International Journal of Pure and Applied Mathematics
Volume 92 No. 1 2014, 133-139
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v92i1.13
ijpam.eu

ON MAXIMAL IDEALS IN TERNARY SEMIGROUPS

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Abstract: It is known that every maximal ideals in a commutative ring with identity is prime; this result is also valid for commutative ordered semigroups and commutative semigroups as well. In this paper, we show that in commutative ternary semigroups with identity every maximal ideals is prime.

AMS Subject Classification: 06F05
Key Words: semigroup, commutative ternary semigroup with identity, ideal, prime ideal, maximal ideal, direct product

1. Preliminaries

It is known that the notion of ideals play an important rule in several algebraic structures such as rings, semigroups and ternary semigroups. Particularly, relations between maximal ideals and prime ideals have been widely studied by many authors. One of important results for a commutative ring with identity is every maximal ideal is a prime ideal ([1], p.128); this result is also valid for a commutative semigroup with identity [8] and also for commutative ordered semigroups with identity [4]. The purpose of this paper is to show that this result is also true for a commutative ternary semigroup with identity. We also give an example to show that the converse of the statement is not valid, in general.

Received: February 3, 2014

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Definition 1. ([5], [7]) A nonempty set T with a ternary operation $(x, y, z) \mapsto [xyz]$ satisfying the associative law, that is, for any x_1, x_2, x_3, x_4, x_5 in T,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]],$$

is called a *ternary semigroup*.

Note that a semigroup S can be made into a ternary semigroup under the associative ternary operation defined by

$$(x, y, z) \mapsto xyz.$$

Definition 2. ([3]) An element e of a ternary semigroup T is called an *identity element* or *unital element* if

$$[eex] = [exe] = [xee] = x$$

for all x in T.

Definition 3. ([2]) A ternary semigroup T is said to be *commutative* if, for all x_1, x_2, x_3 in T,

$$[x_1x_2x_3] = [x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}]$$

for all permutations α on $\{1, 2, 3\}$.

If A, B and C are nonempty subsets of a ternary semigroup T, then the product $[ABC]$ is the set of all elements $[abc] \in T$ where $a \in A, b \in B$ and $c \in C$, i.e.,

$$[ABC] = \{[abc] \mid a \in A, b \in B, c \in C\}.$$

In dealing with singleton sets we write, for example, $[\{a\}BC]$ by $[aBC]$.

Definition 4. ([6]) Let T be a ternary semigroup. A nonempty subset A of T is called a *left* (respectively, *middle*, *right*) *ideal* of T if

$$[TTA] \subseteq A$$

(respectively, $[TAT] \subseteq A$, $[ATT] \subseteq A$). If A is both a left and a right ideal of T, then A is called a *two-sided ideal* of T. If A is a left, a middle and a right ideal of T, then A is called an *ideal* of T.

An ideal A of a ternary semigroup T is said to be a *proper ideal* of T if $A \subset T$ (The symbol $\subset$ stands for proper inclusion for sets). For proper right, left, middle and two-sided ideal are defined similarly.

Theorem 5. ([6]) *If A is a nonempty subset of a ternary semigroup T, then*

$$\langle A\rangle = A \cup [TTA] \cup [ATT] \cup [TAT] \cup [TTATT]$$

is an ideal of T.

Corollary 6. *If A is a nonempty subset of a commutative ternary semigroup T with identity, then*

$$\langle A\rangle = [TTTTA].$$

Proof. Consider:

$$\begin{aligned}\langle A\rangle &= A \cup [TTA] \cup [ATT] \cup [TAT] \cup [TTATT]\\ &= A \cup [TTA] \cup [TTTTA]\\ &= [TTA] \cup [TTTTA]\\ &= [TTTTA].\end{aligned}$$

Thus the assertion holds. □

Definition 7. A proper ideal A of a ternary semigroup T is called a *maximal ideal* of T if, for any ideal B of T, $A \subseteq B \subseteq T$ implies $A = B$ or $B = T$.

Definition 8. ([2]) A proper ideal P of a ternary semigroup T is said to be *prime* if, for any ideal A, B and C of T, $[ABC] \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

The following was proved in [2]:

Theorem 9. *A proper ideal P of a commutative ternary semigroup T is prime if, for all x, y, z in T,*

$$[xyz] \in P \text{ implies } x \in P \text{ or } y \in P \text{ or } z \in P.$$

2. Main Results

We begin with the main result:

Theorem 10. *Let T be a commutative ternary semigroup with identity. If M is a maximal ideal of T, then M is prime.*

Proof. Assume that M is a maximal ideal of T. To show that M is prime, Theorem 9 allows us to show that, for all x, y, z in T, $[xyz] \in M$ implies $x \in M$, $y \in M$ or $z \in M$. Let $x, y, z \in T$ be such that $[xyz] \in M$ and $x, y, z \notin M$. By Corollary 6,

$$\langle M \cup \{x\} \rangle = [TTTT(M \cup \{x\})].$$

Since $x \notin M$, we have

$$M \subset M \cup \{x\} \subseteq [TTTT(M \cup \{x\})].$$

Using the maximality of M, we get

$$T = [TTTT(M \cup \{x\})] \tag{1}$$

Similarly, we obtain two conditions:

$$T = [TTTT(M \cup \{y\})] \tag{2}$$

and

$$T = [TTTT(M \cup \{z\})]. \tag{3}$$

Since $y \in T$ and the equation (3), there exist $a_1, b_1, c_1, d_1 \in T$ and $u \in M \cup \{z\}$ such that $y = [a_1 b_1 c_1 d_1 u]$. If $u \in M$, then $y \in M$. This is a contradiction. Thus

