SLIGHTLY $\pi g\beta$ -CONTINUOUS FUNCTIONS

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Abstract: A new class of functions, called slightly π -generalized β -continuous functions is introduced. Basic properties of slightly π -generalized β -continuous functions are studied. The class of slightly π -generalized β -continuous functions properly includes the class of slightly $g\beta$ -continuous functions and $\pi g\beta$ -continuous functions. Also, by using slightly π -generalized β -continuous functions, some properties of domain/range of functions are characterized.

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1. Introduction and Preliminaries

In the literature, β -open sets are being studied by many authors. Abd El-Monsef [1] introduced β -open sets and β -continuous mappings in 1983. Slightly β -continuous functions were introduced by Noiri [15] in 2001. In 1995, Dontchev [9] introduced the notion of $g\beta$ -closed sets and investigated some of their basic properties. In 2013, Arora and Tahiliani [4] introduced $g\beta$ -continuous functions and slightly $g\beta$ -continuous functions. Recently, Tahiliani [24] introduced and studied $\pi g\beta$ -closed sets in topological spaces.

In this paper, we define slightly π -generalized β -continuous functions and show that the class of slightly π -generalized β -continuous functions properly includes the class of slightly $g\beta$ -continuous functions and $\pi g\beta$ -continuous functions. Secondly we obtain some new results on $\pi g\beta$ -closed sets and investigate basic properties of slightly π -generalized β -continuous functions concerning composition and restriction. Finally, we study the behaviours of some separation axioms and related properties, $\pi G\beta O$ -compactness and $\pi G\beta O$ -connectedness under slightly π -generalized β -continuous functions. Relationships between slightly π -generalized β -continuous functions and $\pi G\beta O$ -connected spaces are investigated. In particular, it is shown that slightly π -generalized β -continuous image of $\pi G\beta O$ -connected spaces is connected.

Throughout this paper, (X, τ) and (Y, σ) (or X and Y) represents a topological space on which no separation axioms are assumed, unless otherwise mentioned. The closure and interior of $A \subseteq X$ will be denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively.

Definition 1.1. 1. A subset A of a space X is called regular open [21] if $A = \text{int}(\text{cl}(A))$.

2. A subset A of a space X is called π -open [25] if the finite union of regular open sets. The complement of π -open set is called π -closed.

3. A subset A of a space X is called β -open [1] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$.

The complement of β -open set is β -closed [1].

The intersection of all β -closed sets containing A is called the β -closure of A and is denoted by $\beta\text{cl}(A)$ [2].

A is said to be β -clopen [15] if it is β -open and β -closed.

The largest β -open set contained in A (denoted by $\beta\text{int}(A)$) is called the β -interior [2] of A .

4. A subset A of a space X is said to be generalized closed [12] (briefly g -closed) if $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X .

The complement of g -closed set is called g -open.

5. A subset A of a space X is said to be π -generalized closed [10] (briefly πg -closed) if $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is π -open in X .

The complement of πg -closed set is πg -open.

6. A subset A of a space X is said to be $g\beta$ -closed [9, 22] if $\beta\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X .

7. A subset A of a space X is said to be $g\beta$ -open [9, 22] if $F \subseteq \beta\text{int}(A)$ whenever $F \subseteq A$ and F is closed in X .

Also it is a complement of $g\beta$ -closed set.

8. A subset A of a space X is said to be $\pi g\beta$ -closed [24] if $\beta\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is π -open in X .

9. A subset A of a space X is said to be $\pi g\beta$ -open [24] if $F \subseteq \beta\text{int}(A)$ whenever $F \subseteq A$ and F is π -closed in X .

Also it is a complement of $\pi g\beta$ -closed set.

If A is both $\pi g\beta$ -closed and $\pi g\beta$ -open, then it is said to be $\pi g\beta$ -clopen.

In this paper, the family of all open (resp. πg -open, $\pi g\beta$ -open, clopen) sets of a space X is denoted by $O(X)$ (resp. $\pi GO(X)$, $\pi G\beta O(X)$, $CO(X)$) and the family of $\pi g\beta$ -open (resp. clopen) sets of X containing x is denoted by $\pi G\beta O(X, x)$ (resp. $CO(X, x)$).

