ON BASES AND MAXIMAL IDEALS IN AN ORDERED SEMIGROUP

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Abstract: In this paper the concepts of left base, right base and two-sided base of an ordered semigroup are introduced. A sufficient condition for an ordered semigroup contains right bases is given.

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1. Preliminaries

Tamura [9] introduced and studied one-sided bases and two-sided bases of semigroups (without order). Fabrici [2] gave a sufficient condition for a semigroup without order contains a right base, and showed that every finite semigroup contains both one-sided bases and two-sided bases. In this paper we introduce the notions of right base, left base and two-sided base of an ordered semigroup, and extend Fabrici’s results to ordered semigroups.
In [1], an ordered semigroup \((S, \cdot, \leq)\) is a semigroup \((S, \cdot)\) together with a partial order \(\leq\) that is compatible with the semigroup operation, meaning that, for any \(x, y, z\) in \(S\),

\[
x \leq y \implies zx \leq zy \text{ and } xz \leq yz.
\]

If \(A, B\) are nonempty subsets of \(S\), we write the set product \(AB\) of \(A\) and \(B\) for the set of all elements \(xy\) of \(S\) with \(x \in A\) and \(y \in B\), and write \((A]\) for the set of all elements \(x\) of \(S\) such that \(x \leq a\) for some \(a\) in \(A\), i.e.,

\[
(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.
\]

In particular, we write \(Ax\) for \(A\{x\}\). It was shown in [6] that \((A \cup B] = (A]\cup(B]\).

A nonempty subset \(A\) of an ordered semigroup \((S, \cdot, \leq)\) is called a left ideal [3] of \(S\) if it satisfies the following conditions:

(i) \(SA \subseteq A\);

(ii) for any \(x \in A\) and \(y \in S\), \(y \leq x\) implies \(y \in A\).

For \(a\) in \(S\), the principal left ideal generated by \(a\) is \(L(a) := (a \cup Sa]\). It is known that the union of two left ideals of \(S\) is a left ideal of \(S\).

A proper left ideal \(A\) of an ordered semigroup \((S, \cdot, \leq)\) is called a maximal left ideal if there is no a proper left ideal \(L'\) of \(S\) such that \(L \subset L'\).

Let \((S, \cdot, \leq)\) be an ordered semigroup. The equivalence relation \(L\) is defined by:

\[
aLb \text{ if and only if } L(a) = L(b)
\]

for any \(a, b\) in \(S\). The \(L\)-class containing \(a\) in \(S\) will be written \(L_a\). Define a preorder \(\preceq\) on the set of all \(L\)-classes by:

\[
L_a \preceq L_b \text{ if and only if } L(a) \subseteq L(b).
\]

The symbol \(L_a \prec L_b\) stands for \(L_a \preceq L_b\), but \(L_a \neq L_b\). The symbol \(a \prec b\) stands for \(a \leq b\), but \(a \neq b\). Note that \(a \leq b\) implies \(L_a \preceq L_b\). In particular, \(a \prec b\) implies \(L_a \prec L_b\).
2. Ordered Semigroups Containing One-Sided and Two-Sided Bases

We define one-sided bases and two-sided bases of an ordered semigroup by:

**Definition 1.** Let \((S, \cdot, \leq)\) be an ordered semigroup. A subset \(A\) of \(S\) is called a **right base** of \(S\) if it satisfies the following conditions:

(i) \(S = (A \cup SA)\);

(ii) if \(B\) is a subset of \(A\) such that \(S = (B \cup SB)\), then \(A = B\).

Dually, one can define for \(A\) to be a **left base** of \(S\).

By a **two-sided base** of \(S\) we mean a subset \(A\) of \(S\) such that

(i) \(S = (A \cup SA \cup AS \cup SAS)\);

(ii) if \(B\) is a subset of \(A\) such that \(S = (B \cup SB \cup BS \cup SBS)\), then \(A = B\).

**Example 2.** ([7]) Let \((S, \cdot, \leq)\) be an ordered semigroup such that the multiplication and the order relation are defined by:

\[
\begin{array}{c|ccccccc}
\cdot & a & b & c & d & e \\
\hline
a & a & e & c & d & e \\
b & a & b & c & d & e \\
c & a & e & c & d & e \\
d & a & e & c & d & e \\
e & a & e & c & d & e \\
\end{array}
\]

\(\leq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (c, e)\}\).

The covering relation and the figure of \(S\) are given by:

\[
< = \{(a, d), (c, e)\}
\]

The left bases of \(S\) are \(\{a\}, \{b\}, \{c\}, \{d\}\) and \(\{e\}\). The right base of \(S\) is \(\{b, d\}\). \(S\) has only one two-sided base: \(\{b\}\).

