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ON THE GENERIC RANK OF LINEAR SPANS OF TANGENT VECTORS

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Abstract: Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. For each $P \in \mathbb{P}^r$ the X-rank $r_X(P)$ is the minimal cardinality of a set of X whose linear span contains P. For each $O \in X_{reg}$ let $\alpha(X, O)$ be the maximal integer $r_X(P)$ for some P in the tangent space of X at O. Let $\alpha(X)_{gen}$ be the integer $\alpha(X, O)$ for a general $O \in X$. Let $\beta(X)$ be the maximum of all $\alpha(X, O)$, $O \in X_{reg}$. The integer $\alpha(X)_{gen}$ is useful to get an upper bound for the integers $r_X(P)$, $P \in \mathbb{P}^r$. We prove that $\alpha(X)_{gen} = \beta(X)$ when X is the degree $d \geq 4$ Veronese embedding of a cubic hypersurface.

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Key Words: X-rank, cubic hypersurface, tangent developable, generic tangent space

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. For each $O \in X_{reg}$ let $T_OX \subset \mathbb{P}^r$ be the Zariski tangent space of X at O (it is a linear subspace of dimension $\dim(X)$). Let $\tau(X) \subseteq \mathbb{P}^r$ denote the closure of the union of all T_OX, $O \in X_{reg}$. The variety $\tau(X)$ is the tangent developable of X. For each $P \in \mathbb{P}^r$ the X-rank $r_X(P)$ of P is the minimal cardinality of a subset $S \subset X$ such that

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$P \in \langle S\rangle$. For each $O \in X_{reg}$ let $\alpha(X, O)$ be the the maximum of all integers $r_X(P)$ for some $P \in T_O X$. The function $\alpha(X, O)$ is constant in a non-empty open subset of X_{reg} and we will call the *generic tangent rank* of X this value. Let $\alpha_{gen}(X)$ denote the generic tangent rank of X. As essentially shown in [1] and [2] we have $r_X(P) \le k\alpha_{gen}(X)$ for all $P \in \mathbb{P}^r$, where k is the minimal integer such that the k-secant variety of X fill in $\mathbb{P}^r$. This is our motivation for the study this concept in specific examples. In summary: we fix a general $O \in X_{reg}$, but then we look at the worst points of $T_O X$, i.e. the ones with higher X-rank. The integer $\alpha(X)_{gen}$ is at least the X-rank of the general point of the tangent developable of X, but it may be higher in some cases. Let $\beta(X)$ be the maximum of all integers $\alpha(X, O)$, $O \in X_{reg}$. If X is smooth, then $\beta(X)$ is the maximal X-rank of a point of the tangent developable of X. The game is now to handle low cardinality sets $A \subset X$ such that $h^1(\mathcal{I}_{v\cup A}(1)) > h^1(\mathcal{I}_{\{O\}\cup A}(1))$, where O is a general point of X and v is an arbitrary degree two connected subscheme of X with $v_{red} = \{O\}$ (a tangent vector of X at its smooth point O). This is rather easy because the zero-dimensional scheme $v\cup A$ is almost reduced: it has a unique non-reduced connected component and this component has only degree two. Moreover, it is sufficient to check general O. It is easy to do this game for several embeddings of several surfaces (Hirzebruch surfaces, the plane blown up in a few points and so on). Here we study cubic hypersurfaces and prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$, $n \ge 3$, be a smooth cubic hypersurface. Fix an integer $d \ge 4$ and see X embedded in $\mathbb{P}^r$, $r := \binom{d+n}{3} - \binom{d+n-3}{3}$, by the complete linear system $|\mathcal{O}_X(d)|$. Then $\alpha(X)_{gen} = \beta(X) = 3d - 2$ for all $O \in X$.*

We work over an algebraically closed base field with characteristic zero.

