POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEM OF SINGULAR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract: In this paper, we consider a class of singular fractional order functional differential equations with delay. By means of a fixed point theorem on cones, some sufficient conditions for the existence of at least one or two positive solutions for the boundary value problem are established.

We also give examples to illustrate the applicability of our main results.

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1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications (see[1], [2], [3], [4], [5]).

In [6], [7], [8], [9], [10], they studied the existence and multiplicity of positive solutions to fractional differential equations, and obtained some results. However, to the best of the author knowledge, there are few articles studying

the functional differential equations of fractional order. In [11], the authors studied the existence of positive solution for boundary value problem of fourth-order FDE. Motivated by the work above, this paper investigates the existence of positive solutions for singular fractional order functional differential equation with boundary conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + r(t)f(u_t) = 0, & 0 < t < 1, \quad 2 < \alpha \leq 3 \\ u(t) = \phi(t), & -\tau \leq t \leq 0, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \quad (1)$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α ; $\phi(t) \in C([-\tau, 0], [0, +\infty))$, $\phi(0) = 0$; for $t \in [0, 1]$, $u_t(\theta) = u(t + \theta)$, $\theta \in [-\tau, 0]$, $0 \leq \tau < \frac{1}{2}$ is a constant.

2. Preliminaries

Definition 1. ([6]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow \mathbf{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha - n + 1}} ds,$$

provided that the right side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α .

Lemma 2. ([10]) Let $h(t) \in C[0, 1]$ be a given function. Then the boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, & 0 < t < 1, \quad 2 < \alpha \leq 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) h(s) ds,$$

where

$$G(t, s) = \begin{cases} \frac{t^{-1}(1-s)^{-2}-(t-s)^{-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{-1}(1-s)^{-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2)$$

Lemma 3. ([10]) The function $G(t, s)$ defined by (2) satisfies

(i) $G(t, s) > 0$, $t, s \in (0, 1)$;

$$(ii) \quad G(t, s) \leq \max_{0 \leq t \leq 1} G(t, s) = G(1, s), \quad t, s \in [0, 1];$$

$$(iii) \quad \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s) \geq \left(\frac{1}{4}\right)^{\alpha-1} G(1, s), \quad s \in (0, 1).$$

Note that $\left(\frac{1}{4}\right)^{\alpha-1} \geq \frac{1}{16}$, so we have $\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s) \geq \frac{1}{16} G(1, s)$ for $s \in (0, 1)$.

Let $C = C([-\tau, 0], \mathbf{R})$ be a Banach space with a norm $\|\varphi\|_C = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$

and

$$C^+ = \{\varphi \in C; \varphi(\theta) \geq 0, \theta \in [-\tau, 0]\}.$$

Define $E = \{t \in [0, 1] : \frac{1}{4} \leq t + \theta \leq \frac{3}{4}, -\tau \leq \theta \leq 0\} = [\frac{1}{4} + \tau, \frac{3}{4}]$.

We assume the following:

(A₁) f is a nonnegative continuous functional defined on C^+ , and f maps bounded set in C^+ into bounded set in $\mathbf{R}$.

(A₂) $r(t)$ is a nonnegative measurable function defined on $[0, 1]$, and satisfies

$$0 < \int_E G(1, s)r(s)ds < \int_0^1 G(1, s)r(s)ds < +\infty.$$

We would mention that $r(t)$ is allowed to be zero on some subset of E and has singularity at points $t = 0$ and $t = 1$.

Suppose that $u(t)$ is a solution of BVP(1), then

$$u(t) = \begin{cases} \int_0^1 G(t, s)r(s)f(u_s)ds, & 0 \leq t \leq 1, \\ \phi(t), & -\tau \leq t \leq 0. \end{cases} \quad (3)$$

Suppose that $\bar{x}(t)$ is the solution of BVP(1) with $f \equiv 0$, then

$$\bar{x}(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ \phi(t), & -\tau \leq t \leq 0. \end{cases} \quad (4)$$

Let $x(t) = u(t) - \bar{x}(t)$, then we have from (3)and(4) that

$$x(t) = \begin{cases} \int_0^1 G(t, s)r(s)f(x_s + \bar{x}_s)ds, & 0 \leq t \leq 1, \\ 0, & -\tau \leq t \leq 0. \end{cases}$$

Let K be a cone in Banach space $C[-\tau, 1]$ defined by

$$K = \{x \in C[-\tau, 1] \mid x(t) = 0, t \in [-\tau, 0]; \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} x(t) \geq \frac{1}{16} \|x\|;$$

$$x(t) > 0, t \in (0, 1)\},$$

where $\|x\| := \max\{|x(t)| : -\tau \leq t \leq 1\}$.