$$y = [a_1 b_1 c_1 d_1 z] \tag{4}$$

Again, since $d_1 \in T$ and the equation (2), there exist $a_2, b_2, c_2, d_2 \in T$ and $v \in M \cup \{y\}$ such that $d_1 = [a_2 b_2 c_2 d_2 v]$. If $v \in M$, then $y \in M$. This is a contradiction. Then $d_1 = [a_2 b_2 c_2 d_2 y]$, and so

$$y = [a_1 b_1 c_1 a_2 b_2 c_2 d_2 y] z] \tag{5}$$

by the equation (4). Finally, since $d_2 \in T$ and the equation (1), there exist $a_3, b_3, c_3, d_3 \in T$ and $w \in M \cup \{x\}$ such that $d_2 = [a_3 b_3 c_3 d_3 w]$. If $w \in M$, then $y \in M$. This is a contradiction. Then $d_2 = [a_3 b_3 c_3 d_3 x]$, and hence

$$y = [a_1 b_1 c_1 [a_2 b_2 c_2 [a_3 b_3 c_3 d_3 x] y] z]$$

by (5). Since $[xyz] \in M$, $y \in M$. This is impossible. Thus M is prime, and the proof completes. □

Let $\{T_i \mid i \in I\}$ be an indexed family of ternary semigroups where

$$(x, y, z) \mapsto [xyz]_i$$

is the ternary operation on T_i for each $i \in I$. We define a ternary operation on the Cartesian product $\prod_{i \in I} T_i$ by

$$[(x_i)_{i \in I}, (y_i)_{i \in I}, (z_i)_{i \in I}] = ([x_i y_i z_i]_i)_{i \in I}$$

for all $(x_i)_{i \in I}, (y_i)_{i \in I}, (z_i)_{i \in I}$ in $\prod_{i \in I} T_i$. It is a routine matter to verify that $\prod_{i \in I} T_i$ is a ternary semigroup. If $I = \{1, 2, ..., n\}$ we write $\prod_{i \in I} T_i$ as $T_1 \times T_2 \times \cdots \times T_3$.

We need the following two lemmas to show that the converse of Theorem 10 is not valid in general.

Lemma 11. *Let $\{T_i \mid i \in I\}$ be an indexed family of ternary semigroups such that $(x, y, z) \mapsto [xyz]_i$ is the ternary operation on T_i for each $i \in I$. If A_i is an ideal of T_i for each $i \in I$, then $\prod_{i \in I} A_i$ is an ideal of $\prod_{i \in I} T_i$.*

Proof. Since $A_i \neq \emptyset$ for all $i \in I$, we have $\prod_{i \in I} A_i \neq \emptyset$. If $(c_i)_{i \in I}$, $(b_i)_{i \in I} \in \prod_{i \in I} T_i$ and $(a_i)_{i \in I} \in \prod_{i \in I} A_i$, then

$$[(b_i)_{i \in I}(a_i)_{i \in I}(c_i)_{i \in I}] = ([b_i a_i c_i]_i)_{i \in I} \in \prod_{i \in I} A_i.$$

Similarly, we have

$$[\prod_{i \in I} T_i \prod_{i \in I} T_i \prod_{i \in I} A_i] \subseteq \prod_{i \in I} A_i$$

and

$$[\prod_{i \in I} A_i \prod_{i \in I} T_i \prod_{i \in I} T_i] \subseteq \prod_{i \in I} A_i.$$

Hence $\prod_{i \in I} A_i$ is an ideal of $\prod_{i \in I} T_i$ as required. □

We now consider the closed interval of real numbers $T = [0, 1]$. Under the ternary operation on T defined by $(x, y, z) \mapsto xyz$, it is clear that T is a ternary semigroup. The next lemma shows that, for any $a \in T$, $[0, a]$ is an ideal of T.

Lemma 12. *For any $a \in T$, $A_a = [0, a]$ is an ideal of T.*

Proof. Let $a \in T$, i.e., $a \in [0, a]$. Then $A_a \neq \emptyset$. If $x, y \in T$ and $z \in [0, a]$, then

$$0 \leq xzy, xyz, zxy \leq a.$$

Hence A_a is an ideal of T. □

The following example shows that the converse of the theorem is not true in general.

Example 13. We consider the ternary semigroup

$$T \times T = [0,1] \times [0,1].$$

Clearly, $T \times T$ is a commutative ternary semigroup with identity $(1,1)$. By Lemma 12, $A_0 = \{0\}$ is an ideal of T. Then by Lemma 11 we have

$$A = [0,1] \times \{0\}$$

is an ideal of $T \times T$.

To show that A is prime we use Theorem 9. Let (x_1, y_1), (x_2, y_2), $(x_3, y_3) \in T \times T$ be such that $[(x_1, y_1)(x_2, y_2)(x_3, y_3)] \in A$; thus

$$[x_1 x_2 x_3, y_1 y_2 y_3] \in A,$$

and so $y_1 y_2 y_3 = 0$. This implies that $y_1 = 0$, $y_2 = 0$ or $y_3 = 0$. Therefore, A is prime.

Since $A_{1/2}$ is an ideal of T, it follows that

$$A \subset [0,1] \times A_{1/2} \subset T \times T,$$

whence A is not maximal.

Acknowledgments

The author was supported by the Faculty of Science, Khon Kaen University.

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