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EQUITABLE DEFECTIVE COLORINGS OF COMPLETE BIPARTITE GRAPHS

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Abstract: A graph G has an equitable k-defective coloring in m colors if its vertices can be colored with m colors such that the maximum degree of any subgraph induced by vertices assigned to the same color is at most k and the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. The equitable k-defective chromatic number of a graph G, denoted by $\chi_{ED,k}(G)$, is the smallest positive integer m for which G has an equitable k-defective coloring in m colors. In this paper, we present the equitable k-defective chromatic numbers of complete bipartite graphs for $k = 1$ and $k = 2$.

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1. Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. In this paper,

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graphs are considered to be finite, undirected, and simple. Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer that is not less than x and the largest integer that is not greater than x, respectively. We refer the reader to [15] for terminology in graph theory.

An *equitable m-coloring* of a graph G is a proper m-coloring of G such that the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. A graph G is said to be *equitably m-colorable* if G has an equitable m-coloring. The *equitable chromatic number* of a graph G, denoted by $\chi_=(G)$, is the smallest positive integer m for which G is equitably m-colorable. The equitable coloring introduced first by Meyer [13] in 1973.

Lih and Wu [9] investigated the equitable chromatic number of a connected bipartite graph. The authors proved that every complete bipartite graph $K_{n,n}$ can be equitably colored using k colors if and only if $\lceil n/\lfloor k/2 \rfloor \rceil - \lfloor n/\lceil k/2 \rceil \rfloor \leq 1$. Moreover, if G is a connected bipartite graph with partite sets X, Y and ϵ edges such that $\epsilon < \lfloor n/(m+1) \rfloor (n-m) + 2n$, then $\chi_=(G) \leq 1 + \lceil n/(m+1) \rceil$, where $|X| = m \leq n = |Y|$.

Lam et. al. [8] determined the equitable chromatic number of a complete r-partite graph $K_{m_1,\ldots,m_r}$. They showed that $\chi_=(K_{m_1,\ldots,m_r}) = \sum_{i=1}^{r} \lceil m_i/(M+1) \rceil$ where M is the largest integer such that $m_i (\text{mod } M) \leq \lceil m_i/M \rceil$ $(i = 1, \ldots, r)$.

Nakprasit and Saigrasun [14] characterized the complete bipartite graph $K_{m,n}$ with $m \leq n$ such that $\chi_=(K_{m,n}) = 1 + \lceil n/(m+1) \rceil$ and found the smallest integer C such that for every integer $n \geq C$ implies $\chi_=(K_{m,n}) = 1 + \lceil n/(m+1) \rceil$.

For more on the equitable coloring of graphs see [6] and [10].

A subset U of $V(G)$ is said to be *k-independent* if the maximum degree of an induced subgraph $G[U]$ is at most k. A *k-defective coloring in m colors* of a graph G is an m-coloring of G such that the set of vertices that are assigned to the same color is k-independent. A graph G is *(m, k)-colorable* if G has an k-defective coloring in m colors. The *k-defective chromatic number* of a graph G, denoted by $\chi_k(G)$, is the smallest positive integer m for which G is (m, k)-colorable. Note that $\chi_0(G)$ is the usual chromatic number of G. It is clear that $\chi_k(G) \leq \lceil n/(k+1) \rceil$, where n is the order of G.

The concept of (m, k)-coloring has been studied by several authors. Hopkins and Staton [7] refered to a k-independent set as a k-small set. Maddox ([11], [12]) and Andrews and Jacobson [2] refered to this set as a k-dependent set. The k-defective chromatic number has been investigated as the k-partition number by Frick [4], Frick and Henning [5], Maddox ([11], [12]), Hopkins and Staton [7] and under the name k-chromatic number by Andrews and Jacobson [2].

Achuthan et al. [1] determined the smallest order of a triangle-free graph such that $\chi_k(G) = m$, denoted by $f(m,k)$. They showed that $f(3,2) = 13$. Moreover, they presented a lower bound for $f(m,k)$ for $m \geq 3$ and also an upper bound for $f(3,k)$.