Definition 1.2. A function $f : X \rightarrow Y$ is called:

1. π -irresolute [24] if $f^{-1}(F)$ is π -closed in X for every π -closed set F of Y .
2. $\pi g\beta$ -continuous [24] if $f^{-1}(F)$ is $\pi g\beta$ -closed in X for every closed set F of Y .
3. slightly continuous [18] (resp. slightly $g\beta$ -continuous [4]) if for each $x \in X$ and each clopen set V of Y containing $f(x)$, there exists an open (resp. $g\beta$ -open) set U such that $f(U) \subseteq V$.
4. $\pi g\beta$ -irresolute [24] if $f^{-1}(F)$ is $\pi g\beta$ -closed in X for every $\pi g\beta$ -closed set F of Y .

5. pre β -closed [13] if the image of each β -closed set in X is β -closed in Y .
6. $\pi g\beta$ c-homeomorphism [5] if it is bijective, $\pi g\beta$ -irresolute and its inverse f^{-1} is $\pi g\beta$ -irresolute.

Definition 1.3. [4] A function $f : X \rightarrow Y$ is called slightly $g\beta$ -continuous if the inverse image of every clopen set in Y is $g\beta$ -open (resp. $g\beta$ -closed, $g\beta$ -clopen) in X .

Lemma 1.4. [24] A subset A of a space X is $\pi g\beta$ -open if and only if $F \subseteq \beta \text{int}(A)$ whenever F is π -closed and $F \subseteq A$.

Theorem 1.5 ([3], Theorem 2.7). If V is open and A is β -open then $V \cap A$ is β -open.

Lemma 1.6. [8] In any topological space (X, τ) , the intersection of an open set and πg -open set in X is a πg -open set in X .

2. Slightly π -Generalized β -Continuous Functions

Definition 2.1. A function $f : X \rightarrow Y$ is called slightly π -generalized β -continuous (briefly slightly $\pi g\beta$ -continuous) if the inverse image of every clopen set in Y is $\pi g\beta$ -open in X .

The proof of the following Theorem is straightforward and hence omitted.

Theorem 2.2. For a function $f : X \rightarrow Y$, the following statements are equivalent:

1. f is slightly $\pi g\beta$ -continuous.
2. Inverse image of every clopen subset of Y is $\pi g\beta$ -open in X .
3. Inverse image of every clopen subset of Y is $\pi g\beta$ -closed in X .
4. Inverse image of every clopen subset of Y is $\pi g\beta$ -clopen in X .

Obviously, slight $g\beta$ -continuity implies slight $\pi g\beta$ -continuity and $\pi g\beta$ -continuity implies slight $\pi g\beta$ -continuity. The following Examples show that the implications are not reversible.

Example 2.3. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined by $f(a) = a$, $f(b) = b$ and $f(c) = c$ is slightly $\pi g\beta$ -continuous but not slightly $g\beta$ -continuous.

Example 2.4. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is slightly $\pi g\beta$ -continuous but not $\pi g\beta$ -continuous.

A space is called locally discrete if every open subset is closed [7]. Also, a space is called as $g\beta$ - $T_{1/2}$ if every $\pi g\beta$ -closed subset of it is $g\beta$ -closed.

The next two Theorems are immediate of the definitions of a locally discrete and $g\beta$ - $T_{1/2}$ space.

Theorem 2.5. *If $f : X \rightarrow Y$ is slightly $\pi g\beta$ -continuous and Y is locally discrete, then f is $\pi g\beta$ -continuous.*

Theorem 2.6. *If $f : X \rightarrow Y$ is slightly $\pi g\beta$ -continuous and X is $g\beta$ - $T_{1/2}$ space, then f is slightly $g\beta$ -continuous.*

Definition 2.7. [20] A topological space X is called hyperconnected if every open set is dense.

Remark 2.8. The following Example shows that slightly $\pi g\beta$ -continuous surjection do not necessarily preserve hyperconnectedness.

Example 2.9. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is slightly $\pi g\beta$ -continuous surjective. Also (X, τ) is hyperconnected. But (X, σ) is not hyperconnected.

3. Basic Properties of Slightly $\pi g\beta$ -Continuous Functions

Definition 3.1. The intersection of all $\pi g\beta$ -closed sets containing a set A is called the $\pi g\beta$ -closure of A and is denoted by $\pi g\beta\text{cl}(A)$.

Remark 3.2. It is obvious that $\pi g\beta\text{cl}(A)$ is $\pi g\beta$ -closed and A is $\pi g\beta$ -closed if and only if $\pi g\beta\text{cl}(A) = A$.

