ON THE DIOPHANTINE EQUATION $5^x + 43^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers.

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1. Introduction

In [1], Acu solved the Diophantine equation $2^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(3, 0, 3)$ and $(2, 1, 3)$.

In [16], Suvarnamani, Singta and Chotchaisthit proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers.

In [7], Sroysang solved the Diophantine equation $3^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 2)$. In [9], Sroysang proved that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In [10], Sroysang proved that the Diophantine equation $5^x + 23^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. Moreover, we refer to [5, 6, 8, 11, 12, 13, 14, 15].
In [4], Rabago proved that the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ have exactly two solutions $(x, y, z)$ where $x, y$ and $z$ are non-negative integers. The solutions are in $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively.

In [2], Chotchaisthit solved the Diophantine equation $2^x + 11^y = z^2$ where $x, y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(3, 0, 3)$.

In this paper, we show that the Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution where $x, y$ and $z$ are non-negative integers.

2. Preliminaries

Proposition 2.1. [3] (Catalan’s conjecture) $(3, 2, 2, 3)$ is a unique solution $(a, b, x, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [10] The Diophantine equation $5^x + 1 = z^2$ has no non-negative integer solution where $x$ and $z$ are non-negative integers.

Lemma 2.3. The Diophantine equation $1 + 43^y = z^2$ has no non-negative integer solution where $y$ and $z$ are non-negative integers.

Proof. Suppose that there are non-negative integers $y$ and $z$ such that $1 + 43^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. It follows that $y \geq 1$. Note that $z$ is even. Then $z^2 \equiv 0 \pmod{4}$. Since $5^x \equiv 1 \pmod{4}$, it follows that $43^y \equiv 3 \pmod{4}$. Then $3^y \equiv 3 \pmod{4}$. This implies that $y$ is odd. Note that $43^y \equiv 2 \pmod{5}$ or $43^y \equiv 3 \pmod{5}$. By Lemma 2.3, we have $x \geq 1$. Since $5^x \equiv 0 \pmod{5}$, it follows that

3. Main Results

Theorem 3.1. The Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution where $y$ and $z$ are non-negative integers.

Proof. Suppose that there are non-negative integers $x, y$ and $z$ such that $5^x + 43^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Note that $z$ is even. Then $z^2 \equiv 0 \pmod{4}$. Since $5^x \equiv 1 \pmod{4}$, it follows that $43^y \equiv 3 \pmod{4}$. Then $3^y \equiv 3 \pmod{4}$. This implies that $y$ is odd. Note that $43^y \equiv 2 \pmod{5}$ or $43^y \equiv 3 \pmod{5}$. By Lemma 2.3, we have $x \geq 1$. Since $5^x \equiv 0 \pmod{5}$, it follows that
\[ z^2 \equiv 0 \pmod{5} \text{ or } z^2 \equiv 1 \pmod{5} \text{ or } z^2 \equiv 4 \pmod{5}. \] This is a contradiction. Hence, the equation \( 5x + 43y = z^2 \) has no non-negative integer solution.

**Corollary 3.2.** Let \( k \) be a positive integer. The Diophantine equation \( 5^x + 43^y = w^{2k+2} \) has no non-negative integer solution where \( x, y \) and \( w \) are non-negative integers.

**Proof.** Suppose that there are non-negative integers \( x, y \) and \( w \) such that \( 5^x + 43^y = w^{2k+2} \). Let \( z = w^{k+1} \). Then \( 5^x + 43^y = z^2 \). This is a contradiction with Theorem 3.1. Hence, the equation \( 5^x + 43^y = w^{2k+2} \) has no non-negative integer solution.

**Corollary 3.3.** The Diophantine equation \( 25^u + 43^y = z^2 \) has no non-negative integer solution where \( u, y \) and \( z \) are non-negative integers.

**Proof.** Suppose that there are non-negative integers \( u, y \) and \( z \) such that \( 25^u + 43^y = z^2 \). Let \( x = 2u \). Then \( 5^x + 43^y = z^2 \). This is a contradiction with Theorem 3.1. Hence, the equation \( 25^u + 43^y = z^2 \) has no non-negative integer solution.

**References**