A graph G has an *equitable k-defective coloring* in m colors if G has a k-defective coloring in m colors and the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. The *equitable k-defective chromatic number* of a graph G, denoted by $\chi_{ED,k}(G)$, is the smallest positive integer m for which G has an equitably k-defective coloring in m colors.

Cummuang and Nakprasit [3] presented the equitable k-defective chromatic numbers of paths, cycles, complete graphs, hypercubes, stars, and wheels for any positive integer k.

Williams, Vandenbussche, and Yu [16] studied the equitable defective coloring of sparse planar graph by using the discharing method. The authors proved that every planar graph with minimum degree at least 2 and girth at least 10 has an equitable 1-defective coloring in m colors for $m \geq 3$.

In this paper, we show that $\chi_{ED,1}(K_{m,n}) = \chi_{=}(K_{m,n})$ for $2 \leq m \leq n$ and $\chi_{ED,2}(K_{m,n}) = \chi_{=}(K_{m,n})$ for all but a finite number of (m,n) pairs.

2. Preliminary Results

Let $K_{m,n}$ be a complete bipartite graph with partite sets X and Y, where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, and $m \leq n$. Lemmas 1 – 4 can be found in [14].

Lemma 1. *Let $K_{m,n}$ be a complete bipartite graph.*
If $V_1, V_2, \ldots, V_{1+\lceil n/(m+1)\rceil}$ are equitable color classes of $K_{m,n}$, then $X = V_i$ for some $i \in \{1, 2, \ldots, 1 + \lceil n/(m+1)\rceil\}$.

For Lemmas 2 to 4, we let $n = a(m+1) + b, 0 \leq b \leq m$ where a and b are integers.

Lemma 2. *The complete bipartite graph $K_{m,n}$ has $1 + \lceil n/(m+1)\rceil$ equitable color classes of size m or $m+1$ if and only if $b = 0$ or $m - a \leq b \leq m$.*

Lemma 3. *The equitable chromatic number $\chi_{=}(K_{m,n}) = 1 + \lceil n/(m+1)\rceil$ if and only if $b = 0$ or $m - 2a - 1 \leq b \leq m$.*

Lemma 4. *Given a positive integer m, let C be the smallest positive integer such that $\chi_{=}(Km,n) = 1 + \lceil n/(m+1)\rceil$ for every integer $n \geq C$. Then $C = (m-1)(\lceil m/2\rceil - 1)$.*

Lemma 5. *Let $k, n \in \mathbb{N}$ and $n \geq k$ such that $n = a(k+1)+b$, $0 \leq b \leq k$. There exist nonnegative integers s and t such that $n = s(k+1)+tk$ if and only if $b = 0$ or $k-a \leq b \leq k$.*

Proof. To prove the theorem, we first assume that there exist nonnegative integers s and t such that $n = s(k+1)+tk$. By choosing s and t such that $s+t$ is minimum, we can show that $(a,b) = (s,0)$ for $t = 0$ and $(a,b) = (s+t-1, k-t+1)$ for $t > 0$.

Suppose that $b = 0$ or $k-a \leq b \leq k$. We can show that $n = s(k+1)+tk$ where (s,t) are $(a,0)$, $(0,a+1)$, and $(a+b-k, 1+k-b)$ for $b = 0$, $b = k-a$, and $k-a+1 \leq b \leq k$, respectively. This completes the proof. □

Lemma 6. *Let $k, n \in \mathbb{N}$ with $n \geq k(k-1)$. Then there exist nonnegative integers s and t such that $n = s(k+1)+tk$ and $s+t = \lceil n/(k+1) \rceil$.*

Proof. By the division algorithm, $n = a(k+1)+b$ where $0 \leq b \leq k$. We consider two cases.

Case 1 : $a = k-2$. Then $b \geq 2$. Consequently, $k-a = 2 \leq b \leq k$.

Case 2 : $a \geq k-1$. For $b = 0$, we have $s = a$ and $t = 0$.