Lemma 3.3. *Let A be a $\pi g\beta$ -open set and B be any set in X . If $A \cap B = \emptyset$, then $A \cap \pi g\beta cl(B) = \emptyset$.*

Proof. Suppose that $A \cap \pi g\beta cl(B) \neq \emptyset$ and $x \in A \cap \pi g\beta cl(B)$. Then $x \in A$ and $x \in \pi g\beta cl(B)$ and from the definition of $\pi g\beta cl(B)$, $A \cap B \neq \emptyset$. This is contrary to hypothesis.

Proposition 3.4. *Let A be $\pi g\beta$ -open in X and B an open in X . Then $A \cap B$ is $\pi g\beta$ -open in X .*

Proof. Let F be any π -closed subset of X such that $F \subseteq A \cap B \subseteq A$. By Lemma 1.4 $F \subseteq \beta int(A) = \cup \{G \subseteq X : G \text{ is } \beta\text{-open in } X \text{ and } G \subseteq A\}$. Then $F \subseteq (\cup G) \cap B = \cup (G \cap B)$. Since $G \cap B$ is a β -open set in X by Theorem 1.5 and $G \cap B \subseteq A \cap B$ for each β -open set $G \subseteq A$, $F \subseteq \cup (A \cap B) \subseteq \beta int(A \cap B)$ and by Lemma 1.4, $A \cap B$ is $\pi g\beta$ -open in X .

For a subset A of space X , the π -kernel of A [17], denoted by $\pi\text{-ker}(A)$, is the intersection of all π -open supersets of A .

Proposition 3.5. *A subset A of X is $\pi g\beta$ -closed if and only if $\beta cl(A) \subseteq \pi\text{-ker}(A)$.*

Proof. Since A is $\pi g\beta$ -closed, $\beta cl(A) \subseteq U$ for any π -open set U with $A \subseteq U$ and hence $\beta cl(A) \subseteq \pi\text{-ker}(A)$. Conversely, let U be any π -open set such that $A \subseteq U$. By hypothesis, $\beta cl(A) \subseteq \pi\text{-ker}(A) \subseteq U$ and hence A is $\pi g\beta$ -closed.

The union of two $\pi g\beta$ -closed sets is generally not a $\pi g\beta$ -closed set and the intersection of two $\pi g\beta$ -open sets is generally not a $\pi g\beta$ -open set.

Example 3.6. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then $A = \{a, b\}$ and $B = \{a, c\}$ are $\pi g\beta$ -open sets but their intersection $A \cap B = \{a\}$ is not a $\pi g\beta$ -open set.

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then $A = \{b\}$ and $B = \{c\}$ are $\pi g\beta$ -closed sets but their union $A \cup B = \{b, c\}$ is not a $\pi g\beta$ -closed set.

Proposition 3.8. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. If f is slightly*

$\pi g\beta$ -continuous, then for each point $x \in X$ and each clopen set V containing $f(x)$, there exists a $\pi g\beta$ -open set U containing x such that $f(U) \subseteq V$.

Proof. Let $x \in X$ and V be a clopen set such that $f(x) \in V$. Since f is slightly $\pi g\beta$ -continuous, $f^{-1}(V)$ is $\pi g\beta$ -open set in X . If we put $U = f^{-1}(V)$, we have $x \in U$ and $f(U) \subseteq V$.

Let (X, τ) be a topological space. The quasi-topology on X is the topology having as base all clopen subsets of (X, τ) . The open (resp. closed) subsets of the quasi-topology are said to be quasi-open (resp. quasi-closed). A point x of a space X is said to be quasi closure of a subset A of X , denoted by $cl_q A$, if $A \cap U \neq \emptyset$ for every clopen set U containing x . A subset A is said to be quasi closed if and only if $A = cl_q A$ [19]. If the closure of A in topological space coincides with $\pi g\beta cl(A)$, then it is denoted by (X, η) .

Proposition 3.9. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following are equivalent:*

1. *For each point $x \in X$ and each clopen set V containing $f(x)$, there exists a $\pi g\beta$ -open set U containing x such that $f(U) \subseteq V$.*
2. *For every subset A of X , $f(\pi g\beta cl(A)) \subseteq cl_q(f(A))$.*
3. *The map $f : (X, \eta) \rightarrow (Y, \sigma)$ is slightly continuous.*

Proof. (1) $\Rightarrow$ (2). Let $y \in f(\pi g\beta cl(A))$ and V be any clopen nbd of y . Then there exists a point $x \in X$ and a $\pi g\beta$ -open set U containing x such that $f(x) = y$, $x \in \pi g\beta cl(A)$ and $f(U) \subseteq V$. Since $x \in \pi g\beta cl(A)$, $U \cap A \neq \emptyset$ holds and hence $V \cap f(A) \neq \emptyset$. Therefore we have $y = f(x) \in cl_q(f(A))$.

