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AN IMPROVED POISSON TO APPROXIMATE THE NEGATIVE BINOMIAL DISTRIBUTION

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Abstract: This paper gives an improved Poisson distribution with mean λ for approximating the negative binomial distribution with parameters n and p, where $\lambda = nq = n(1-p)$. The improved approximation is more appropriate than the well-known Poisson approximation when n is sufficiently large and q is sufficiently small.

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1. Introduction

The negative binomial distribution is an important discrete distribution as same as other discrete distributions. Its applications appear in fields such as automobile insurance, inventory analysis, telecommunications networks analysis and population genetics. Let X be the negative binomial random variable with pa-

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rameters $n > 0$ and $p \in (0,1)$, then the probability function in our attention is of the form

$$\mathbf{nb}_{n,p}(x) = \frac{\Gamma(n+x)}{\Gamma(n)x!} q^x p^n, \; x = 0, 1, ..., \tag{1.1}$$

and the mean and variance of X are $E(X) = \frac{nq}{p}$ and variance $Var(X) = \frac{nq}{p^2}$, respectively. Under parametrization, $\lambda = nq$ and $p = \frac{n-\lambda}{n}$, it can be expressed as

$$\mathbf{nb}_{n,p}(x) = \frac{\lambda^x}{x!} \frac{\Gamma(n+x)}{\Gamma(n)n^x} \left(1 - \frac{\lambda}{n}\right)^n, \; x = 0, 1, ... \tag{1.2}$$

Observe that if $n \to \infty$ and $q \to 0$ while $\lambda = nq$ remains fixed, then $\mathbf{nb}_{n,p}(x) \to \mathbf{p}_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x \in \mathbb{N} \cup \{0\}$. Therefore, the Poisson probability function with mean $\lambda = nq$ can be used as an estimate of the negative binomial probability function if n is large and q is small. In this case, Teerapabolarn [2] gave a non-uniform bound on $|\mathbf{nb}_{n,p}(x) - \mathbf{p}_\lambda(x)|$ for $x \in \mathbb{N} \cup \{0\}$.

In this paper, we are interested to determine an improved Poisson probability function, $\widehat{\mathbf{p}}_\lambda(x)$, for approximating the negative binomial probability function, and the accuracy of the approximation is measured in the form of $|\mathbf{nb}_{n,p}(x) - \widehat{\mathbf{p}}_\lambda(x)|$ for $x \in \mathbb{N} \cup \{0\}$. The result of this study is in Section 2. In Section 3, some numerical examples are given to illustrate the improved approximation and the conclusion of this study is presented in the last section.

2. Result

Before giving an improved Poisson distribution, we also need the following lemma, which similar to that of [1].

Lemma 2.1. *For $x \in \mathbb{N}$ and $n > 0$, then*

$$\prod_{i=0}^{x-1} \left(1 + \frac{i}{n}\right) = 1 + \frac{x(x-1)}{2n} + O\left(\frac{1}{n^2}\right). \tag{2.1}$$

Proof. For $x = 1$, $\prod_{i=0}^{1-1}\left(1 + \frac{i}{n}\right) = 1 = 1 + \frac{1(1-1)}{2n} + O\left(\frac{1}{n^2}\right)$. Assuming $x = k$ for $k \in \mathbb{N}$ such that $\prod_{i=0}^{k-1}\left(1 + \frac{i}{n}\right) = 1 + \frac{k(k-1)}{2n} + O\left(\frac{1}{n^2}\right)$. Thus, for $x = k+1$, we have $\prod_{i=0}^{k}\left(1 + \frac{i}{n}\right) = \left\{1 + \frac{k(k-1)}{2n} + O\left(\frac{1}{n^2}\right)\right\}\left(1 + \frac{k}{n}\right) =$

$1+\frac{k(k-1)}{2n}+\frac{k}{n}+O\left(\frac{1}{n^2}\right) = 1+\frac{(k+1)k}{2n}+O\left(\frac{1}{n^2}\right)$. Therefore, by mathematical induction, (2.1) holds. □

Theorem 2.1. *Let* $x \in \mathbb{N}\cup\{0\}$, $\lambda = nq$ *and* $\widehat{\mathbf{p}}_\lambda(x) = \mathbf{p}_\lambda(x)e^\lambda p^n\left\{1+\frac{x(x-1)}{2n}\right\}$. *Then we have the following:*

$$\mathbf{nb}_{n,p}(x) = \widehat{\mathbf{p}}_\lambda(x) + O\left(\frac{1}{n^2}\right) \tag{2.2}$$

and for large n *and small* q,

$$\widehat{\mathbf{p}}_\lambda(x) = \mathbf{nb}_{n,p}(x). \tag{2.3}$$

Proof. For $x=0$, it is clear that $\mathbf{nb}_{n,p}(0) = p^n = \widehat{\mathbf{p}}_\lambda(0) + O\left(\frac{1}{n^2}\right)$. Next, we have to show that (2.2) holds for $x \in \mathbb{N}$. Using (1.2), we obtain

$$\begin{aligned}
\mathbf{nb}_{n,p}(x) &= \frac{\lambda^x}{x!}p^n\prod_{i=0}^{x-1}\left(1+\frac{i}{n}\right)\\
&= \mathbf{p}_\lambda(x)e^\lambda p^n\left\{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)\right\} \quad \text{(by (2.1))}\\
&= \widehat{\mathbf{p}}_\lambda(x) + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Also, if n is large and q is small, then $O\left(\frac{1}{n^2}\right) = 0$. Hence $\widehat{\mathbf{p}}_\lambda(x) = \mathbf{nb}_{n,p}(x)$. □

