline{R}_r(X \cap Y) \neq \overline{R}_r(X) \cap \overline{R}_r(X)$. Also if $A = \{a, c, d\}$ and $B = \{b, d\}$, then $A \cap B = \{d\}$ and $A \cup B = \{a, b, c, d\}$. Thus $\underline{R}_l(A) = \{a, d\}, \underline{R}_l(B) = \{b, d\}, \underline{R}_l(A \cup B) = \{a, b, c, d\}, \overline{R}_l(A) = \{a, c, d\}$, and $\overline{R}_l(A \cap B) = \{d\}$. Clearly, $\underline{R}_l(A \cup B) \neq \underline{R}_l(A) \cup \underline{R}_l(B)$ and $\overline{R}_l(A \cap B) \neq \overline{R}_l(A) \cap \overline{R}_l(B)$.

Remark 3.7. Although many authors have been introduced sorts to generalize Pawlak approximation space, but most of them could not applied and satisfied the all properties of Pawlak approximations see: [13]-[16] and [18]-[23], but our approach satisfied most of properties of Pawlak approximations. So, we can say that our approach is the actual generalizations of Pawlak approximation space. The following example illustrates the comparison between our approaches and Yao's method, see [19].

Example 3.3. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, a), (c, d), (d, b)\}$ be a binary relation on U. Then $aR = \{a\}, bR = \{c\}, cR = \{a, d\}$ and $dR = \{b\}$. Also $Ra = \{a, c\}$, $Rb = \{d\}, Rc = \{b\}$and $Rd = \{c\}$ thus we get $N_r(a) = \{a\}, N_r(b) = \{b\}, N_r(c) = \{c\}$ and $N_r(d) = \{a, d\}$. Also $N_l(a) = \{a, c\}, N_l(b) = \{b\}, N_l(c) = \{c\}$ and $N_l(d) = \{d\}$. Yao (see [19]-[21]) defines the approximations of any subset $X \subseteq U$ as follow: $\underline{apr}(X) = \{x \in U : xR \subseteq X\}$ and $\overline{apr}(X) = \{x \in U : xR \cap X \neq \phi\}$.The following table shows the differences between our approach ***GAS*** and Yao method:

From the above table, we can notice that:

(i) $\underline{apr}(X) \nsubseteq X \nsubseteq \overline{apr}(X)$, for some subsets e.g. the subsets $\{b\}$ and $\{c\}$ but in our approaches $\underline{R}_i(X) \subseteq X \subseteq \overline{R}_i(X)$ for any $X \subseteq U$.

(ii) The subsets $\{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, d\}$ and $\{a, c, d\}$ are right-exact and also$\{b\}, \{d\}$, $\{a, c\}, \{b, d\}, \{a, b, c\}$ and $\{a, c, d\}$ are left-exact in our approaches. But they are rough sets in Yao's approaches.

(iii)The subsets $\{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, d\}$ and $\{a, c, d\}$ has accuracy $\mu_r(X) = 1$ and the subsets$\{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}$ and $\{a, c, d\}$ has accuracy $\mu_l(X) = 1$, that is, in our approaches the accuracy ratio be 100%. But in Yao the accuracy is less than 1.

4. New Approach for Closure Operators Derived from Relations

Pawlak in [12], noted that the approximation space $A = (U, R)$ defines an uniquely topology that represents a quasi-discrete topology (every open set is also closed). According to this remark, Pawlak opens the way for using the

	Yao Method		Our Method			
$P(U)$	$\underline{apr}(X)$	$\overline{apr}(X)$	$\underline{R}_r(X)$	$\overline{R}_r(X)$	$\underline{R}_l(X)$	$\overline{R}_l(X)$
$\{a\}$	$\{a\}$	$\{a,c\}$	$\{a\}$	$\{a,d\}$	ϕ	$\{a,c\}$
$\{b\}$	$\{d\}$	$\{d\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\{b\}$	ϕ	$\{c\}$	$\{c\}$	$\{c\}$	$\{a,c\}$
$\{d\}$	ϕ	ϕ	ϕ	$\{d\}$	$\{d\}$	$\{d\}$
$\{a,b\}$	$\{a,d\}$	$\{a,c,d\}$	$\{a,b\}$	$\{a,b,d\}$	$\{b\}$	$\{a,b\}$
$\{a,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,c\}$	$\{a,c,d\}$	$\{a,c\}$	$\{a,c\}$
$\{a,d\}$	$\{a,c\}$	$\{a,c\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$	$\{a,d\}$
$\{b,c\}$	$\{b,d\}$	$\{b,d\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{b,d\}$	$\{d\}$	$\{c,d\}$	$\{b\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
$\{c,d\}$	$\{b\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{a,c,d\}$
$\{a,b,c\}$	$\{a,b,d\}$	U	$\{a,b\}$	U	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{b,d\}$	$\{a,b,d\}$
$\{a,c,d\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
$\{b,c,d\}$	$\{b,d\}$	U	$\{b,c\}$	U	$\{b,c,d\}$	U
U	U	U	U	U	U	U
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