(2) $\Rightarrow$ (1). Let $x \in X$ and V be a clopen set with $f(x) \in V$. Let $A = f^{-1}(Y \setminus V)$, then $x \notin A$. Since $f(\pi g\beta cl(A)) \subseteq cl_q(f(A)) \subseteq cl_q(Y \setminus V) = Y \setminus V$, it is shown that $\pi g\beta cl(A) = A$. Then since $x \notin \pi g\beta cl(A)$, there exists a $\pi g\beta$ -open set U containing x such that $U \cap A = \emptyset$ and hence $f(U) \subseteq f(X \setminus A) \subseteq V$.

(2) $\Rightarrow$ (3). Suppose that (2) holds and let V be any clopen subset of Y . Since $f(\pi g\beta cl(f^{-1}(V))) \subseteq cl_q(f(f^{-1}(V))) \subseteq cl_q(V) = V$, it is shown that $\pi g\beta cl(f^{-1}(V)) = f^{-1}(V)$ and hence we have $f^{-1}(V)$ is $\pi g\beta$ -closed in (X, τ) and hence $f^{-1}(V)$ is closed in (X, η) .

(3) $\Rightarrow$ (2). Let $y \in f(\pi g\beta cl(A))$ and V be any clopen nbd of y . Then there exists a point $x \in X$ such that $f(x) = y$ and $x \in \pi g\beta cl(A)$. Since f is slightly continuous, $f^{-1}(V)$ is open in (X, η) and so $\pi g\beta$ -open set containing x . Since x

$\in \pi g\beta\text{cl}(A)$, $f^{-1}(V) \cap A \neq \emptyset$ holds and hence $V \cap f(A) \neq \emptyset$. Therefore, we have $y = f(x) \in \text{cl}_q(f(A))$.

Now we investigate some basic properties of slightly $\pi g\beta$ -continuous functions concerning composition and restriction. The proofs of the first three results are straightforward and hence omitted.

Theorem 3.10. *If $f : X \rightarrow Y$ is $\pi g\beta$ -irresolute and $g : Y \rightarrow Z$ is slightly $\pi g\beta$ -continuous, then $g \circ f : X \rightarrow Z$ is slightly $\pi g\beta$ -continuous.*

Theorem 3.11. *If $f : X \rightarrow Y$ is slightly $\pi g\beta$ -continuous and $g : Y \rightarrow Z$ is continuous, then $g \circ f : X \rightarrow Z$ is slightly $\pi g\beta$ -continuous.*

Corollary 3.12. *Let $\{X_i : i \in I\}$ be any family of topological spaces. If $f : X \rightarrow \prod X_i$ is slightly $\pi g\beta$ -continuous, then $P_i \circ f : X \rightarrow X_i$ is slightly $\pi g\beta$ -continuous for each $i \in I$, where P_i is the projection of $\prod X_i$ onto X_i .*

Lemma 3.13. *Let $f : X \rightarrow Y$ be bijective, π -irresolute and pre β -closed. Then for every $\pi g\beta$ -closed set A of X , $f(A)$ is $\pi g\beta$ -closed in Y .*

Proof. Let A be any $\pi g\beta$ -closed set of X and V a π -open set of Y containing $f(A)$. Since $f^{-1}(V)$ is a π -open set of X containing A , $\beta\text{cl}(A) \subseteq f^{-1}(V)$ and hence $f(\beta\text{cl}(A)) \subseteq V$. Since f is pre β -closed and $\beta\text{cl}(A)$ is β -closed in X , $f(\beta\text{cl}(A))$ is β -closed in Y . Since $\beta\text{cl}(f(A)) \subseteq \beta\text{cl}(f(\beta\text{cl}(A))) \subseteq V$, $\beta\text{cl}(f(A)) \subseteq V$. Therefore $f(A)$ is $\pi g\beta$ -closed in Y .

Theorem 3.14. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. If f is bijective, π -irresolute and pre β -closed and if $g \circ f : X \rightarrow Z$ is slightly $\pi g\beta$ -continuous, then g is slightly $\pi g\beta$ -continuous.*

Proof. Let V be a clopen subset of Z . Then $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\pi g\beta$ -closed in X . Then by the above Lemma, $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ is $\pi g\beta$ -closed in Y .