2. The Proofs

For any projective variety X, any effective Cartier divisor D of X and any closed subscheme $Z \subset X$ let $\mathrm{Res}_D(Z)$ be the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$. If Z is zero-dimensional, then $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. For each line bundle $\mathcal{L}$ on X we have an exact sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L} \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z\cap D, D} \otimes \mathcal{L}|D \to 0 \qquad (1)$$

(usually called the residual exact sequence).

Lemma 1. *Let $v \subset \mathbb{P}^3$ be a connected degree two zero-dimensional scheme. Set $\{O\} := v_{red}$. Let $E \subset \mathbb{P}^3$ be a finite subset such that $\sharp(E) \le 3d-3$, $h^1(\mathcal{I}_{\{O\}\cup E}(d)) = 0$ and $h^1(\mathcal{I}_{v\cup E}(d)) > 0$. Then either there is a line $J \subset \mathbb{P}^3$ such that $v \subset J$ and $\deg(J \cap (v \cup E)) \ge d+2$ or there is a reduced conic T such that $v \subset T$ and $\deg(T \cap (v \cup E)) \ge 2d+2$.*

Proof. Set $F_0 := v \cup E$. Let $H_1 \subset \mathbb{P}^3$ be a plane such that $a_1 := \deg(H_1 \cap F_0)$ is maximal and set $F_1 := \mathrm{Res}_{H_1}(F_0)$. Define recursively for all integers $i \ge 2$ the plane H_i, the integer a_i and the scheme F_i in the following way. Let H_i be a plane such that $a_i := \deg(H_i \cap F_{i-1})$ is maximal and set $F_i := \mathrm{Res}_{H_i}(F_{i-1})$. Since $F_i \subseteq F_{i-1}$ for all i, the sequence $\{a_i\}_{i\ge 1}$ is non-increasing. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, if $a_i \le 2$, then $F_i = \emptyset$. Since $\sum_{i\ge 1} \deg(v \cup E) \le 3d-1$, we get $a_{d+1} = 0$ and $F_{d+1} = \emptyset$. For all $i \ge 1$ we have the residual exact sequence

$$0 \to \mathcal{I}_{F_i}(d-i) \to \mathcal{I}_{F_{i-1}}(d-i+1) \to \mathcal{I}_{H_i\cap F_{i-1},H_i}(d-i+1) \to 0 \qquad (2)$$

Since $h^1(\mathcal{I}_{v\cup E}(d)) > 0$, (3) gives the existence of an integer $i \ge 1$ such that $h^1(H_i, \mathcal{I}_{H_i\cap F_{i-1},H_i}(d-i+1)) > 0$. Call e the minimal such an integer. Since $F_{d+1} = \emptyset$, we have $e \le d$.

(a) Assume $e = 1$. Since $\deg(v \cup E) \le 3d-1$, either there is an integer $i \in \{1,2\}$ and a degree i curve $C \subset H_1$ such that $\deg(C \cap (v \cup E)) \ge id+2$ ([6], Remarques at page 116). Since $h^0(C, \mathcal{O}_C(d)) = id+1$, we get $h^1(\mathcal{I}_{C\cap(v\cup E)}(d)) > 0$. Since $h^1(\mathcal{I}_{\{O\}\cup E}(d)) = 0$, we get $v \subset C$. Assume $i = 2$ and that there is no line $J \subset H_1$ with $\deg(J \cap (v \subset E)) \ge d+2$. Since E is a finite set, we get that C is reduced. Hence Lemma 1 is true in this case.