For $1 \leq b \leq k$, we have $k-a+1 \leq b \leq k$. Lemma 5 implies there exist nonnegative integers s and t such that $n = s(k+1)+tk$. We choose s and t with minimum $s+t$ to obtain $t \leq k$ which implies $s+t = \lceil n/(k+1) \rceil$. □

3. The Equitable 1-Defective Coloring of $K_{m,n}$

In this section, we investigate the equitable 1-defective chromatic numbers of complete bipartite graphs $K_{m,n}$ with $m \leq n$.

Theorem 7. *For a complete bipartite graph $K_{1,n}$, $\chi_{ED,1}(K_{1,n}) = 1 + \lceil (n-1)/3 \rceil$.*

Proof. The proof follows from Corollary 2 in [3]. □

Theorem 8. *For a complete bipartite graph $K_{m,n}$ with $m \geq 2$, $\chi_{ED,1}(K_{m,n}) = \lceil m/(M+1) \rceil + \lceil n/(M+1) \rceil$ where M is the largest integer such that $m (\text{mod } M) < \lceil m/M \rceil$ and $n (\text{mod } M) < \lceil n/M \rceil$.*

Proof. We first define $f(t) = t+\lceil (m-t)/3\rceil+\lceil (n-t)/3\rceil$ for $t = 0, 1, 2, \ldots, m$. Observe that the equitable 1-defective coloring with t non-independent color classes, where $t \geq 1$, has at least $f(t)$ color classes.

One can verify that $\min\{f(t) : t = 0, 1, 2, \ldots, m\} = f(1)$ for $m \equiv n \equiv 1 (\text{mod } 3)$, and otherwise $\min\{f(t) : t = 0, 1, 2, \ldots, m\} = f(0)$. We consider two cases.

Case 1 : $m \equiv n \equiv 1(\text{mod } 3)$.

In this case, the minimum of $f(t)$ is attained at $t = 1$ and $f(1) = 1 + \lceil (m - 1)/3\rceil + \lceil (n-1)/3\rceil \geq (m + n + 1)/3$.

Since $m \equiv 1(\text{mod } 3) < \lceil m/3\rceil$ and $n \equiv 1(\text{mod } 3) < \lceil n/3\rceil$, we obtain $M \geq 3$. This implies that $\lceil m/(M + 1)\rceil + \lceil n/(M + 1)\rceil \leq \lceil m/4\rceil + \lceil n/4\rceil \leq [(m + 3)/4] + [(n + 3)/4]$.

Since $[(m+n+1)/3] - [((m+3)/4) + ((n+3)/4)] = [(m+n-14)/12] \geq 0$ for $m + n \geq 14$, the theorem holds. The remaining (m, n) are $(4, 4)$ and $(4, 7)$ for which $\chi_{ED,1}(K_{m,n}) = \lceil m/(M + 1)\rceil + \lceil n/(M + 1)\rceil$.

Case 2 : $m \not\equiv 1(\text{mod } 3)$ or $n \not\equiv 1(\text{mod } 3)$.

In this case, the minimum of $f(t)$ is attained at $t = 0$. Since $\lceil m/(M+1)\rceil + \lceil n/(M+1)\rceil \leq \lceil m/3\rceil + \lceil n/3\rceil = f(0)$, therefore $\chi_{ED,1}(K_{m,n}) = \lceil m/(M+1)\rceil + \lceil n/(M + 1)\rceil$. □

4. The Equitable 2-Defective Coloring of $K_{m,n}$

In this section, we investigate the equitable 2-defective chromatic numbers of complete bipartite graphs $K_{m,n}$ with $m \leq n$. First, we introduce some definitions that will be used in later arguments.

Definition 9. Let P be a color class in an equitable 2-defective coloring of $K_{m,n}$. We say that

1. P is a **color class of type *A*** if it comprises two vertices of X and two vertices of Y;
2. P is a **color class of type *B*** if it comprises two vertices of X and one vertex of Y;
3. P is a **color class of type *C*** if it comprises one vertex of X and two vertices of Y.

