ON THE DIOPHANTINE EQUATION $483^x + 485^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $483^x + 485^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 22)$.

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1. Introduction

In [9], Sroysang showed that $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $3^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. In [10], he showed that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In [13], he showed that the Diophantine equation $47^x + 49^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In [14], he showed that the Diophantine equation $89^x + 91^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. Many equations of type $a^x + b^y = z^2$ were solved [1, 2, 3, 4, 6, 7, 8, 11, 12]. In this paper, $(1, 0, 22)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 485^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.
2. Preliminaries

**Proposition 2.1.** [5] (Catalan’s conjecture) \( (3, 2, 2, 3) \) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers such that \(\min\{a, b, x, y\} > 1\).

**Lemma 2.2.** \((1, 22)\) is a unique solution \((x, z)\) for the Diophantine equation \(483^x + 1 = z^2\) where \(x\) and \(z\) are non-negative integers.

**Proof.** Let \(x\) and \(z\) be non-negative integers such that \(483^x + 1 = z^2\). If \(x = 0\), then \(z^2 = 2\) which is impossible. Thus, \(x \geq 1\). Note that \(z^2 = 483^y + 1 \geq 483^1 + 1 = 484\). It follows that \(z \geq 22\). Now, we consider on the equation \(z^2 - 483^x = 1\). By Proposition 2.1, we obtain that \(x = 1\). Hence, \(z^2 = 484\) and then \(z = 22\). \(\square\)

**Lemma 2.3.** The Diophantine equation \(1 + 485^y = z^2\) has no non-negative integer solution where \(y\) and \(z\) are non-negative integers.

**Proof.** Suppose that there are non-negative integers \(y\) and \(z\) such that \(1 + 485^y = z^2\). If \(y = 0\), then \(z^2 = 2\) which is impossible. Thus, \(y \geq 1\). Note that \(z^2 = 1 + 485^y \geq 1 + 485^1 = 486\). It follows that \(z \geq 23\). Now, we consider on the equation \(z^2 - 485^y = 1\). By Proposition 2.1, we obtain that \(y = 1\). Hence, \(z^2 = 486\). This is a contradiction. \(\square\)

3. Main Results

**Theorem 3.1.** \((1, 0, 22)\) is a unique non-negative integer solution \((x, y, z)\) for the Diophantine equation \(483^x + 485^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

**Proof.** Let \(x, y\) and \(z\) be non-negative integers such that \(483^x + 485^y = z^2\). Since \(z\) is even, we have \(z^2 \equiv 0 \pmod{4}\). Thus, \(483^x \equiv 3 \pmod{4}\). By Lemma 2.3, we obtain that \(x \geq 1\). Then \(x\) is odd. Now, we will divide the number \(y\) into two cases.

Case \(y = 0\). By Lemma 2.2, it follows that \(x = 1\) and \(z = 22\).

Case \(y \geq 1\). Note that \(483^y \equiv 2 \pmod{5}\) or \(483^y \equiv 3 \pmod{5}\). Since \(485^y \equiv 0 \pmod{5}\), we have \(z^2 \equiv 2 \pmod{5}\) or \(z^2 \equiv 3 \pmod{5}\). Thus, \(z\) is odd. This is a contradiction.

Hence, the solution \((x, y, z)\) is \((1, 0, 22)\). \(\square\)
Corollary 3.2. The Diophantine equation $483^x + 485^y = w^4$ has no non-negative integer solution where $x, y$ and $w$ are non-negative integers.

Proof. Suppose that there are non-negative integers $x, y$ and $w$ such that $483^x + 485^y = w^4$. Let $z = w^2$. We obtain that $483^x + 485^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 22)$. It follows that $w^2 = z = 22$. This is a contradiction. □

Corollary 3.3. $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 485^y = 121u^2$ where $x, y$ and $u$ are non-negative integers.

Proof. Let $x, y$ and $u$ be non-negative integers such that $483^x + 485^y = 4u^4$. Let $z = 11u$. This implies that $483^x + 485^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 22)$. Hence, $u = 2$. □

Corollary 3.4. $(1, 0, 11)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 485^y = 4v^4$ where $x, y$ and $v$ are non-negative integers.

Proof. Let $x, y$ and $v$ be non-negative integers such that $483^x + 485^y = 4v^4$. Let $z = 2v^2$. This implies that $483^x + 485^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 22)$. Hence, $v = 11$. □

References