(b) Assume $e \ge 2$. Since $h^1(H_i, \mathcal{I}_{H_i\cap F_{i-1},H_i}(d-i+1)) > 0$, we have $a_e \ge d-i+3$ ([5], Lemma 34). Since $a_i \ge a_e$ for all $i \ge e$, we get $e(d-e+3) \ge 3d-1$. Set $\psi(t) = t(d+3-t)$. Since the function $\psi(t)$ is increasing if $t \le (d+3)/2$ and decreasing $t > (d+3)/2$ and $\psi(3) = \psi(d) = 3d$, we get $e = 2$. Since $a_1 \ge a_2$, $a_1 + a_2 \le 3d-1$, we get $a_2 \le 2(d-1)+1$. Hence there is a line $L \subset H_2$ such that $\deg(L \cap F_1) \ge d+1$. If $\deg(L \cap (v \cup E)) \ge d+2$, then we conclude as in step (a). Hence we may assume $\deg(L \cap (v \cup E)) = d+1$. Set $G_0 := F_0$. Let $M_1 \subset \mathbb{P}^3$ be a plane such that $b_1 := \deg(M_1 \cap F_0)$ is maximal among the planes containing L and set $G_1 := \mathrm{Res}_{M_1}(G_0)$. Define recursively for all integers $i \ge 2$ the plane M_i, the integer b_i and the scheme G_i in the following way. Let M_i be a plane such that $b_i := \deg(M_i \cap G_{i-1})$ is maximal and set $G_i := \mathrm{Res}_{M_i}(G_{i-1})$. We have $b_i \ge b_{i+1}$ for all $i \ge 2$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, we have $b_1 \ge d+2$ and if $b_i \le 2$, then $G_i = \emptyset$. Since $\sum_{i\ge 1} b_i = \deg(v \cup E) \le 3d-1$, we get $b_d = 0$ and $G_d = \emptyset$. For all $i \ge 1$ we have the residual exact sequence

$$0 \to \mathcal{I}_{G_i}(d-i) \to \mathcal{I}_{G_{i-1}}(d-i+1) \to \mathcal{I}_{M_i\cap G_{i-1},H_i}(d-i+1) \to 0 \qquad (3)$$

Since $h^1(\mathcal{I}_{v\cup E}(d)) > 0$, (3) gives the existence of an integer $i \ge 1$ such that $h^1(M_i, \mathcal{I}_{M_i\cap G_{i-1},H_i}(d-i+1)) > 0$. Call c the minimal such an integer. If $c = 1$ we conclude as in step (a). Since $b_1 \ge d+2$, as above we exclude the case $c > 2$. Now assume $c = 2$. Since $b_1 \ge d+2$, we have $b_2 \le 2(d-1)+1$. Hence there is a line $R \subset M_2$ such that $\deg(R\cap G_1) \ge d+1$. If $\deg(R\cap F_0) \ge d+2$, then we are done as in step (a). Assume $\deg(R\cap F_0) = d+1$. We have $R \ne L$, because $\mathrm{Res}_{M_1}(E) \cap L = \emptyset$. Since $\deg((R\cup L)\cap F_0) = 2d+2$, we are done as in step (a) if $R\cap L \ne \emptyset$. Assume $R \cap L = \emptyset$. Let $Q \subset \mathbb{P}^3$ be a general quadric surface containing $R\cup L$. Since $\deg(\mathrm{Res}_Q(F_0)) \le 3d-1-2d-2 \le (d-2)+1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(F_0)}(d-2) = 0$. Since $R\cup L$ is the scheme-theoretic base locus of $|\mathcal{I}_{L\cup R}(2)|$, Q is general and F_0 is curvilinear, we get $Q\cap F_0 = (R\cup L)\cap F_0$ as schemes. Therefore $h^1(Q, \mathcal{I}_{Q\cap F_0}(d)) = 0$. The residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_Q(F_0)}(d-2) \to \mathcal{I}_{F_0}(d) \to \mathcal{I}_{Q\cap F_0,Q}(d) \to 0$$

gives a contradiction. □

Lemma 2. *Fix an integer $d \ge 4$. Let $X \subset \mathbb{P}^3$ be a smooth degree 3 surface. Let $v \subset X$ be a degree 2 connected subscheme of X. Set $O := v_{red}$. Let ρ be the minimal cardinality of a finite subset A of X such that $h^1(\mathcal{I}_{v\cup A}(d)) > h^1(\mathcal{I}_{\{O\}\cup A}(d))$.*

(i) If there is a line $L \subset X$ such that $v \subset L$, then $\rho = d$.