Definition 10. Let c be an equitable 2-defective coloring of $K_{m,n}$. We say that

1. c is a **coloring of type *A5*** if its maximum color class size is 5 and it contains a color class of type ***A*** as its only non-independent color class;
2. c is a **coloring of type *A4*** if its maximum color class size is 4 and it contains a color class of type ***A*** as its only non-independent color class;
3. c is a **coloring of type *B4*** if its maximum color class size is 4 and it contains a color class of type ***B*** as its only non-independent color class;
4. c is a **coloring of type *B3*** if its maximum color class size is 3 and it contains a color class of type ***B*** as its only non-independent color class;
5. c is a **coloring of type *C4*** if its maximum color class size is 4 and it contains a color class of type ***C*** as its only non-independent color class;
6. c is a **coloring of type *C3*** if its maximum color class size is 3 and it contains a color class of type ***C*** as its only non-independent color class.

Lemma 11. *The following statements hold for the equitable 2-defective coloring of $K_{m,n}$.*

(i) Every equitable 2-defective coloring of $K_{m,n}$ with $\chi_{ED,2}(K_{m,n})$ color classes has a color class which induces K_2 if and only if $(m,n) = (1,1)$ or $(1,3)$.

(ii) If $K_{m,n}$ has an equitable 2-defective coloring in c colors, then it has an equitable 2-defective coloring in c colors that has at most one non-independent color classes.

Proof. We shall prove only the sufficiency of (i) because the necessity can be easily verified.
Consider an equitable 2-defective coloring with $\chi_{ED,2}(K_{m,n})$ color classes with a color class that induces K_2.

Let $P = \{x_1, y_1\}$ and Q be non-independent color classes that result from this coloring. We repartition $P \cup Q$ into two equitable independent sets and leave all other color classes unchanged. By continuing this process, we obtain an equitable 2-defective coloring that has at most one non-independent set that is a color class that induces K_2.

Let $\{x_1, y_1\}, A_1, A_2, \ldots, A_r, B_1, B_2, \ldots, B_s$ be color classes that result from the above coloring where $A_i \subseteq X, B_j \subseteq Y$ and $|A_1| \le |A_i|, |B_1| \le |B_j|$ for $1 \le i \le r, 1 \le j \le s$.

Note that $1 + r + s = \chi_{ED,2}(K_{m,n})$. Since $m \ge 2$, we have $r \ge 1$ and $s \ge 1$. Then $A_1 \cup \{x_1\}, A_2, \ldots, A_r, B_1 \cup \{y_1\}, B_2, \ldots, B_s$ are equitable color classes of $K_{m,n}$. This contradicts $\chi_{ED,2}(K_{m,n}) = 1 + r + s$. Thus $m = 1$.

For $n = 2$ or $n \ge 4$, it is straightforward to verify that a coloring of type ***C3*** or ***C4*** has $\chi_{ED,2}(K_{1,n})$ color classes. Thus, the only possible (m,n) are $(1,1)$ and $(1,3)$.

Next, we prove (ii) by considering an equitable 2-defective coloring with (at least) two non-independent color classes, say P and Q. We can repartition $P \cup Q$ into two color classes with fewer non-independent color classes and leave all other color classes unchanged. By continuing this process, we obtain an equitable 2-defective coloring with at most one non-independent color class. □

Lemma 12. *For a complete bipartite graph* $K_{1,n}$, $\chi_{ED,2}(K_{1,n}) = 1+\lceil (n-2)/4 \rceil$.

Proof. The proof follows from Corollary 2 in [3]. □

Lemma 13. *For a complete bipartite graph* $K_{2,n}$, $\chi_{ED,2}(K_{2,n}) = 1+\lceil (n-2)/5 \rceil$.

Proof. For $1 \le n \le 6$, it is easy to see that $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$. We consider the case when $n \ge 7$. By the division algorithm, $n - 2 = 5q + r, 0 \le r \le 4$. Therefore, $r = 0$ or $3 - 2q \le r \le 4$. Lemma 3 implies that $\chi_{=}(K_{4,n-2}) = 1 + \lceil (n-2)/5 \rceil$.