Table 1

topology structure on rough set theory for measuring exactness or roughness of sets. The use of the topology structure can helps for further developments in theoretical and application of rough sets.

The present section is divided to subsections; the first section is devoted to introduce a new definition for the closure and interior operators using general binary relation depending on Galton definition, see [6]. Moreover, we are study the topological structures and the properties of the operators. In second section, we introduce new definitions for closure operators induced from binary relations. Topological structures and their properties are investigated. In addition, the introduced closure spaces represent topological spaces without any conditions on the relation. Comparisons between Galton approaches (see [6]) and our approach are investigated.

4.1. Topological Structures for Galton Definition

Galton in [6] introduced *closure operator* derived from binary relation using the after set concept. Moreover, he was put some conditions on the relation that (The relation must be transitive) in order to closure space is topological space and the introduced closure operator being topological closure. Galton, also in [6], used only the after set concepts to define the closure operator. The present subsection is devoted to introduce some topological properties of Galton definition. Moreover, we introduce new definition to closure operator derived from relation depending on fore set concepts. The properties of operator are examined.

Definition 4.1. Let U be any set and let $R \subseteq U \times U$ be any binary relation on U. The relation R gives rise to a closure operation $\boldsymbol{cl}_R$ on U as follow:

$$\boldsymbol{cl}_R(X) = X \cup \{x \in U : \exists y \in X \wedge xRy\}.$$

We can reformulate the above definition as follow:

$$\boldsymbol{cl}_R(X) = X \cup \{x \in U : xR \cap X \neq \phi\}.$$

According to Galton (see [6]) definition, A.A. Allam et al in [1] give another closure operator on U using the concept of fore set as follow.

Definition 4.2. Let U be any set and let $R \subseteq U \times U$ be any binary relation on U. The relation R gives rise to a closure operation $\boldsymbol{cl}^*_R$ on U as follow:

$$\boldsymbol{cl}^*_R(X) = X \cup \{x \in U : Rx \cap X \neq \phi\}.$$

Lemma 4.1. *Let U be any set and let $R \subseteq U \times U$ be any binary relation on U. Then, $(U, \boldsymbol{cl}^*_R)$ is a closure space.*

Proof. Straightforward. □

Lemma 4.2. *The closure space $(U, \boldsymbol{cl}^*_R)$ is topological if and only if R is transitive.*

Proof. First suppose that R is transitive, we must prove that $\boldsymbol{cl}^*_R(\boldsymbol{cl}^*_R(X)) \subseteq \boldsymbol{cl}^*_R(X)$ as follow:

Let $x \in \boldsymbol{cl}^*_R(\boldsymbol{cl}^*_R(X))$, then $x \in \boldsymbol{cl}^*_R(X) or Rx \cap \boldsymbol{cl}^*_R(X) \neq \phi$ which means that $\exists y \in U$, such that yRx and $y \in \boldsymbol{cl}^*_R(X)$. Then yRx and $y \in X$ or yRx

and $Ry \cap X \neq \phi$ and this implies to $Rx \cap X \neq \phi$ or yRx and $\exists z \in U$, such that $z \in X$ and $z \in Ry$. Thus, $x \in \boldsymbol{cl}^*_R(X)$or zRy and yRx such that $z \in X$. By transitivity, zRx and this means that $Rx \cap X \neq \phi$.Thus $x \in \boldsymbol{cl}^*_R(X)$ and this implies $\boldsymbol{cl}^*_R(\boldsymbol{cl}^*_R(X)) \subseteq \boldsymbol{cl}^*_R(X)$. Conversely, let $(U, \boldsymbol{cl}^*_R)$ is topological and let $x, y \in U$ such that xRy and yRz.Then $x \in Ry$ and $y \in Rz$ which implies $x \in \boldsymbol{cl}^*_R(\{y\})$ and $y \in \boldsymbol{cl}^*_R(\{z\})$. Thus $\boldsymbol{cl}^*_R(\{y\}) \subseteq \boldsymbol{cl}^*_R(\boldsymbol{cl}^*_R(\{z\}))$. Since $(U, \boldsymbol{cl}^*_R)$ is topological, then $\boldsymbol{cl}^*_R(\boldsymbol{cl}^*_R(\{z\})) = \boldsymbol{cl}^*_R(\{z\})$. Hence $x \in \boldsymbol{cl}^*_R(\{y\})$ this means that xRz and then R is transitive. □