Combining Theorems 3.10 and 3.14, we obtain the following result.

Corollary 3.15. *Let $f : X \rightarrow Y$ be a bijective $\pi g\beta$ c-homeomorphism and let $g : Y \rightarrow Z$ be a function. Then $g \circ f : X \rightarrow Z$ is slightly $\pi g\beta$ -continuous if*

and only if g is slightly $\pi g\beta$ -continuous.

Lemma 3.16. [16] *Let $A \subseteq X$. If A is open, then if V is π -open in X , then $V \cap A$ is π -open in A .*

Let $B \subseteq A \subseteq X$. Then we say that B is $\pi g\beta$ -closed relative to A if $\beta \text{cl}_A(B) \subseteq U$ where $B \subseteq U$ and U is π -open in A .

Theorem 3.17. *Let $B \subseteq A \subseteq X$ where A is $\pi g\beta$ -closed and π -open set. Then B is $\pi g\beta$ -closed relative to A if and only if B is $\pi g\beta$ -closed in X .*

Proof. We first note that $B \subseteq A$ and A is both $\pi g\beta$ -closed and π -open set, then $\beta \text{cl}(A) \subseteq A$ and $\beta \text{cl}(B) \subseteq \beta \text{cl}(A) \subseteq A$. Now from the fact that $A \cap \beta \text{cl}(B) = \beta \text{cl}_A(B)$, we have $\beta \text{cl}(B) = \beta \text{cl}_A(B) \subseteq A$. If B is $\pi g\beta$ -closed relative to A and U is π -open subset of X such that $B \subseteq U$, then $B = B \cap A \subseteq U \cap A$ where $U \cap A$ is π -open in A . Hence as B is $\pi g\beta$ -closed relative to A , $\beta \text{cl}(B) = \beta \text{cl}_A(B) \subseteq U \cap A \subseteq U$. Therefore B is $\pi g\beta$ -closed in X .

Conversely, if B is $\pi g\beta$ -closed in X and U is a π -open subset of A such that $B \subseteq U$, then $U = V \cap A$ for some π -open subset V of X . As $B \subseteq V$ and B is $\pi g\beta$ -closed in X , $\beta \text{cl}(B) \subseteq V$. Thus $\beta \text{cl}_A(B) = \beta \text{cl}(B) \cap A \subseteq V \cap A = U$. Therefore B is $\pi g\beta$ -closed relative to A .

Theorem 3.18. *If $f: X \rightarrow Y$ is slightly $\pi g\beta$ -continuous and A is a $\pi g\beta$ -closed and π -open subset of X , then $f/A: A \rightarrow Y$ is slightly $\pi g\beta$ -continuous.*

Proof. Let V be a clopen subset of Y . Then $(f/A)^{-1}(V) = f^{-1}(V) \cap A \subseteq f^{-1}(V)$. Since $f^{-1}(V)$ is $\pi g\beta$ -closed in X , $(f/A)^{-1}(V)$ is $\pi g\beta$ -closed in X . Since $(f/A)^{-1}(V) = f^{-1}(V) \cap A \subseteq A \subseteq X$, by Theorem 3.17, $(f/A)^{-1}(V)$ is $\pi g\beta$ -closed in the relative topology of A . Therefore f/A is slightly $\pi g\beta$ -continuous.

4. Some Application Theorems

Definition 4.1. A space is called

1. $\pi g\beta$ - T_2 (resp. ultra Hausdorff or UT_2 [18]) if every two distinct points of X can be separated by disjoint $\pi g\beta$ -open (resp. clopen) sets.
2. $\pi G\beta O$ -compact (resp. mildly compact [19]) if every $\pi g\beta$ -open (resp. clopen) cover has a finite subcover.

Example 4.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then (X, τ) is $\pi g\beta$ - T_2 .

The following Theorem gives a characterization of $\pi g\beta$ - T_2 spaces and is an analogous to that in general topology, hence its proof is omitted.

Theorem 4.3. A space X is $\pi g\beta$ - T_2 if and only if for every point x in X , $\{x\} = \cap \{F : F \text{ is } \pi g\beta\text{-closed nbd of } x\}$.

Theorem 4.4. If $f : X \rightarrow Y$ is slightly $\pi g\beta$ -continuous injection and Y is UT_2 , then X is $\pi g\beta$ - T_2 .