Consider a coloring of type ***C4*** or ***C3***. Note that the minimum number of color classes that result from this coloring is $\chi_{=}(K_{4,n-2}) = 1+\lceil (n-2)/5 \rceil$, but $\chi_{=}(K_{2,n}) = 1 + \lceil n/3 \rceil$. Thus, $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$ by Lemma 11. □

Lemma 14. *[coloring of type **A5**] Suppose that* $m, n \ne 2$. *The number of color classes in a coloring of type* **A5** *is less than* $\chi_{=}(K_{m,n})$ *if and only if* $(m,n) = (7,10), (7,11), (7,17),$ $(11,14),$ *or* $(11,19)$.

Proof. We begin by assuming that the number of color classes in a coloring of type ***A5*** is less than $\chi_{=}(K_{m,n})$. Then there exist nonnegative integers $q, r, s,$ and t such that $m = 2+4q+5r$, $n = 2+4s+5t$, and $\chi_{=}(K_{m,n}) > q+r+s+t+1$.

Let $A = \{x \in \mathbb{N} : x \ge 20\} \cup \{5, 6, 10, 11, 12, 15, 16, 17, 18\}$.

Observe that if the smallest color class size of an equitable coloring is at least 5, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***A5***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. Thus, we consider only the cases when $m \in \{7, 14, 19\}$ or $n \in \{7, 14, 19\}$.

Case 1 : $m \in \{7, 14, 19\}$. We consider two subcases.

Subcase 1.1 : $m = 7$. If $n \ge 18$, then Lemmas 1 and 4 indicte that $K_{7,n}$ has an equitable coloring with color classes of size 7, which is a contradiction. The remaining cases are $n = 7, 10, 11, 12, 14, 15, 16,$ or 17. We can verify directly that the only possible (m,n) are $(7,10),(7,11)$, and $(7,17)$.

Subcase 1.2 : $m = 14$ or 19. It is easy to show that $\chi_=(K_{m,n}) \le q+r+s+t+1$ which is a contradiction.

Case 2 : $n \in \{7,14,19\}$. We consider two subcases.

Subcase 2.1 : $n = 7$. We consider only $(m,n) = (6,7)$ and $(7,7)$. The number of color classes that result from a coloring of type ***A5*** on $K_{m,7}$ is 3. Because this result is greater than $\chi_=(K_{m,7}) = 2$, we have a contradiction.

Subcase 2.2 : $m = 14$ or 19. If $q \ge 2$ or $r \ge 2$, then $\chi_=(K_{m,n}) \le q+r+s+t+1$ which is a contradiction. It is sufficient to consider $q \le 1$ and $r \le 1$. That is $m = 6, 7$, or 11. With the exception of the cases $(m,n) = (11,14)$ and $(11,19)$, the numbers of color classes resulting from a coloring of type ***A5*** is greater than $\chi_=(K_{m,n})$.

Conversely, we can easily verify that the numbers of color classes resulting from a coloring of type ***A5*** are $4,4,5,6$ and 7 for $(m,n) = (7,10)$, $(7,11)$, $(7,17)$, $(11,14)$, and $(11,19)$, respectively. Each of the numbers of color classes is less than the equitable chromatic number of the corresponding graph. □

Lemma 15. *[coloring of type* ***A4****] Suppose that $m,n \ne 2$. The number of color classes in a coloring of type* **A4** *is less than $\chi_=(K_{m,n})$ if and only if $(m,n) = (6,9)$.*

Proof. We begin by assuming that the number of color classes in a coloring of type ***A4*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 2+3q+4r$, $n = 2+3s+4t$, and $\chi_=(K_{m,n}) > q+r+s+t+1$.

Let $A = \{x \in \mathbb{N} : x \ge 12\} \cup \{4,5,8,9,10\}$.