Lemma 4.3. *In any closure space $(U, \boldsymbol{cl}_R)$ (resp. closure space $(U, \boldsymbol{cl}^*_R)$): For any subsets $X_k \subseteq U, \forall k \in K$; $\boldsymbol{cl}_R(\bigcap_{k \in K} X_k) \subseteq \bigcap_{k \in K} \boldsymbol{cl}_R(X_k)$ (resp. $\boldsymbol{cl}^*_R(\bigcap_{k \in K} X_k) \subseteq \bigcap_{k \in K} \boldsymbol{cl}^*_R(X_k)$).*

Proof. From the definition of closure operators, the proof is obvious. □

Now, according to the closure operator $\boldsymbol{cl}^*_R$, we can define an interior operation on any set generating by relations as follow.

Definition 4.3. Let U be any set and let $R \subseteq U \times U$ be any binary relation on U. Then the interior operation corresponding to $\boldsymbol{cl}^*_R$ is given by
$int^*_R(X) = [\boldsymbol{cl}^*_R(X^c)]^c = [X^c \cup \{x \in U \colon Rx \cap X^c \neq \phi\}]^c$
$= X \cap \{x \in U \colon Rx \cap X^c \neq \phi\}^c = X \cap \{x \in U : Rx \cap X^c = \phi\} = \{x \in X : Rx \subseteq X\}$.

Remark 4.1. Galton (see [6]) introduced another interior operator on any set generating by relations using the concept of after set as follow: $int_R(X) = \{x \in X : xR \subseteq X\}$.

Moreover, using the concept of closure operator, we can generate two topologies by using the binary relation on any set as the following discussion.

Definition 4.4. Let the closure space $(U, \boldsymbol{cl}_R)$ (resp.the closure space $(U, \boldsymbol{cl}^*_R)$), and let $X \subseteq U$. Then:

(i) The subset $X \subseteq U$ is called R-*closed* (resp. R^*-*closed*) if $X = \boldsymbol{cl}_R(X)$(resp. $X = \boldsymbol{cl}^*_R(X)$).

(ii) The subset $X \subseteq U$ is called $R-$ *open* (resp. R^*- *open*) if $X = int_R(X)$ (resp. $X = int^*_R(X)$).

Lemma 4.4. *Let the closure space $(U, \boldsymbol{cl}_R)$ (resp.the closure space $(U, \boldsymbol{cl}^*_R)$), then the intersection of any family of $R-$ closed (resp. R^*- closed) sets is also $R-$ closed (resp. R^*- closed) set.*

Proof. Let $X_k \subseteq U, \forall k \in K$ are $R-$ closed .Then $X_k = \boldsymbol{cl}_R(X_k), \forall k \in K$ which implies $\bigcap\limits_{k\in K} X_k = \bigcap\limits_{k\in K} \boldsymbol{cl}_R(X_k)$. But $\boldsymbol{cl}_R(\bigcap\limits_{k\in K} X_k) \subseteq \bigcap\limits_{k\in K} \boldsymbol{cl}_R(X_k)$, that is $\boldsymbol{cl}_R(\bigcap\limits_{k\in K} X_k) \subseteq \bigcap\limits_{k\in K} X_k$ and since $\bigcap\limits_{k\in K} X_k \subseteq \boldsymbol{cl}_R(\bigcap\limits_{k\in K} X_k)$ .Then $\bigcap\limits_{k\in K} X_k = \boldsymbol{cl}_R(\bigcap\limits_{k\in K} X_k)$ and this means that $\bigcap\limits_{k\in K} X_k$ is $R-$ closed. By the same way, we can prove that, if $X_k \subseteq U, \forall k \in K$ is R^*- closed. Then $\bigcap\limits_{k\in K} X_k$ is alsoR^*- closed. □