**Example 3.** ([4]) Let \((S, \cdot, \leq)\) be an ordered semigroup such that the multiplication and the order relation are defined by:
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
  & a & b & c & d & e \\
\hline
 a & a & a & c & a & c \\
 b & a & a & c & a & c \\
 c & a & a & c & a & c \\
 d & d & d & e & d & e \\
 e & d & d & e & d & e \\
\hline
\end{array}
\]

\[\leq = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}.\]

The covering relation and the figure of $S$ are given by:
\[
< = \{(a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, e), (d, e)\}
\]

![Diagram](image)

The right bases of $S$ are $\{e\}$ and $\{c\}$. The left bases of $S$ are $\{d\}$ and $\{e\}$. The two-sided bases of $S$ are $\{c\}$, $\{d\}$ and $\{e\}$.

**Lemma 4.** Let $A$ be a subset of an ordered semigroup $(S, \cdot, \leq)$. Then $A$ is a right base of $S$ if and only if it satisfies the following conditions:

(i) for any $x$ in $S$ there exists $a$ in $A$ such that $L_x \leq L_a$;

(ii) if $a, b \in A$ such that $a \neq b$, then neither $L_a \leq L_b$ nor $L_b \leq L_a$;

**Proof.** Assume that $A$ is a right base of $S$. Let $x \in S$. Since $S = (A \cup SA]$, so $x \in (A]$ or $x \in (SA]$. If $x \in (A]$, then $x \leq a$ for some $a$ in $A$; hence $L_x \leq L_a$. If $x \in (SA]$, then $x \leq sa'$ for some $s$ in $S$ and $a'$ in $A$; hence $L_x \leq L_a$. This proves that (i) holds. Let $a, b \in A$ such that $a \neq b$. Suppose $L_a \leq L_b$. We set $B = A \setminus \{a\}$. Let $x \in S$. By (i), there exists $c$ in $A$ such that $L_x \leq L_c$. If $c \neq a$, then $c \in B$; thus

\[x \in L(x) \subseteq L(c) \subseteq (B \cup SB]\]

If $c = a$, then, by $b \in B$, we have
\[ x \in L(x) \subseteq L(b) \subseteq (B \cup SB). \]

Thus \( S = (B \cup SB) \). This is a contradiction. The case \( L_b \preceq L_a \) is proved similarly. Thus (ii) holds true.

Conversely, assume that the conditions (i) and (ii) hold. By (i), \( S = (A \cup SA) \). Suppose that \( S = (B \cup SB) \) for some a proper subset \( B \) of \( A \). Let \( a \in A \setminus B \). If \( a \preceq b \) for some \( b \) in \( B \), then \( L_a \preceq L_a' \). This contradicts to (ii). Similarly, if \( a \preceq sb' \) for some \( s \) in \( S \) and \( b' \) in \( B \), then \( L_a \preceq L_{b'} \). This is a contradiction. Hence \( A \) is a right base of \( S \).

**Theorem 5.** Assume that an ordered semigroup \((S, \cdot, \leq)\) contains a left ideal. Then \( L \) is a maximal left ideal of \( S \) if and only if \( S \setminus L \) is a maximal \( L \)-class.

**Proof.** Assume that \( L \) is a maximal left ideal of \( S \). Let \( x, y \in S \setminus L \). Since \( L \subset L \cup L(x) \subseteq S \), we have \( L \cup L(x) = S \), and so \( y \in L(x) \). Similarly, \( x \in L(y) \). Thus \( L(x) = L(y) \). This proves that \( S \setminus L \) is an \( L \)-class. If \( S \setminus L \prec L_a \) for some \( a \) in \( S \), then \( S \setminus L \subseteq L(a) \subseteq L \). This is a contradiction. Therefore, \( S \setminus L \) is a maximal \( L \)-class.

Conversely, assume that \( S \setminus L \) is a maximal \( L \)-class such that \( S \setminus L = L_a \) for some \( a \) in \( S \). If \( x \in SL \setminus L \), then \( x \in L_a \); hence \( L(x) = L(a) \). By \( x \in SL \), \( x = sb \) for some \( s \) in \( S \) and \( b \) in \( L \). We have \( L_a \prec L_b \). This is a contradiction. Hence \( SL \subseteq L \). Let \( y \in L \) and \( c \in S \) be such that \( c \preceq y \). Suppose that \( c \in L_a \). Then \( c \preceq y \), and so \( L_a \prec L_y \). This is a contradiction. Thus \( c \in L \). This shows that \( L \) is a left ideal of \( S \). Let \( L' \) be a left ideal of \( S \) such that \( L \subseteq L' \subseteq S \). Then there is \( z \in S \setminus L' \). We have \( L_a = L_z \). Similarly, there exists \( w \in L' \setminus L \) such that \( L_w = L_a \). Then

\[ z \in L(z) = L(a) = L(w) \subseteq L'. \]

This is a contradiction. Therefore, \( L \) is a maximal left ideal of \( S \).