Observe that if the smallest color class size of an equitable coloring is at least 4, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***A4***. Combining this with Lemma 6, we have a contradiction for $m,n \in A$. Therefore, we consider only the cases when $m \in \{6,11\}$ or $n \in \{6,11\}$.

Case 1 : $m \in \{6,11\}$. We consider two subcases.

Subcase 1.1 : $m = 6$. For $t \ge 2$, Lemmas 1 and 4 imply that $\chi_=(K_{6,n}) < q+r+s+t+1$ which is a contradiction. Because $\chi_=(K_{6,n}) \le q+r+s+t+1$ when$s \ge 2$, we consider only $(m,n) = (6,6)$ and $(6,9)$. The number of color classes that result from a coloring of type ***A4*** on $K_{6,6}$ is 3, which is greater than $\chi_=(K_{6,6}) = 2$. Thus, the remaining (m,n) is $(6,9)$.

Subcase 1.2 : $m = 11$. Since $\chi_=(K_{11,n}) \le q+r+s+t+1$ for all n, we have a contradiction.

Case 2 : $n \in \{6,11\}$. We can show that $\chi_=(K_{m,n}) \le q+r+s+t+1$ except the cases $0 \le q \le 1$ and $r = 0$. That is, $m = 2$ or 5. However, each of these cases is eliminated.

Hence, the only possible (m,n) in this case is $(6,9)$.

Conversely, we can verify that the number of color classes that result from a coloring of type ***A4*** on $K_{6,9}$ is 4, which is less than $\chi_=(K_{6,9}) = 5$. □

Lemma 16. *[coloring of type **B4**] Suppose that $m,n \neq 2$. The number of color classes in a coloring of type* **B4** *is less than $\chi_=(K_{m,n})$ if and only if $(m,n) = (5,7)$ or $(6,9)$.*

Proof. We begin by assuming that the number of color classes in a coloring of type ***B4*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q,r,s, and t such that $m = 2+3q+4r$, $n = 1+3s+4t$, and $\chi_=(K_{m,n}) > q+r+s+t+1$.

Let $A = \{x \in \mathbb{N} : x \geq 12\} \cup \{4,5,8,9,10\}$.

Observe that if the smallest color class size of an equitable coloring is at least 4, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***B4***. Combining this with Lemma 6, we have a contradiction for $m,n \in A$. Therefore, we consider only the cases when $m \in \{6,11\}$ or $n \in \{1,7,11\}$.

Case 1 : $m \in \{6,11\}$. We consider two subcases.

Subcase 1.1 : $m = 6$. By Lemmas 1 and 4, $\chi_=(K_{6,n}) < q+r+s+t+1$ where $n \geq 10$. Therefore, $n = 7, 8$, or 9. However, $\chi_=(K_{6,7}) = 2$ and $\chi_=(K_{6,8}) = 4$ are not less than the number of color classes that result from a coloring of type ***B4***. Thus, the only possibility is $(m,n) = (6,9)$.

Subcase 1.2 : $m = 11$. Since $\chi_=(K_{11,n}) \leq q+r+s+t+1$ for all n, we have a contradiction.

Case 2 : $n \in \{1,7,11\}$. Because $K_{1,1}$ cannot be assigned by a coloring of type ***B4***, we consider only $n = 7$ or 11. We can show that $\chi_=(K_{m,n}) \leq q+r+s+t+1$ if $m \neq 5$.

The number of color classes that result from a coloring of type ***B4*** on $K_{5,11}$ is 5, which is greater than $\chi_=(K_{5,11}) = 3$. Thus, the only possible (m,n) in this case is $(5,7)$.

Conversely, for both $(m,n) = (5,7)$ and $(6,9)$, the number of color classes that result from a coloring of type ***B4*** on $K_{m,n}$ is 4, which is less than $\chi_=(K_{m,n}) = 5$. □

Lemma 17. *[coloring of type **B3**] Suppose $m,n \neq 2$. The number of color classes in a coloring of type* **B3** *is less than $\chi_=(K_{m,n})$ if and only if $(m,n) = (5,7)$.*

Proof. We first assume that the number of color classes in a coloring of type ***B3*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q,r,s, and t

such that $m = 2 + 2q + 3r$, $n = 1 + 2s + 3t$, and $\chi_=(K_{m,n}) > q + r + s + t + 1$.