Proposition 4.1. *Let the closure space* $(U, \boldsymbol{cl}_R)$ *(resp.the closure space* $(U, \boldsymbol{cl}^*_R)$*), then the class* $\boldsymbol{\tau}_R = \{X \subseteq U : X^c \text{ is } R\text{−closed}\}$ *(resp.*$\boldsymbol{\tau}^*_R = \{X \subseteq U : X^c \text{ is } R^*\text{− closed}\}$*) represents a topology on* U*.*

Proof. First, we shall prove that the class $\boldsymbol{\tau}_R$ forms a topology on U and similarly the class$\boldsymbol{\tau}^*_R$:

(T1) Clearly, $U = \boldsymbol{cl}_R(U)$ and $\phi = \boldsymbol{cl}_R(\phi)$. Then U and ϕ are $R-$closed and thus U and $\phi \in \boldsymbol{\tau}_R$.

(T2) Let $A, B \in \boldsymbol{\tau}_R$. Then $A^c = \boldsymbol{cl}_R(A^c)$ and $B^c = \boldsymbol{cl}_R(B^c)$, and thus $(A\cap B)^c = A^c \cup B^c = \boldsymbol{cl}_R(A^c) \cup \boldsymbol{cl}_R(B^c) = \boldsymbol{cl}_R(A^c \cup B^c) = \boldsymbol{cl}_R(A \cap B)^c$ which implies $A \cap B \in \boldsymbol{\tau}_R$.

(T3) Let $\{A_i : i \in I\} \subseteq \boldsymbol{\tau}_R$, Then $\forall i \in I$, $A^c_i = \boldsymbol{cl}_R(A^c_i)$, and thus $(\bigcup\limits_{i\in I} A_i)^c = \bigcap\limits_{i\in I} A^c_i = \bigcap\limits_{i\in I} \boldsymbol{cl}_R(A^c_i)$. Since A^c_i are $R-$closed, then $\boldsymbol{cl}_R(\bigcap\limits_{i\in I} A^c_i) = \bigcap\limits_{i\in I} \boldsymbol{cl}_R(A^c_i)$. Hence $(\bigcup\limits_{i\in I} A_i)^c = \boldsymbol{cl}_R\ (\bigcap\limits_{i\in I} A^c_i) = \boldsymbol{cl}_R\ (\bigcup\limits_{i\in I} A_i)^c$ and this implies $\bigcup\limits_{i\in I} A_i \in \boldsymbol{\tau}_R$. □

Corollary 4.1. *Let the closure space*$(U, \boldsymbol{cl}_R)$ *(resp.the closure space* $(U, \boldsymbol{cl}^*_R)$*), and let* $X \subseteq U$*. Then:*

(i) X *is* R^*-*open if and only if* $X \in \boldsymbol{\tau}^*_R$*. (ii)* X *is* $R-$*open if and only if* $X \in \boldsymbol{\tau}_R$*.*

Proof. Straightforward. □

Example 4.1. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (c, b), (c, d), (d, c), (d, a)\}$.Then $aR = \{a\}, bR = \{b\}, cR = \{b, d\}$ and $dR = \{a, c\}$. Also, $Ra = \{a, d\}, Rb = \{b, c\}, Rc = \{d\}$and $Rd = \{c\}$.Thus, $\boldsymbol{\tau}_R = \{U, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\boldsymbol{\tau}^*_R = \{U, \phi, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Proposition 4.2. *Let the closure space* $(U, \boldsymbol{cl}_R)$ *(resp.the closure space* $(U, \boldsymbol{cl}^*_R)$*), and let* $X \subseteq U$*. Then:*

(i) X is $R-$open if and only if $\forall x \in X, xR \subseteq X$.

(ii) X is R^*-open if and only if $\forall x \in X, Rx \subseteq X$.

Proof. (i) First, let X is $R-$open. Then X^c is $R-$closed which means that $\forall x \in X^c$, $xR \cap X^c \neq \phi$. Thus if $x \in X, xR \cap X^c = \phi$ and this implies $xR \subseteq X, \forall x \in X$.Now, if $\forall x \in X, xR \subseteq X$. Then $\forall x \in X$, $xR \cap X^c = \phi$ which means that $\forall x \in X^c$,$xR \cap X^c \neq \phi$. Thus, X^c is $R-$closed and then X is $R-$open.