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then since f is injective and Y is UT_2 , $f(x_1) \neq f(x_2)$ and there exist $V_1, V_2 \in CO(Y)$ such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since f is slightly $\pi g\beta$ -continuous, $x_i \in f^{-1}(V_i) \in \pi G\beta O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus X is $\pi g\beta$ - T_2 .

Theorem 4.5. If $f : X \rightarrow Y$ is slightly $\pi g\beta$ -continuous surjection, and X is $\pi G\beta O$ -compact, then Y is mildly compact.

Proof. Let $\{V_\alpha : V_\alpha \in CO(Y), \alpha \in I\}$ be a cover of Y . Since f is slightly $\pi g\beta$ -continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ be $\pi g\beta$ -open cover of X so there is a finite subset I_0 of I such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Therefore, $Y = \cup \{V_\alpha : \alpha \in I_0\}$ since f is surjective. Thus Y is mildly compact.

We shall continue to work by generalizing the well known Theorems in general topology.

Recall that a space X is submaximal if every dense set is open and it is said to be extremally disconnected if the closure of every open set is open.

Lemma 4.6. [11] If X is submaximal and extremally disconnected, then every β -open set in X is open.

Remark 4.7. By Lemma 4.6 and the following diagram, we can say that every $\pi g\beta$ -open set in X is πg -open.

$$\begin{array}{ccc}
\beta\text{-open} & \longrightarrow & \text{open} \\
\downarrow & & \downarrow \\
g\beta\text{-open} & \longrightarrow & g\text{-open} \\
\downarrow & & \downarrow \\
\pi g\beta\text{-open} & \longrightarrow & \pi g\text{-open}
\end{array}$$

Theorem 4.8. *If $f: X \rightarrow Y$ is a slightly continuous, $g: X \rightarrow Y$ is a slightly $\pi g\beta$ -continuous, Y is UT_2 , X is submaximal and extremally disconnected, then $A = \{x \in X : f(x) = g(x)\}$ is πg -closed.*

Proof. Let $x \notin A$, then $f(x) \neq g(x)$. Since Y is UT_2 , there exist $V_1, V_2 \in \text{CO}(Y)$ such that $f(x) \in V_1$ and $g(x) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since f is slightly continuous and g is slightly $\pi g\beta$ -continuous, $f^{-1}(V_1)$ is open in X and $g^{-1}(V_2)$ is $\pi g\beta$ -open and hence πg -open in X since X is submaximal and extremally disconnected (Remark 4.7) with $x \in f^{-1}(V_1) \cap g^{-1}(V_2)$. Let $U = f^{-1}(V_1) \cap g^{-1}(V_2)$. Then U is a πg -open set (Lemma 1.6). If $z \in U$, then $z \in f^{-1}(V_1)$ and $z \in g^{-1}(V_2)$. Hence $f(z) \neq g(z)$. This shows that $U \subseteq A^c$ and therefore A is πg -closed.

Definition 4.9. A subset of a space X is said to be $\pi g\beta$ -dense if its $\pi g\beta$ -closure equals X .

The next Corollary is a generalization of the well known principle of extension of the identity.

Corollary 4.10. *Let f, g be slightly $\pi g\beta$ -continuous from a space X into a UT_2 -space Y . If f and g agree on $\pi g\beta$ -dense set of X , then $f = g$ everywhere.*

Definition 4.11. Let A be a subset of X . A mapping $r: X \rightarrow A$ is called slightly $\pi g\beta$ -continuous retraction if r is slightly $\pi g\beta$ -continuous and the restriction $r|_A$ is the identity mapping on A .

Theorem 4.12. *Let A be a subset of X and $r: X \rightarrow A$ be a slightly $\pi g\beta$ -continuous retraction. If X is UT_2 , then A is $\pi g\beta$ -closed set of X .*

Proof. Suppose that A is not $\pi g\beta$ -closed. Then there exists a point x in X such that $x \in \pi g\beta\text{cl}(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is slightly $\pi g\beta$ -continuous retraction. Since X is UT_2 , there exist disjoint clopen sets U

and V such that $x \in U$ and $r(x) \in V$. Since $r(x) \in A$, $r(x) \in V \cap A$ and $V \cap A$ is clopen set in A . Now let W be arbitrary $\pi g\beta$ -nbhd of x . Then $W \cap U$ is a $\pi g\beta$ -nbhd of x . Since $x \in \pi g\beta cl(A)$, $(W \cap U) \cap A \neq \emptyset$. Therefore, there exists a point y in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V \cap A$. This implies $r(W) \not\subseteq V \cap A$ because $y \in W$. This is contrary to slight $\pi g\beta$ -continuity of r from Proposition 3.8. Hence A is $\pi g\beta$ -closed.