Define $T : K \rightarrow C[-\tau, 1]$ as

$$(Tx)(t) = \begin{cases} \int_0^1 G(t, s)r(s)f(x_s + \bar{x}_s)ds, & 0 \leq t \leq 1, \\ 0, & -\tau \leq t \leq 0. \end{cases} \quad (5)$$

We define $\|x\|_{[0,1]} = \sup\{|x(t)|; 0 \leq t \leq 1\}$, then we have $\|x\| = \|x\|_{[0,1]}$ and $\|Tx\| = \|Tx\|_{[0,1]}$ for any $x \in K$.

For $0 < t < 1$, $x \in K$, we obtain from (5) and Lemma 3 that $(Tx)(t) > 0$, and

$$\begin{aligned} \|Tx\| &\leq \int_0^1 \max_{0 \leq t \leq 1} G(t, s)r(s)f(x_s + \bar{x}_s)ds, \\ \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} (Tx)(t) &= \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \int_0^1 G(t, s)r(s)f(x_s + \bar{x}_s)ds \\ &\geq \frac{1}{16} \int_0^1 \max_{0 \leq t \leq 1} G(t, s)r(s)f(x_s + \bar{x}_s)ds \\ &\geq \frac{1}{16} \|Tx\|. \end{aligned}$$

For $-\tau \leq t \leq 0$, $x \in K$, one has $(Tx)(t) = 0$. That is $T(K) \subset K$.

Then we have the following lemma.

Lemma 4. *The operator $T : K \rightarrow K$ is completely continuous.*

Proof. We can obtain the continuity of T from the continuity of f . In face, suppose $x^{(n)}, x \in K$ and $\|x^{(n)} - x\| \rightarrow 0$ as $n \rightarrow \infty$, then we get

$$\|x_s^{(n)} - x_s\|_C = \sup_{-\tau \leq \theta \leq 0} |x^{(n)}(s + \theta) - x(s + \theta)| \rightarrow 0, \quad s \in [0, 1].$$

Thus, for $t \in [-\tau, 1]$ we have from (5) and Lemma 3 that

$$\begin{aligned} &|(Tx^{(n)})(t) - (Tx)(t)| \\ &\leq \max_{0 \leq s \leq 1} |f(x_s^{(n)} + \bar{x}_s) - f(x_s + \bar{x}_s)| \int_0^1 G(1, s)r(s)ds. \end{aligned}$$

This implies that $\|Tx^{(n)} - Tx\| \rightarrow 0$ as $n \rightarrow \infty$.

Now let $\Omega \subset K$ be a bounded subset of K and $M_1 > 0$ is the constant such that $\|x\| \leq M_1$ for $x \in \Omega$. Suppose that $\|\bar{x}\| = M_2$, then $\|x + \bar{x}\| \leq$

$M_1 + M_2 =: M$ for $x \in \Omega$. Define a set $S = \{\varphi \in C^+; \|\varphi\|_C \leq M\}$. Let $L = \max_{0 \leq t \leq 1, \varphi \in S} |f(\varphi) + 1|$, then we have

$$\|Tx\| \leq L \int_0^1 G(1, s)r(s)ds < +\infty,$$

which implies the boundedness of $T(\Omega)$. Furthermore, we have for $0 \leq t \leq 1$,

$$\begin{aligned} (Tx)'(t) &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} r(s) f(x_s + \bar{x}_s) ds \\ &\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} r(s) f(x_s + \bar{x}_s) ds. \end{aligned}$$

So,

$$|(Tx)'(t)| \leq \frac{2L}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} r(s) ds =: L_1$$

For $-\tau < t < 0$, we have $(Tx)'(t) = 0$. Thus with $x \in \Omega$, $\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{L_1}$, for $t_1, t_2 \in [-\tau, 0]$, $|t_1 - t_2| < \delta$, we get $|(Tx)(t_1) - (Tx)(t_2)| \leq L_1 |t_1 - t_2| < \epsilon$. By means of the Arzela-Ascoli theorem, $T : K \rightarrow K$ is completely continuous. $\square$

Lemma 5. ([12]) *Let X be a Banach space, and let $K \subset X$ be a cone in X . Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $T : K \rightarrow K$ be a completely continuous operator such that, either*