**Corollary 6.** If an ordered semigroup \((S, \cdot, \leq)\) contains a right base, then \( S \) contains a maximal left ideal.

**Proof.** This is a consequence of Lemma 4 and Theorem 5.

**Definition 7.** A proper left ideal \( L \) of an ordered semigroup \((S, \cdot, \leq)\) is called a covered left ideal if \( L \subseteq S(S \setminus L) \).

**Example 8.** ([8]) Let \((S, \cdot, \leq)\) be an ordered semigroup such that \( S = \{a, b, c, d, e\} \) and
$S$ is given by:

$\leq = \{(a, a), (a, b), (a, e), (b, b), (c, c), (c, e), (d, d), (d, b), (d, e), (e, e)\}$.

The covering relation and the figure of $S$ are given by:

$\preceq = \{(a, b), (a, e), (c, e), (d, b), (d, e)\}$

The left ideals of $S$ are $\{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a, b, c, d\}, \{a, c, d, e\}$ and $S$. The covered left ideals of $S$ are $\{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}$.

**Example 9.** ([5]) Let $(S, \cdot, \leq)$ be an ordered semigroup such that $S = \{a, b, c, d, f\}$ and

$\leq = \{(a, a), (b, b), (b, c), (b, e), (c, c), (d, a), (d, d), (d, e), (e, e)\}$.

The covering relation and the figure of $S$ are given by:

$\preceq = \{(b, c), (b, e), (d, a), (d, e)\}$
The left ideals of $S$ are $\{b, d, e\}$, $\{a, b, d, e\}$ and $S$. The covered left ideal of $S$ is $\{b, d, e\}$.

**Corollary 10.** Let $(S, \cdot, \leq)$ be an ordered semigroup containing a right base. Then the following statements hold.

1. $S$ contains maximal left ideals.
2. Every maximal left ideal $L_i$ of $S$ is $L_i = S \setminus L_{a_i}$ for some $a_i$ in $S$.

**Proof.** It is a consequence of Theorem 4 and 5.

**Theorem 11.** Let $(S, \cdot, \leq)$ be an ordered semigroup containing maximal left ideals. If the intersection of all maximal left ideals of $S$ is empty or a covered left ideal, then $S$ contains a right base.

**Proof.** Let $\{L_i \mid i \in I\}$ be the set of all maximal left ideals of $S$. By Theorem 5, for each $i \in I$, $S \setminus L_i$ is a maximal $\mathcal{L}$-class. Setting $S \setminus L_i := L_{a_i}$ for each $i$ in $I$ then

$$L := \bigcap_{i \in I} L_i = \bigcap_{i \in I} (S \setminus L_{a_i}) = S \setminus \bigcup_{i \in I} L_{a_i}.$$  

Let $A$ denote the set of all elements $a_i$. We assert that $A$ is a right base of $S$, and hence the theorem is proved. We consider two cases:

**Case 1:** $L = \emptyset$. Then $S = \bigcup_{i \in I} L_{a_i}$. If $x \in S$, then $x \in L_{a_i}$ for some $i$ in $I$, and so $L(x) = L(a_i)$. Thus $L_x \leq L_{a_i}$. Since $L_{a_i}$ is a maximal $\mathcal{L}$-class for all $i$ in $I$, it follows that, for any different $i, j$ in $I$, neither $L_{a_i} \leq L_{a_j}$ nor $L_{a_j} \leq L_{a_i}$. By Lemma 4, $A$ is a right base of $S$.

**Case 2:** $L$ is a covered left ideal of $S$. That is, $L \subseteq (S(S \setminus L)]$. If $x \in S \setminus L$, then $x \in \bigcup_{i \in I} L_{a_i}$, and so $x \in L_{a_{i_0}}$ for some $i_0$ in $I$. Since

$$L(x) = L(a_{i_0}) \subseteq (A \cup SA),$$

we have $x \in (A \cup SA)$. This proves that

$$S \setminus L \subseteq (A \cup SA).$$

By

$$L \subseteq (S(S \setminus L]) \subseteq (S(A \cup SA)) \subseteq (SA \cup SSA) \subseteq (A \cup SA)$$

it follows that
This implies that if \( x \in S \) then there exists \( a_i \in A \) such that \( L_x \preceq L_{a_i} \). It follows by Lemma 4 that \( A \) is a right base of \( S \).

It is not true in general that an ordered semigroup contains one-sided bases implies the semigroup contains two-sided bases. This was shown by Example 2 in [2]. However, it is easy to see that this statement holds true for any finite ordered semigroups.

**Theorem 12.** If an ordered semigroup \((S, \cdot, \leq)\) contains a left or a right base which is finite, then \( S \) contains a two-sided base.

**Corollary 13.** Every finite ordered semigroup contains both one-sided bases and two-sided bases.

**References**