Definition 4.13. A space X is called $\pi G\beta O$ -connected provided X is not the union of two disjoint, non-empty $\pi g\beta$ -open sets.

Example 4.14. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then (X, τ) is not $\pi G\beta O$ -connected.

Theorem 4.15. If $f : X \rightarrow Y$ is slightly $\pi g\beta$ -continuous surjection, and X is $\pi G\beta O$ -connected, then Y is connected.

Proof. Assume that Y is disconnected. Then there exist disjoint, non-empty clopen sets U and V for which $Y = U \cup V$. Therefore, $X = f^{-1}(U) \cup f^{-1}(V)$ is the union of two disjoint, $\pi g\beta$ -open nonempty sets and hence is not $\pi G\beta O$ -connected.

Slight $\pi g\beta$ -continuity turns out to be a very natural tool for relating $\pi G\beta O$ -connected spaces to connected spaces. In Theorem 4.15, we have seen that the slightly $\pi g\beta$ -continuous image of a $\pi G\beta O$ -connected space is connected but that a slightly $\pi g\beta$ -continuous function is not necessarily a $\pi G\beta O$ -connected function which is defined below.

Definition 4.16. A function $f : X \rightarrow Y$ is called $\pi G\beta O$ -connected if the image of every $\pi G\beta O$ -connected subset of X is a connected subset of Y .

The following Example shows that a slightly $\pi g\beta$ -continuous function is not necessarily $\pi G\beta O$ -connected.

Example 4.17. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = f(c) = a$ and $f(b) = b$. Then f is slightly $\pi g\beta$ -continuous but it is not $\pi G\beta O$ -connected.

Next we show by the Example that a $\pi G\beta O$ -connected function need not

be slightly $\pi g\beta$ -continuous.

Example 4.18. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and τ be the usual relative topology on X . Let $Y = \{0, 1\}$ and σ be the discrete topology on Y . Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(\frac{1}{n}) = 0$ for every $n \in \mathbb{N}$ and $f(0) = 1$. It can be seen that the $\pi g\beta$ -open sets in (X, τ) are precisely the open sets. It then follows that f is $\pi G\beta O$ -connected but not slightly $\pi g\beta$ -continuous.

Thus we have established that slight $\pi g\beta$ -continuity and $\pi G\beta O$ -connectedness are independent.

Definition 4.19. A space X is said to be $\pi G\beta O$ -connected between the subsets A and B of X provided there is no $\pi g\beta$ -clopen set F for which $A \subseteq F$ and $F \cap B = \emptyset$.

Example 4.20. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, \{b, c\}, X\}$. Take $A = \{a, b\}$ and $B = \{b, c\}$. Then (X, τ) is $\pi G\beta O$ -connected between the subsets A and B .

Definition 4.21. A function $f : X \rightarrow Y$ is said to be set $\pi G\beta O$ -connected if whenever X is $\pi G\beta O$ -connected between subsets A and B of X , then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology on $f(X)$.

Example 4.22. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{c\}, X\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. By the above Example, (X, τ) is $\pi G\beta O$ -connected between the subsets A and B . It implies that $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology on $f(X)$.

Theorem 4.23. A function $f : X \rightarrow Y$ is set $\pi G\beta O$ -connected if and only if $f^{-1}(F)$ is $\pi g\beta$ -clopen in X for every clopen set F of $f(X)$ (with respect to the relative topology on $f(X)$).

Proof. The proof is obtained by following similar arguments as in ([6, Theorem 7]) or ([26, Theorem 20]) or ([4, Theorem 4.9]) or ([8, Theorem 3.23]).

Obviously, every slightly $\pi g\beta$ -continuous surjective function is set $\pi G\beta O$ -connected. On the other hand, it can be easily shown that every set $\pi G\beta O$ -connected function is slightly $\pi g\beta$ -continuous. Thus we have seen that in the

class of surjective functions, slight $\pi g\beta$ -continuity and set $\pi G\beta O$ -connectedness coincide. The following Example shows that in general slight $\pi g\beta$ -continuity is not equivalent to set $\pi G\beta O$ -connectedness.