(I) $\|Tx\| \leq \|x\|, \forall x \in K \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_2$, or

(II) $\|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|, \forall x \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

For convenience, we introduce the following notations.

$$f_0 = \limsup_{\|\varphi\|_C \rightarrow 0} \frac{f(\varphi)}{\|\varphi\|_C}, \quad f_\infty = \limsup_{\|\varphi\|_C \rightarrow +\infty} \frac{f(\varphi)}{\|\varphi\|_C},$$

$$\bar{f}_0 = \liminf_{\varphi \in C^*, \|\varphi\|_C \rightarrow 0} \frac{f(\varphi)}{\|\varphi\|_C}, \quad \bar{f}_\infty = \liminf_{\varphi \in C^*, \|\varphi\|_C \rightarrow +\infty} \frac{f(\varphi)}{\|\varphi\|_C},$$

$$A = \frac{1}{3} \left[\int_0^1 G(1, s) r(s) ds \right]^{-1}, \quad B = \left[\frac{1}{16} \int_E G(1, s) r(s) ds \right]^{-1},$$

$$p_0 = \max_{-\tau \leq t \leq 0} |\phi(t)|, \quad \Omega_\alpha = \{x \in C[-\tau, 1]; \|x\| < \alpha\}.$$

In the next, we let $C^* = \{\varphi \in C^+; 0 < \gamma_0 \|\varphi\|_C \leq \varphi(\theta), \theta \in [-\tau, 0]\}$, where $0 < \gamma_0 \leq \frac{1}{16}$ is a constant.

Theorem 6. Assume that one of the following conditions is satisfied:

(H₁) $f_0 < A$, $\bar{f}_\infty > B\gamma_0^{-1}$ (particularly, $f_0 = 0$, $\bar{f}_\infty = +\infty$), $\phi(t) \equiv 0$;

(H₂) $\bar{f}_0 > B\gamma_0^{-1}$, $f_\infty < A$ (particularly, $\bar{f}_0 = +\infty$, $f_\infty = 0$).

Then BVP (1) has at least one positive solution.

Proof. Suppose that (H₁) is satisfied. By $\phi(t) \equiv 0$, we know $\bar{x}_t = 0, t \in [0, 1]$. From $f_0 < A$, there is a $\rho_1 > 0$ such that $f(\varphi) \leq A\|\varphi\|_C, \varphi \in C^+, 0 \leq \|\varphi\|_C \leq \rho_1$. For any $x \in K$, $\|x\| = \rho_1$, we deduce that $\|x_s\|_C \leq \|x\| = \rho_1, s \in [0, 1]$ and thus

$$\begin{aligned} 0 \leq (Tx)(t) &\leq A\|x_s\|_C \int_0^1 G(1, s) r(s) ds \\ &\leq A\|x\| \int_0^1 G(1, s) r(s) ds < \|x\|, \quad (0 \leq t \leq 1) \end{aligned}$$

which implies $\|Tx\| = \|Tx\|_{[0,1]} < \|x\|, \forall x \in K \cap \partial\Omega_{\rho_1}$.

Since $\bar{f}_\infty > B\gamma_0^{-1}$, there exists a $\rho_2 > \rho_1$ such that $f(\varphi) \geq B\gamma_0^{-1}\|\varphi\|_C, \varphi \in C^*, \|\varphi\|_C > \gamma_0\rho_2$. For any $x \in K$, $\|x\| = \rho_2$, we have

$$\gamma_0\|x_s\|_C \leq \gamma_0\|x\| \leq \frac{1}{16}\|x\| \leq \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} x(t) \leq x(t), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], s \in E.$$

which implies that $x_s \in C^*$ for $s \in E$ and

$$\|x_s\|_C \geq \gamma_0\|x\| = \gamma_0\rho_2, \quad s \in E. \quad (6)$$

Note that when $s \in E$ we have $\bar{x}_s = 0$. Thus, we obtain

$$\begin{aligned} (Tx)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) r(s) f(x_s) ds \geq \frac{1}{16} \int_E G(1, s) r(s) f(x_s) ds \\ &\geq \frac{1}{16} B\gamma_0^{-1} \|x_s\|_C \int_E G(1, s) r(s) ds \\ &\geq \frac{1}{16} B\gamma_0^{-1} \gamma_0 \rho_2 \int_E G(1, s) r(s) ds = \rho_2 = \|x\|, \end{aligned} \quad (7)$$

which leads to $\|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_{\rho_2}$. According to the first part of Lemma 5, it follows that T has a fixed point $x \in K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$ such that $0 < \rho_1 \leq \|x\| \leq \rho_2$.