(ii) If v is not contained in a line of X, but v is contained in a reduced conic $D \subset X$, then $\rho = 2d$.

(iii) If v is contained neither in a line $L \subset X$ nor in a reduced conic $D \subset X$, then $\rho = 3d-2$.

Proof. Since $h^1(\mathcal{I}_v(d)) = 0$, the minimality of the integer ρ gives $h^1(\mathcal{I}_{\{O\}\cup A}(d)) = 0$ and $h^1(\mathcal{I}_{v\cup A}(d)) = 1$. Take any $A \subset X$ evincing ρ. Since $X \subset \mathbb{P}^3$ we have $v\cup A \subset \mathbb{P}^3$ and $h^1(\mathbb{P}^3, \mathcal{I}_{v\cup A}(d)) > 0$. Either $\deg(v\cup A) = \rho+2$ (case $O \notin A$) or $\deg(v\cup A) = \rho+1$ (case $O \in A$). Let Θ be the set of all smooth conics contained in X. Let $C \subset X$ be any smooth conic. Since $\omega_X \cong \mathcal{O}_X(-1)$, the adjunction formula $-2 = 2p_a(C) - 2 = \omega_X \cdot C + C^2$ gives that the normal bundle $\mathcal{O}_C(C)$ of C in X has degree zero. Since $C \cong \mathbb{P}^1$, we get $h^1(\mathcal{O}_C(C)) = 0$ and $h^0(\mathcal{O}_C(C)) = 1$. Therefore Θ is smooth and of dimension 1 and the union of all $C \in \Theta$ coves a non-empty open subset of X. Since a flat limit of a family of $C \in \Theta$ must be a conic, we get that for each $P \in X \setminus \Gamma$ the set Θ_P of all $C \in \Theta$ containing P is finite and non-empty. There are only finitely line bundles $\mathcal{O}_X(C)$, $C \in \Theta$, up to isomorphisms. If $\mathcal{O}_X(C) \cong \mathcal{O}_X(C')$ and $C\cap C' \ne \emptyset$, then $C = C'$. Therefore for each $P \in X$ the set Θ_P is finite.

(i) Assume $v \subset L$ for some line L. For each $B \subset L \setminus \{O\}$ we have $h^1(\mathcal{I}_{v\cup B}(d)) = 0$ if $\sharp(B) \le b-1$ and $h^1(\mathcal{I}_{v\cup B}(d)) = 1 + h^1(\mathcal{I}_{\{O\}\cup B}(d))$ if $\sharp(B) \ge d$. Hence $\rho \le d$. Fix any $E \subset \mathbb{P}^3$ such that $\sharp(E) \le d-1$. Since $\deg(v\cup E) \le d+1$, we have $h^1(\mathbb{P}^3, \mathcal{I}_{v\cup E}(d)) = 0$. Hence $\rho \ge d$. Assume $\rho < d$. Since $\rho + 2 \le d+1$, [5], Lemma 34, gives $h^1(\mathbb{P}^3, \mathcal{I}_{v\cup A}(d)) = 0$, a contradiction.

(ii) Assume $v \nsubseteq L$ for any line L, but that O is contained in two different lines, L and R. Since L and R are not tangent at O and X is a smooth surface, we have $v \subset L\cup R$. Since $h^0(L\cup R, \mathcal{O}_{L\cup R}(d)) = 2d+1$ and $\deg(v) = 2$, we have $\rho \le 2d$. Assume $\rho < 2d$. Since $\deg(v\cup A) \le 2d+1$, there is a line $J \subset \mathbb{P}^3$ such that $\sharp(J\cap (v\cup A) \ge d+2$. Since $v\cup A \subset X$ and $d+2 > 3$, Bezout theorem gives $J \subset X$. As in [4], Lemma 5.1, we also get $h^1(\mathcal{I}_{v\cup A}(d)) = h^1(\mathcal{I}_{J\cap(v\cup A)}(d))$. Since $h^1(\mathcal{I}_{\{O\}\cup A}(d)) = 0$, we get $v \subset J$, a contradiction.