(ii)By the same way as in (i). □

Example 4.2. In Example 4.1, it is clear that for each $X \in \boldsymbol{\tau}_R$ (resp.for each $X \in \boldsymbol{\tau}^*_R$), $\forall x \in X, xR \subseteq X$ (resp.$\forall x \in X, Rx \subseteq X$).

Remark 4.2. According to the above proposition, we can redefine the topologies $\boldsymbol{\tau}_R$ and $\boldsymbol{\tau}^*_R$ as follow: $\boldsymbol{\tau}_R = \{X \subseteq U : \forall x \in X, xR \subseteq X\}$ and $\boldsymbol{\tau}^*_R = \{X \subseteq U : \forall x \in X, Rx \subseteq X\}$.

Proposition 4.3. *Let the non empty set* U *and* R *be any relation on* U*. Then the class* $\boldsymbol{\tau}_R$ *is the dual topology of the class* $\boldsymbol{\tau}^*_R$*.*

Proof. We must prove that $X \in \boldsymbol{\tau}_R$ if and only if $X^c \in \boldsymbol{\tau}^*_R$ as follow:

First, let $X \in \boldsymbol{\tau}_R$. Then, $\forall x \in X, xR \subseteq X$. If $x \in X^c$, then the fore set of element x is given by: $Rx = \{y \in U : yRx\}$.

Thus there are two different cases are:

Case (1): If $Rx \cap X \neq \phi$, then $\exists z \in Rx$ and $z \in X$ which implies zRx and $z \in X$. Thus $x \in zR$ and $z \in X$. This means that $x \in X$ which is a contradiction to assumption that $x \in X^c$. Hence, the following case is true:

Case (2): $Rx \subseteq X^c, \forall x \in X^c$, then $X^c \in \boldsymbol{\tau}^*_R$.

Conversely, by the same way, we can prove that: if $X^c \in \boldsymbol{\tau}^*_R$, then $X \in \boldsymbol{\tau}_R$. □

Example 4.3. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (b, c), (c, d), (d, a)\}$. Then $aR = \{a\}, bR = \{b, c\}, cR = \{d\}$ and $dR = \{a\}$. Also, $Ra = \{a, d\}, Rb = \{b\}, Rc = \{b\}$and $Rd = \{c\}$. Thus, $\boldsymbol{\tau}_R = \{U, \phi, \{a\}, \{a, d\}, \{a, c, d\}\}$ and $\boldsymbol{\tau}^*_R = \{U, \phi, \{b\}, \{b, c\}, \{b, c, d\}\}$.

4.2. New Approaches to Generate Closure Operators from Relations

The present subsection is devoted to introduce new closure operators induced from binary relation using coverings rough sets by relations. We generate closure spaces from relations which represent topological spaces without any conditions on the relation. Moreover, the properties of operators are investigated.

Definition 4.5. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, then it clear that $\forall i = r, l$ the class $\boldsymbol{E}_i$ of all $i-$elementary sets closed under finite intersection and arbitrary union. So, $\boldsymbol{E}_i$ represent two different topologies on U and we can say that $\boldsymbol{T}_i = (U, \boldsymbol{E}_i)$ is a *topological approximation space*, in briefly, $\boldsymbol{TAS}$.

Proposition 4.4. *Let $\boldsymbol{T}_i = (U, \boldsymbol{E}_i)$ be $\boldsymbol{TAS}$, then the subset $X \subseteq U$ is $i-$open set if and only if $\forall x \in X$, $N_i(x) \subseteq X$.*

Proof. First, it is clear that if $X \subseteq U$ is open set (i.e. $X \in \boldsymbol{E}_i$), then $\forall x \in X, N_i(x) \subseteq X, \forall i = r, l$. Now, suppose that $\forall x \in X, N_i(x) \subseteq X$. Then $X = \bigcup N_i(x), \forall x \in X$ (Since $\forall x \in U$, $x \in N_i(x)$ and R coverings U). □

Definition 4.6. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, it is clear that the approximations $\underline{R}_i(X)$ and $\overline{R}_i(X)$ represent the interior and the closure of a subset $X \subseteq U$ in the topological approximation space $\boldsymbol{T}_i = (U, \boldsymbol{E}_i)$,respectively . Then we can define *closure and interior operators* as follows:

$$\boldsymbol{cl}_i(X) = \overline{R}_i(X) = \{x \in U : N_i(x) \cap X \neq \phi\} \text{and}$$
$$\boldsymbol{int}_i(X) = \underline{R}_i(X) = \{x \in X : N_i(x) \subseteq X\}, \forall i = r, l.$$

Definition 4.7. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, then the subset $X \subseteq U$ is $i-$closed (resp. $i-$open) set if $\boldsymbol{cl}_i(X) = X$ (resp.$\boldsymbol{int}_i(X) = X$), $\forall i = r, l$.