Now, suppose that (H_2) is satisfied. Since $\bar{f}_0 > B\gamma_0^{-1}$, there is a $\rho_1 > 0$ such that $f(\varphi) \geq B\gamma_0^{-1}\|\varphi\|_C, \varphi \in C^*, \|\varphi\|_C \leq \rho_1$. For $x \in K$, $\|x\| = \rho_1$, we have $\|x_s\|_C \leq \|x\| = \rho_1, s \in [0, 1]$. Furthermore, by a similar argument as (6), we have $x_s \in C^*, \|x_s\|_C \geq \gamma_0\|x\| = \gamma_0\rho_1, s \in E$. Thus, we have an analogous result to (7): $(Tx)(t_1) \geq \rho_1 = \|x\|$, which implies that $\|Tx\| = \|Tx\|_{[0,1]} \geq \|x\|, \forall x \in K \cap \partial\Omega_{\rho_1}$.

On the other hand, since $f_\infty < A$, there is a $N > \rho_1$ such that $f(\varphi) \leq A\|\varphi\|_C, \varphi \in C^+, \|\varphi\|_C > N$. Choose a positive constant ρ_2 such that

$$\rho_2 > \max\{p_0, 3 \max\{f(\varphi); 0 \leq \|\varphi\|_C \leq N + p_0\} \int_0^1 G(1, s)r(s)ds\}.$$

For $x \in K, \|x\| = \rho_2$, we have from the facts: $\bar{x}(t) \geq 0, x(t) \geq 0, t \in [-\tau, 1]$, that for $s \in [0, 1]$,

$$\begin{aligned} \|x_s + \bar{x}_s\|_C &\geq \|x_s\|_C > N, \quad \|x_s\|_C > N; \\ \|x_s + \bar{x}_s\|_C &\leq \|x_s\|_C + \|\bar{x}_s\|_C \leq N + \|\bar{x}\|, \quad \|x_s\|_C \leq N. \end{aligned}$$

Thus, we have

$$\begin{aligned} (Tx)(t) &\leq \int_{\|x_s\|_C > N} G(1, s)r(s)f(x_s + \bar{x}_s)ds \\ &\quad + \int_{0 \leq \|x_s\|_C \leq N} G(1, s)r(s)f(x_s + \bar{x}_s)ds \\ &\leq A(\|x\| + \|\bar{x}\|) \int_0^1 G(1, s)r(s)ds \\ &\quad + \max\{f(\varphi); 0 \leq \|\varphi\|_C \leq N + p_0\} \int_0^1 G(1, s)r(s)ds \\ &< \frac{1}{3}\|x\| + \frac{1}{3}p_0 + \frac{1}{3}\rho_2 < \rho_2 = \|x\|, \quad 0 \leq t \leq 1, \end{aligned}$$

which implies that $\|Tx\| < \|x\|, \forall x \in K \cap \partial\Omega_{\rho_2}$. By the second part of Lemma 5, it follows that T has a fixed point $x \in K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$ such that $0 < \rho_1 \leq \|x\| \leq \rho_2$.

Suppose that $x(t)$ is the fixed point of T in $K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$, then

$$x(t) = \begin{cases} \int_0^1 G(t, s)r(s)f(x_s + \bar{x}_s)ds, & 0 \leq t \leq 1, \\ 0, & -\tau \leq t \leq 0. \end{cases}$$

Let $u(t) = x(t) + \bar{x}(t)$. Since $0 < \rho_1 \leq \|x\| = \|x\|_{[0,1]} \leq \rho_2, x(t) \in K$ and $\bar{x}(t) \geq 0$, we conclude that $u(t)$ is a positive solution of BVP(1). This completes the proof. $\square$

Theorem 7. *If the following conditions are satisfied:*

(H₃) $\bar{f}_0 > B\gamma_0^{-1}$, $\bar{f}_\infty > B\gamma_0^{-1}$ (particularly, $\bar{f}_0 = +\infty$, $\bar{f}_\infty = +\infty$);

(H₄) $\exists p > 0$ such that for $\forall \varphi$, $0 \leq \|\varphi\|_C \leq p + p_0$, one has $|f(\varphi)| < \eta p$,
where

$$\eta = \left(\int_0^1 G(1, s)r(s)ds \right)^{-1}.$$

Then BVP (1) has at least two positive solution $u_1, u_2 \in K$ such that $0 < \|u_1\|_{[0,1]} < p < \|u_2\|_{[0,1]}$.