Example 4.24. Let $\mathbb{R}$ be real numbers with the usual topology and let $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Since $\mathbb{R}$ is connected, f is slightly $\pi g\beta$ -continuous. However, $\{1\}$ is clopen in $f(\mathbb{R})$ but $f^{-1}(\{1\}) = [0, +\infty)$ is not $\pi g\beta$ -open i.e., f is not set $\pi G\beta O$ -connected.

References

- [1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β -open sets and β -continuous mappings, Bull. Fac. Sci. Assiut Univ., 12(1983), 77-90.
- [2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, β -closure and β -interior, Rep. J. of Fac. of Edu. Ain. Shams. Univ., 10(1986), 235-245.
- [3] D. Andrijevic, Semi-preopen sets, Mat. Vesnik, 38(1986), 24-32.
- [4] S. C. Arora and Sanjay Tahiliani, Slightly generalized β -continuous functions, Sarajevo Journal of Mathematics, 9(21)(2013), 143-152.
- [5] S. C. Arora, Sanjay Tahiliani and H. Maki, On π -generalized β -closed sets in topological spaces II, Scientiae Mathematicae Japonicae, (e-2009), 637-648.
- [6] C. W. Baker, Slightly precontinuous functions, Acta Math. Hungar., 94(1-2)(2002), 45-52.
- [7] J. Cao, M. Ganster and I. Reilly, On sg-closed sets and $g\alpha$ -closed sets, Mem. Fac. Sci. Kochi. Univ. Math., 20(1999), 1-5.
- [8] C. Carpintero, N. Rajesh and E. Rosas, On slightly πg -continuous functions, Journal of Advanced Research in Pure Mathematics, 3(3)(2011), 115-127.
- [9] J. Dontchev, On generalizing semi-preopen sets, Mem. Fac. Sci. Kochi. Univ. Ser. A. Math., 16(1995), 35-48.
- [10] J. Dontchev and T. Noiri, Quasi-normal spaces and πg -closed sets, Acta Math. Hungar., 89(3)(2000), 211-219.

- [11] M. Ganster and D. Andrijevic, On some questions concerning semi-pre open sets, J. Inst. Math. Compu. Sci. Math., 1(1988), 65-75.
- [12] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo., 19(2)(1970), 89-96.
- [13] R. A. Mahmoud and M. E. Abd El-Monsef, β -irresolute and β -topological invariant, Proc. Pakistan Acad. Sci., 27(1990), 285-296.
- [14] H. Maki, The special issue in commemoration of Prof. Kazusada IKEDA's retirement, 1 October 1986, 139-146.
- [15] T. Noiri, Slightly β -continuous functions, Int. J. Math. Math. Sci., 28(8)(2001), 469-478.
- [16] J. K. Park, On πgp -closed sets in topological spaces, Indian J. Pure Appl. Math., (To appear).
- [17] M. S. Sarsak and N. Rajesh, π -generalized semi-preclosed sets, International Mathematical Forum, 5(12)(2010), 573-578.
- [18] A. R. Singal and R. C. Jain, Slightly continuous mappings, J. Indian. Math. Soc., 64(1997), 195-203.
- [19] R. Staum, The algebra of a bounded continuous functions into a nonarchimedean field, Pacific J. Math., 50(1974), 169-185.
- [20] L. A. Steen and J. A. Seebach Jr., Counterexamples in topology, Holt, Rinenhart and Winston, New York, 1970.
- [21] M. H. Stone, Applications of the theory of boolean rings to general topology, Trans. Amer. Math. Soc., 41(1937), 357-381.
- [22] S. Tahiliani, More on $g\beta$ -closed sets and β - $g\beta$ -continuous functions, Bulletin of Allahabad Mathematical Society, 23(2)(2008), 273-283.
- [23] S. Tahiliani, More on slightly β -continuous functions, University of Bacau, Faculty of Sciences, Scientific Studies and Research, Series Mathematics and Informatics, 19(1)(2009), 231-238.
- [24] S. Tahiliani, On $\pi g\beta$ -closed sets in topological spaces, Note di Matematica, 30(1)(2010), 49-55.
- [25] V. Zaitsav, On certain classes of topological spaces and their bicompaifications, Dokl. Akad Nauk SSSR, 178(1979), 778-779.

- [26] I. Zorlutuna, M. Kucuk and Y. Kucuk, Slightly generalized continuous functions, Kochi J. Math., 3(2008), 99-108.