Proposition 4.5. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, then $\forall i = r, l$ the space $(U, \boldsymbol{cl}_i)$ represent closure spaces.*

Proof. Straightforward. □

Example 4.4. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (a, b), (b, b), (c, d), (d, a)\}$ be a binary relation on U. Then $aR = \{a, b\}, bR = \{b, c\}, cR = \{d\}$ and $dR = \{a\}$. Also $Ra = \{a, d\}$, $Rb = \{a, b\}$, $Rc = \{b\}$ and $Rd = \{c\}$.Thus $N_r(a) = \{a\}, N_r(b) = \{b\}, N_r(c) = \{b, c\}$ and $N_r(d) = \{d\}$. Also, $N_l(a) = \{a\}, N_l(b) = \{b\}, N_l(c) = \{c\}$and $N_l(d) = \{a, d\}$. Then the $r-$closed sets are given by: $X, \phi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}$. Also, the $l-$closed sets are given by: $X, \phi, \{b\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

Proposition 4.6. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, then the space $(U, \boldsymbol{cl}_i)$ represents topological space.*

Proof. We must prove that $\boldsymbol{cl}_i(\boldsymbol{cl}_i(X)) = \boldsymbol{cl}_i(X), \forall X \subseteq U$ as follow:

First, it is clear that $\boldsymbol{cl}_i(X) \subseteq \boldsymbol{cl}_i(\boldsymbol{cl}_i(X))$. Now, let $x \in \boldsymbol{cl}_i(\boldsymbol{cl}_i(X))$. Then $N_i(x) \cap \boldsymbol{cl}_i(X) \neq \phi$ and this implies $\exists y \in N_i(x)$ and $y \in \boldsymbol{cl}_i(X)$. Thus $N_i(y) \subseteq N_i(x)$ and $N_i(y) \cap X \neq \phi$, which means that $N_i(x) \cap X \neq \phi$. Hence, $x \in \boldsymbol{cl}_i(X)$ and then $\boldsymbol{cl}_i(\boldsymbol{cl}_i(X)) = \boldsymbol{cl}_i(X), \forall X \subseteq U$. □

According to the above proposition, we can generate two different topologies on any set as follow:

Definition 4.8. Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$. Then the class $\boldsymbol{\tau}_i = \{X \subseteq U : X^c$ is $i-$closed$, \forall i = r, l$ represents a topology on U.

Example 4.5. In Example4.4, it is clear that, both of $r-$closed sets and $l-$closed sets represent two different topologies on U.

Lemma 4.5. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, and let $X \subseteq U$. Then $\boldsymbol{int}_i(X) = (\boldsymbol{cl}_i(X^c))^c$.*

Proof. Let $x \in (\boldsymbol{cl}_i(X^c))^c = \{x \in U \colon N_i(x) \cap N_i \neq \phi\}^c$. Then $x \notin X^c$ and $N_i \cap X^c = \phi$, which means that $x \in X$ and $N_ic\ (x) \subseteq X$. Thus, $x \in \boldsymbol{int}_i(X)$. Similarly, if $x \in \boldsymbol{int}_i(X)$, then $x \in (\boldsymbol{cl}_i(X^c))^c$. Hence, $\boldsymbol{int}_i(X) = (\boldsymbol{cl}_i(X^c))^c, \forall i = r, l$. □

Corollary 4.2. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, and let $X \subseteq U$. Then, the subset X is $i-$open if and only if $X^c$is $i-$closed.*

Proposition 4.7. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, and let $X, Y \subseteq U$. Then, the operator $\boldsymbol{int}_i$ satisfies the following properties:*

(i) $\boldsymbol{int}_i(X)$ is $i-$open set.
(ii)$\boldsymbol{int}_i(X) \subseteq X$.
(iii)$\boldsymbol{int}_i(U) = U$.
(iv)$\boldsymbol{int}_i(X \cap Y) = \boldsymbol{int}_i(X) \cap \boldsymbol{int}_i(Y)$.
(v)$\boldsymbol{int}_i(\boldsymbol{int}_i(X)) = \boldsymbol{int}_i(X)$.