Let $A = \{x \in \mathbb{N} : x \geq 6\} \cup \{3, 4\}$.

Observe that if the smallest color class size of an equitable coloring is at least 3, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***B3***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. We consider only the cases when $m = 5$ or $n = 5$.

Case 1 : $m = 5$. Lemmas 1 and 4 imply that $\chi_=(K_{5,n}) < q + r + s + t + 1$ for $n \geq 8$. If $n = 5$ or 6, then Lemma 2 indicates that $\chi_=(K_{5,n}) < q+r+s+t+1$. Thus, the only possibility is $(m, n) = (5, 7)$.

Case 2 : $n = 5$. For both $(m, n) = (4, 5)$ and $(5, 5)$, the number of color classes that result from a coloring of type ***B3*** is 4, which is greater than $\chi_=(K_{m,n}) = 2$. This is a contradiction.

Conversely, the number of color classes the result from a coloring of type ***B3*** on $K_{5,7}$ is 4, which is less than $\chi_=(K_{5,7}) = 5$. □

Lemma 18. *[coloring of type* ***C4****] Suppose that* $m, n \notin \{1, 2\}$. *The number of color classes in a coloring of type* **C4** *is not less than* $\chi_=(K_{m,n})$.

Proof. Suppose that the number of color classes in a coloring of type ***C4*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 1 + 3q + 4r$, $n = 2 + 3s + 4t$, and $\chi_=(K_{m,n}) > q + r + s + t + 1$.

Let $A = \{x \in \mathbb{N} : x \geq 12\} \cup \{4, 5, 8, 9, 10\}$.

Observe that if the smallest color class size of an equitable coloring is at least 4, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***C4***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. We consider only the cases when $m \in \{7, 11\}$ or $n \in \{6, 11\}$.

Case 1 : $m \in \{7, 11\}$. We can show that $\chi_=(K_{m,n}) \leq q+r+s+t+1$ if $n \neq 2, 5$. However, $n = 2, 5$ violates the condition $m \leq n$.

Case 2 : $n \in \{6, 11\}$. Because $\chi_=(K_{4,6}) = 3$ and $\chi_=(K_{5,6}) = 2$ are not greater than the number of color classes that result from a coloring of type ***C4***, we consider only $n = 11$.

Because $m = 7$ and 11 have been considered, only the cases when $m = 4, 5, 8, 9$, or 10 remain to be considered. We can show that the numbers of color classes that result from a coloring of type ***C4*** are $5, 5, 6, 6$, and 7 for $(m, n) = (4, 11)$, $(5, 11)$, $(8, 11)$, $(9, 11)$, and $(10, 11)$, respectively. Each of the numbers of color classes is not less than the equitable chromatic number of the corresponding graph.

Hence, the numbers of color classes that result from a coloring of type ***C4*** are not less than $\chi_=(K_{m,n})$. □

Lemma 19. *[coloring of type **C3**] Suppose* $m, n \notin \{1, 2\}$. *The number of color classes in a coloring of type* **C3** *is not less than* $\chi_=(K_{m,n})$.

Proof. Suppose that the number of color classes in a coloring of type ***C3*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 1 + 2q + 3r$, $n = 2 + 2s + 3t$, and $\chi_=(K_{m,n}) > q + r + s + t + 1$.

Let $A = \{x \in \mathbb{N} : x \geq 6\} \cup \{3, 4\}$.

Observe that if the smallest color class size of an equitable coloring is at least 3, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***C3***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. We consider only the cases when $m = 5$ or $n = 5$.