(iii) Assume $v \nsubseteq L$ for any line L, but the existence of $C \in \Theta$ such that $v \subset C$. For each set $B \subset C\setminus\{O\})$ with $\sharp(B) = 2d$ we have $h^1(C, \mathcal{I}_{v\cup B}(d)) > 0$. Hence $\rho \le 2d$. Assume $\rho < 2d$. Since $\deg(v\cup A) \le 2d+1$, there is a line $J \subset \mathbb{P}^3$ such that $\sharp(J\cap(v\cup A) \ge d+2$. Since $v\cup A \subset X$ and $d+2 > 3$, Bezout gives $J \subset X$. As in step (ii) we get a contradiction.

(iv) Assume that v is not as in one of the cases (i), (ii), (iii), i.e. assume that v is neither contained in a line nor in a reduced conic. Let $H \subset \mathbb{P}^3$ be a general hyperplane containing the line $\langle v\rangle$. Set $C := X\cap H$. Bertini's theorem implies that C is smooth outside $\langle v\rangle \cap X$. Since $\langle v\rangle \nsubseteq X$ by our first assumption on v we have $\deg(\langle v\rangle \cap X) = 3 < \deg(v)+2$. Hence C is smooth outside O. Since H is general, we have $H \ne T_OX$. Hence C is smooth at O. Hence C is smooth. Since C is connected, C is a smooth elliptic curve. Since v is a Cartier divisor of C. In this case $\mathcal{O}_C(d)(-v)$ is a degree $3d-2$ line bundle on C which is very ample. Hence there is $E \in |\mathcal{O}_C(3d-2)|$ which is reduced. Since $h^1(C, \mathcal{O}_C(d)(-v-E)) = h^1(\mathcal{O}_C) > 0$, we get $\rho \le 3d-2$.

Assume $\rho \le 3d-3$. Hence $\deg(v\cup A) \le 3d-1$. By Lemma 1 either there is a line $J \subset \mathbb{P}^3$ such that $\deg(J\cap(v\cup A)) \cap d+2$ or there is a reduced conic $T \subset \mathbb{P}^3$ such that $\deg(T\cap(v\cup A)) \ge 2d+2$. Since $2d+2 > 6$, Bezout theorem gives that the line or the conic are contained in X. Call J' the line J or the conic T. Since $h^1(\mathcal{I}_{J'\cap(v\cup A)}(d)) > 0$ and $h^1(\mathcal{I}_{\{O\}\cup A}(d)) = 0$, we get $v \subset J'$, a contradiction. □

Proposition 1. *Fix an integer $d \ge 4$. Let $X \subset \mathbb{P}^3$ be a smooth degree 3 surface. Let $\phi : X \to \mathbb{P}^r$, $r = \binom{d+3}{3} - \binom{d}{3}$, be the embedding associated to the linear system $|\mathcal{O}_X(d)|$. Fix $P \in \tau(\phi(X)) \setminus \phi(X)$ and let $v \subset X$ the only degree two connected zero-dimensional scheme such that $P \in \langle\phi(v)\rangle$. Set $\{O\} := v_{red}$ and $\rho := r_{\phi(X)}(P)$.*

(i) If there is a line $L \subset X$ such that $v \subset L$, then $\rho = d$.

(ii) If v is not contained in a line of X, but v is contained in a reduced conic $D \subset X$, then $\rho = 2d$.

(iii) If v is contained neither in a line $L \subset X$ nor in a reduced conic $D \subset X$, then $\rho = 3d - 2$.