Proof. Obvious. □

Proposition 4.8. *Let $\boldsymbol{G} = (U, R)$ be a $\boldsymbol{GAS}$, then the topologies $\boldsymbol{\tau}_i$ are exactly the topologies $\boldsymbol{E}_i, \forall i = r, l$.*

Proof. By Proposition4.4 and Corollary4.2, the proof is obvious. □

Example 4.6. In Example 4.1, it is clear that, $\boldsymbol{\tau}_r = \{X, \phi, \{b\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\boldsymbol{\tau}_l = \{X, \phi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$.Thus, $\boldsymbol{\tau}_r = \boldsymbol{E}_r$ and $\boldsymbol{\tau}_l = \boldsymbol{E}_l$.

Remark 4.3. The topology $\boldsymbol{\tau}_r$ and $\boldsymbol{\tau}_l$ are not comparable as the following example illustrates.

Example 4.7. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (b, a), (b, b), (c, c), (c, d), (d, a)\}$ be a binary relation on U. Then $aR = \{a\}, bR = \{a, b\}, cR = \{c, d\}$ and $dR = \{c, d\}$. Also, $Ra = \{a, b, d\}, Rb = \{b\}, Rc = \{c\}$ and $Rd = \{c\}$.Thus $N_r(a) = \{a\}, N_r(b) = \{a, b\}, N_r(c) = \{c, d\}$ and $N_r(d) = \{c, d\}$ and also $N_l(a) = \{a, b, d\}, N_l(b) = \{b\}, N_l(c) = \{c\}$ and $N_l(d) = \{a, b, d\}$. Then $\boldsymbol{\tau}_r = \{X, \phi, \{a\}, \{a, b\}, \{b, d\}, \{a, c, d\}\}$ and $\boldsymbol{\tau}_l = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b, d\}\}$. It is clear that $\{a\} \in \boldsymbol{\tau}_r$ and its complement does not belongs to $\boldsymbol{\tau}_l$, which means that $\boldsymbol{\tau}_r$ is not the dual topology of $\boldsymbol{\tau}_l$.

Remark 4.4. The topologies $\boldsymbol{\tau}_i$ and topologies $\boldsymbol{\tau}_R$ and $\boldsymbol{\tau}^*_R$ that are given in Subsection 4.2 are not comparable as the following example illustrates.

Example 4.8. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (c, c), (c, d), (d, a)\}$. Then $\boldsymbol{\tau}_R = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$ and $\boldsymbol{\tau}^*_R = \{U, \phi, \{b\}, \{c\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Also $\boldsymbol{\tau}_r = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\boldsymbol{\tau}_l = \{U, \phi, \{b\}, \{c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}\}$. It is clear that:

(i) $\{a, d\}$ and $\{a, b, d\} \in \boldsymbol{\tau}_R$, but $\{a, d\}$ and $\{a, b, d\} \notin \boldsymbol{\tau}^*_R$. Also,$\{c, d\}$ and $\{b, c, d\} \in \boldsymbol{\tau}^*_R$, but $\{c, d\}$ and $\{b, c, d\} \notin \boldsymbol{\tau}_R$.

(ii) $\{c, d\}$ and $\{b, c, d\} \in \boldsymbol{\tau}_r$, but $\{c, d\}$ and $\{b, c, d\} \notin \boldsymbol{\tau}_l$. Also, $\{a, d\}$ and $\{a, b, d\} \in \boldsymbol{\tau}_l$, but $\{a, d\}$ and $\{a, b, d\} \notin \boldsymbol{\tau}_l$.

5. Conclusion

In this paper, we further investigate the covering rough sets based on neighborhoods by approximation operations. The used technique depends basically on a general binary relation to define the neighborhoods and approximation operations. Our approach represent a generalization of Pawlak definition, see [12]. In addition, we give several methods to define a closure operators induced from relations. Moreover, the properties of introduced operations, and their connections are examined. Also, we generate new methods for computing the topology from general binary relation. In light of this, one may investigate further coverings with some links to topology that emphasize and generalize the application field.

Acknowledgments

This research work was supported by Grant No. 3021/1434 from the deanship of the scientific research at Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia.

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