Proof. Since $\bar{f}_0 > B\gamma_0^{-1}$, there exists a $0 < R_1 < p$ such that $f(\varphi) \geq B\gamma_0^{-1}\|\varphi\|_C$, $\varphi \in C^*$, $0 < \|\varphi\|_C \leq R_1$. For $x \in K$, $\|x\| = R_1$, similar to (6) one has $x_s \in C^*$, $R_1 \geq \|x_s\|_C \geq \gamma_0\|x\| = \gamma_0 R_1$, $s \in E$. Hence, we obtain an analogous inequality: $\|Tx\| \geq \|x\|$, $\forall x \in K \cap \partial\Omega_{R_1}$.

Since $\bar{f}_\infty > B\gamma_0^{-1}$, there is a $R_2 > p$ such that $f(\varphi) \geq B\gamma_0^{-1}\|\varphi\|_C$, $\varphi \in C^*$, $\|\varphi\|_C \geq \gamma_0 R_2$. For $x \in K$, $\|x\| = R_2$, similar to (6) one has $x_s \in C^*$, $R_2 \geq \|x_s\|_C \geq \gamma_0\|x\| = \gamma_0 R_2$, $s \in E$. Hence, we obtain an analogous inequality: $\|Tx\| \geq \|x\|$, $\forall x \in K \cap \partial\Omega_{R_2}$.

Now, by (H₄), for $x \in K$, $\|x\| = p$, since $0 \leq \|x_s + \bar{x}_s\|_C \leq \|x_s\|_C + \|\bar{x}_s\|_C \leq p + p_0$, we have

$$\begin{aligned} (Tx)(t) &\leq \int_0^1 G(1, s)r(s)f(x_s + \bar{x}_s)ds \\ &\leq \eta p \int_0^1 G(1, s)r(s)ds = p = \|x\|, \quad 0 \leq t \leq 1, \end{aligned}$$

which implies that $\|Tx\| \leq \|x\|$, $\forall x \in K \cap \partial\Omega_p$. According to Lemma 5, it follows that T has two fixed points x_1, x_2 such that $x_1 \in K \cap (\bar{\Omega}_p \setminus \Omega_{R_1})$, $x_2 \in K \cap (\bar{\Omega}_{R_2} \setminus \Omega_p)$, that is $0 < \|x_1\| < p < \|x_2\|$. Since $x_i(t) \in K$, we have $x_i(t) > 0$, $\forall t \in (0, 1)$, $i = 1, 2$. Let $u_1(t) = x_1(t) + \bar{x}(t)$, $u_2(t) = x_2(t) + \bar{x}(t)$, then u_1, u_2 are positive solutions of BVP(1) satisfying $0 < \|u_1\|_{[0,1]} < p < \|u_2\|_{[0,1]}$. This completes the proof. $\square$

Similarly, we have the following result.

Theorem 8. *If the following conditions are satisfied:*

(H₅) $\bar{f}_0 < A$, $\bar{f}_\infty < A$ (particularly, $\bar{f}_0 = 0$, $\bar{f}_\infty = 0$), $\phi(t) \equiv 0$;

(H₆) $\exists p > 0$ such that for $\forall \varphi$, $\gamma_0 p \leq \|\varphi\|_C \leq p$, one has $|f(\varphi)| \geq Bp$.

Then BVP (1) has at least two positive solution $u_1, u_2 \in K$ such that $0 < \|u_1\|_{[0,1]} < p < \|u_2\|_{[0,1]}$.

4. An Example

Example 9. Consider the BVP

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t) + (1+t^2)u^{\frac{1}{2}}(t - \frac{1}{3}) = 0, & 0 \leq t \leq 1, \\ u(t) = -\sin \pi t, & -\frac{1}{3} \leq t \leq 0, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \quad (8)$$