3. Numerical Examples

The following examples are given to illustrate how well the improved Poisson distribution with mean $\lambda = nq$ approximates the negative binomial distribution with parameters n and p (when n is sufficiently large and q is sufficiently small).

3.1. Let $n = 30$ and $p = 0.9$, then $\lambda = 3.0$ and the numerical results are as follows:

x	$\mathbf{nb}_{n,p}(x)$	$\widehat{\mathbf{p}}_\lambda(x)$	$\mathbf{p}_\lambda(x)$	$\lvert\mathbf{nb}_{n,p}(x) - \widehat{\mathbf{p}}_\lambda(x)\rvert$	$\lvert\mathbf{nb}_{n,p}(x) - \mathbf{p}_\lambda(x)\rvert$
0	0.04239116	0.04239116	0.04978707	0.00000000	0.00739591
1	0.12717347	0.12717347	0.14936121	0.00000000	0.02218773
2	0.19711889	0.19711889	0.22404181	0.00000000	0.02692292
3	0.21026015	0.20983623	0.22404181	0.00042391	0.01378166
4	0.17346462	0.17168419	0.16803136	0.00178043	0.00543326
5	0.11795594	0.11445613	0.10081881	0.00349981	0.01713713
6	0.06880763	0.06438157	0.05040941	0.00442606	0.01839823
7	0.03538678	0.03127105	0.02160403	0.00411573	0.01378275
8	0.01636639	0.01333618	0.00810151	0.00303020	0.00826488
9	0.00691025	0.00505855	0.00270050	0.00185170	0.00420975
10	0.00269500	0.00172451	0.00081015	0.00097049	0.00188485
11	0.00098000	0.00053303	0.00022095	0.00044697	0.00075905
12	0.00033483	0.00015050	0.00005524	0.00018433	0.00027960
13	0.00010818	0.00003907	0.00001275	0.00006910	0.00009543

3.2. Let $n = 100$ and $p = 0.95$, then $\lambda = 5.0$ and the numerical results are as follows:

x	$\mathbf{nb}_{n,p}(x)$	$\widehat{\mathbf{p}}_\lambda(x)$	$\mathbf{p}_\lambda(x)$	$\lvert\mathbf{nb}_{n,p}(x) - \widehat{\mathbf{p}}_\lambda(x)\rvert$	$\lvert\mathbf{nb}_{n,p}(x) - \mathbf{p}_\lambda(x)\rvert$
0	0.00592053	0.00592053	0.00673795	0.00000000	0.00081742
1	0.02960265	0.02960265	0.03368973	0.00000000	0.00408709
2	0.07474668	0.07474668	0.08422434	0.00000000	0.00947766
3	0.12706936	0.12704469	0.14037390	0.00002467	0.01330454
4	0.16360180	0.16343128	0.17546737	0.00017052	0.01186557
5	0.17014587	0.16959849	0.17546737	0.00054738	0.00532150
6	0.14887764	0.14775626	0.14622281	0.00112137	0.00265483
7	0.11272164	0.11104663	0.10444486	0.00167501	0.00827678
8	0.07538260	0.07341926	0.06527804	0.00196334	0.01010456
9	0.04522956	0.04333776	0.03626558	0.00189180	0.00896398
10	0.02465011	0.02310285	0.01813279	0.00154726	0.00651732
11	0.01232505	0.01122552	0.00824218	0.00109953	0.00408288
12	0.00570034	0.00500924	0.00343424	0.00069110	0.00226610
13	0.00245553	0.00206590	0.00132086	0.00038963	0.00113467
14	0.00099098	0.00079171	0.00047174	0.00019927	0.00051925
15	0.00037657	0.00028325	0.00015725	0.00009333	0.00021933
16	0.00013533	0.00009499	0.00004914	0.00004034	0.00008619

For approximating the negative binomial distribution in the examples 3.1 and 3.2, it can be seen that the improved Poisson distribution is more appropriate than the Poisson distribution.

4. Conclusion

In this study, an improved Poisson distribution with mean $\lambda = nq$ was obtained by using some mathematical manipulations. This improved approximation is more accurate than the well-known Poisson approximation, thus the improved Poisson distribution can also be used as an estimate of the negative binomial distribution when n is sufficiently large and q is sufficient small.

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