Because $\chi_=(K_{5,n}) \leq q+r+s+t+1$ for all n, we consider only the case when $n = 5$. The numbers of color classes that result from a coloring of type ***C3*** on $K_{3,5}$, $K_{4,5}$, and $K_{5,5}$ are $3, 3$, and 4, respectively. Each of the numbers of color classes is not less than the equitable chromatic number of the corresponding graph.

Hence, the number of color classes in a coloring of type ***C3*** is not less than $\chi_=(K_{m,n})$. □

Now, we are ready to prove our main Theorem.

Theorem 20. *For a complete bipartite graph* $K_{m,n}$ *with* $m \leq n$, $\chi_{ED,2}(K_{m,n}) = \chi_=(K_{m,n})$ *except*

1. $\chi_{ED,2}(K_{1,n}) = 1 + \lceil (n-2)/4 \rceil$;

2. $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$;

3. $\chi_{ED,2}(K_{m,n}) = 1 + \lceil (m-2)/5 \rceil + \lceil (n-2)/5 \rceil$,
   where $(m, n) = (7, 10), (7, 11), (7, 17), (11, 14),$ *or* $(11, 19)$;

4. $\chi_{ED,2}(K_{5,7}) = 4$;

5. $\chi_{ED,2}(K_{6,9}) = 4$.

Proof. We have by Lemma 12, $\chi_{ED,2}(K_{1,n}) = 1 + \lceil (n-2)/4 \rceil$. By Lemma 13, $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$. In addition, $\chi_{=}(K_{1,n}) = 1 + \lceil n/2 \rceil$ and $\chi_{=}(K_{2,n}) = 1 + \lceil n/3 \rceil$.

Consider $K_{m,n}$ where $(m,n) = (7,10), (7,11), (7,17), (11,14)$, or $(11,19)$. By Lemma 14, the number of color classes that result from a coloring of type ***A5*** on $K_{m,n}$ is less than $\chi_{=}(K_{m,n})$. By Lemmas 15 – 19, colorings of other types on $K_{m,n}$ do not exist or the resulting numbers of color classes are less than $\chi_{=}(K_{m,n})$. From the proof of Lemma 14, the numbers of color classes that result from colorings of type ***A5*** on $K_{m,n}$ are $4, 4, 5, 6$, and 7 for $(m,n) = (7,10), (7,11), (7,17), (11,14)$, and $(11,19)$, respectively. Hence, $\chi_{ED,2}(K_{m,n}) = 1 + \lceil (m-2)/5 \rceil + \lceil (n-2)/5 \rceil$ for (m,n) in this case.

For $K_{5,7}$, Lemma 17 implies that the number of color classes that result from a coloring of type ***B3*** on $K_{5,7}$ is less than $\chi_{=}(K_{5,7})$. By Lemmas 14 – 19, colorings of other types on $K_{5,7}$ do not exist or the resulting numbers of color classes are less than $\chi_{=}(K_{5,7})$. From the proof of Lemma 17, the number of color classes that result from a coloring of type ***B3*** on $K_{5,7}$ is 4. Hence, $\chi_{ED,2}(K_{5,7}) = 4$.

For $K_{6,9}$, Lemmas 15 and 16 imply that the numbers of color classes that result from colorings of types ***A4*** and ***B4*** on $K_{6,9}$ are less than $\chi_{=}(K_{6,9})$. By Lemmas 14 – 19, colorings of other types on $K_{6,9}$ do not exist or the resulting numbers of color classes are less than $\chi_{=}(K_{6,9})$. The proofs of Lemmas 15 and 16 show that the number of color classes resulting from a coloring of types ***A4*** and ***B4*** are equal to 4. Hence, $\chi_{ED,2}(K_{6,9}) = 4$.

If $K_{m,n}$ is not addressed by the previous cases, then Lemmas 14 – 19 indicate that $\chi_{=}(K_{m,n})$ is not greater than the number of color classes that result from a coloring of type ***A5***, ***A4***, ***B4***, ***B3***, ***C4***, or ***C3***. Hence, $\chi_{ED,2}(K_{m,n}) = \chi_{=}(K_{m,n})$. □

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