Then $\tau = \frac{1}{3}$, $f(\varphi) = \varphi^{\frac{1}{2}}(-\frac{1}{3})$, $E = [\frac{7}{12}, \frac{3}{4}]$. As $\|\varphi\|_C \rightarrow +\infty$ we have

$$\frac{f(\varphi)}{\|\varphi\|_C} = \frac{\varphi^{\frac{1}{2}}(-\frac{1}{3})}{\|\varphi\|_C} \leq \frac{\|\varphi\|_C^{\frac{1}{2}}}{\|\varphi\|_C} = \|\varphi\|_C^{-\frac{1}{2}} \rightarrow 0,$$

that is to say that $f_\infty = 0$ holds. On the other hand, suppose $\varphi \in C^*$, then $\varphi(\theta) \geq \gamma_0 \|\varphi\|_C$, thus, as $\|\varphi\|_C \rightarrow 0$, we get

$$\frac{f(\varphi)}{\|\varphi\|_C} = \frac{\varphi^{\frac{1}{2}}(-\frac{1}{3})}{\|\varphi\|_C} \geq \frac{\gamma_0^{\frac{1}{2}} \|\varphi\|_C^{\frac{1}{2}}}{\|\varphi\|_C} = \gamma_0^{\frac{1}{2}} \|\varphi\|_C^{-\frac{1}{2}} \rightarrow +\infty,$$

which means that $\bar{f}_0 = +\infty$ holds. According to Theorem 6, it follows that BVP (8) has at least a positive solution $u(t)$.

Example 10. Consider the BVP

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t) + c[u^{\frac{1}{2}}(t - \frac{1}{3}) + u^{\frac{3}{2}}(t - \frac{1}{3})] = 0, & 0 \leq t \leq 1, \\ u(t) = \phi(t), & -\frac{1}{3} \leq t \leq 0, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \quad (9)$$

where $c > 0$ is a constant, $\phi(t)$ is continuous on $[-\frac{1}{3}, 0]$, $\phi(t) \geq 0$, $\phi(0) = 0$ and $f(\varphi) = \varphi^{\frac{1}{2}}(-\frac{1}{3}) + \varphi^{\frac{3}{2}}(-\frac{1}{3})$, $\tau = \frac{1}{3}$, $E = [\frac{7}{12}, \frac{3}{4}]$. Suppose $\varphi \in C^*$, then $\varphi(\theta) \geq \gamma_0 \|\varphi\|_C$ for $\theta \in [-\frac{1}{3}, 0]$ thus, as $\|\varphi\|_C \rightarrow 0$ or $\|\varphi\|_C \rightarrow +\infty$, we get

$$\begin{aligned} \frac{f(\varphi)}{\|\varphi\|_C} &= \frac{\varphi^{\frac{1}{2}}(-\frac{1}{3}) + \varphi^{\frac{3}{2}}(-\frac{1}{3})}{\|\varphi\|_C} \\ &\geq \frac{\gamma_0^{\frac{1}{2}} \|\varphi\|_C^{\frac{1}{2}} + \gamma_0^{\frac{3}{2}} \|\varphi\|_C^{\frac{3}{2}}}{\|\varphi\|_C} = \gamma_0^{\frac{1}{2}} \|\varphi\|_C^{-\frac{1}{2}} + \gamma_0^{\frac{3}{2}} \|\varphi\|_C^{\frac{1}{2}} \rightarrow +\infty, \end{aligned}$$

Deducing $\eta = (\int_0^1 G(1, s)r(s)ds)^{-1} = \frac{45\sqrt{\pi}}{16c}$, then $\forall p > 0$ and $0 \leq \|\varphi\|_C \leq p + p_0$, we have

$$0 \leq f(\varphi) \leq (p + p_0)^{\frac{1}{2}} + (p + p_0)^{\frac{3}{2}} = (p + p_0)^{\frac{1}{2}}(1 + \frac{1}{p} + \frac{p_0}{p})p.$$

Define $H(p) = (p + p_0)^{\frac{1}{2}}(1 + \frac{1}{p} + \frac{p_0}{p})$, then

$$\lim_{p \rightarrow 0} H(p) = +\infty, \quad \lim_{p \rightarrow +\infty} H(p) = +\infty. \quad (10)$$

Suppose that c and p_0 satisfy $\sqrt{p_0}(2 + \frac{1}{p_0}) < \frac{45\sqrt{\pi}}{16\sqrt{2}c}$, then $H(p_0) = \sqrt{2}\sqrt{p_0}(2 + \frac{1}{p_0}) < \frac{45\sqrt{\pi}}{16c} = \eta$ holds. By the continuity of $H(p)$ and (10), we can find a $p > 0$ such that $|f(\varphi)| < \eta p$ for $0 \leq \|\varphi\|_C \leq p + p_0$. By Theorem 7 we know that BVP (9) has at least two positive solutions.

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