Proof. Fix $A \subset X$ such that $\phi(A)$ evinces $r_{\phi(X)}(P)$. Since $P \notin \phi(X)$, P has scheme rank 2. Since $d \geq 4$, each degree 3 zero-dimensional subscheme of $\phi(X)$ is linearly independent. Hence $\deg(L \cap \phi(X)) \leq 2$ for each line $L \subset \mathbb{P}^r$. Therefore v is unique. The same observation gives $\rho > 1$. Since A is reduced,$v \neq A$. Hence $h^1(X, \mathcal{I}_{v \cup A}(d)) = 0$ ([3], Lemma 1), i.e. $h^1(\mathbb{P}^3, \mathcal{I}_{v \cup A}(d)) > 0$. Lemma 1 gives that in each case ρ is at least as claimed in cases (ii), (iii) and (iii). Assume that v is contained in a degree i reduced and connected curve D, i.e. either in a line or in a reduced conic. We have $\dim(\langle \phi(D) \rangle) = id$ and $P \in \langle \phi(D) \rangle$. Hence it is sufficient to notice that the proof of [7], Proposition 4.1, works verbatim for a connected curve, not just an irreducible one. Now take v as in case (iv) and let H be a general plane containing $\langle v \rangle$. We saw that the curve $C := X \cap H$ is a smooth curve. Hence $\phi(C)$ is a linearly normal elliptic curve of degree $3d$. Apply, for instance, [5], Theorem 28. □

Proof of Theorem 1. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, be the order d Veronese embedding. The case $n = 3$ is true by Proposition 1. Hence we may assume $n \geq 4$ and that Theorem 1 is true for the smooth cubic hypersurfaces of $\mathbb{P}^{n-1}$. We are computing rank with respect to the variety $\nu_d(X)$. Fix $O \in X$. Let $v \subseteq T_O X \cap X$ be any degree 2 connected zero-dimensional scheme. If X contains the line $\langle v \rangle$, then $r_X(P) = 1$ for all $P \in \langle v \rangle$. Hence to get an upper bound for the integer $\alpha(X, O)$ it is sufficient to test the schemes v such that $\langle v \rangle \nsubseteq X$. Let $H \subset \mathbb{P}^n$ be a general hyperplane containing $\langle v \rangle$. Set $Y := X \cap H$ (scheme-theoretic intersection). Since $\langle v \rangle \nsubseteq X$, the scheme $\langle v \rangle \cap X$ is a degree 3 scheme with a connected component of degree at least 2. Since there are infinitely many hyperplanes containing $\langle v \rangle$, while $\langle v \rangle \cap X$ contains finitely many points, the generality of H means that Y is smooth at each of these points. Since these points are the base points of $|\mathcal{I}_v(1)|$, Bertini's theorem gives that Y is a smooth hypersurface. Taking $\phi_d|Y$ as the embedding of Y the inductive assumption gives $\alpha(Y) = \beta(Y) = 3d - 2$. Since $\alpha(X, P) \leq \alpha(Y, P)$, we get $\beta(X) \leq 3d - 2$. Now assume that O is general, so that it is contained in no line contained in v. Take as v a general degree 2 zero-dimensional subscheme of X with $v_{red} = \{O\}$. The line $\langle v \rangle$ intersects in at most two points, For general v there is no smooth conic $D \subset X$ such that $D \subset X$. It is sufficient to prove that $r_{\nu_d(X)}(P) \geq 3d - 2$

for all $P \in \langle \nu_d(v)\rangle \setminus \nu_d(X \cap \langle v\rangle)$. By [3], Lemma 1, this is done almost verbatim as in step (iv) of the proof of Lemma 1 with the following differences. Now H_i and M_i are hyperplanes of $\mathbb{P}^n$ and we quote induction on n for the cases $e = 1$ and $c = 1$ instead of [6]. We work directly in $\mathbb{P}^n$, not in X, and hence we didn't need to worry if $H_1 \cap X$ is singular. □

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