FORMULATION AND SOLUTION OF TIME-FRACTIONAL DEGASPERIS-PROCESI EQUATION VIA VARIATIONAL METHODS

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Abstract: This paper presents the formulation of the time-fractional Degasperis-Procesi equation using the Euler-Lagrange variational technique in the Riemann-Liouville derivatives sense and derives an approximate solitary wave solution. Our results witness that He's variational iteration method was very efficient and powerful technique in finding the solutions of the proposed equations.

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1. Introduction

In this paper, He's variational iteration method [1, 2, 3, 4] is applied to solve time-fractional Degasperis-Procesi equation

$$\begin{aligned} {}_a D_t^\alpha u(x, t) - u_{xxt}(x, t) + 4u^2(x, t)u_x(x, t) - 3u_x(x, t)u_{xx}(x, t) \\ - u(x, t)u_{xxx}(x, t) = 0, \end{aligned}$$

where $u(x, t)$ is a field variable, $x \in \mathbb{R}$ is a space coordinate in the propagation

direction of the field and $t \in T(=[0, T])$ is the time, ${}_a D_t^\alpha$ are left-hand side Riemann-Liouville fractional derivative.

The Degasperis-Procesi equation was discovered by Degasperis and Procesi [5] in a search for integrable equations similar in form to the Camassa-Holm equation, and is widely used in fluid dynamics, aerodynamics, continuum mechanics, optimal fiber, biology, solid state physics, geometry and oceanology. As an abstract bi-Hamiltonian evolution equations with infinitely many conservation laws, the Degasperis-Procesi equation has obtained by Johnson [6], Dullin et al [7] has proved to be an approximate model of shallow water wave propagation in the small amplitude and long wavelength regime, Fokas and Fuchssteiner [8], Lenells [9], Camassa and Holm [10] put it forward the derivation of solution as a model for dispersive shallow water waves and discovered that it is formally integrable dimensional Hamiltonian system and that its solitary waves are solitons. Most of classical mechanics techniques have studied conservative systems, but almost of the processes observed in the physical real world are nonconservative [11]. During the past three decades or so, fractional calculus has obtained considerable popularity and importance as generalizations of integer-order evolution equations, and used to model problems in neurons, hydrology, viscoelasticity and rheology, image processing, mechanics, mechatronics, physics, finance and control theory, see [12, 13, 14, 15, 16, 17, 18, 19]. If the Lagrangian of conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations rather than integer-order equations [20]. Bateman [21] obtained an Euler-Lagrange equation, Riewe [22, 23] formulated a version of the Euler-Lagrange equation for problems of calculus of variation with fractional derivatives, other results we can see Agrawal [24, 25, 26], Baleanu et al [27] and Inokuti et al [28]. In view of most of physical phenomena may be considered as nonconservative, then they can be described using fractional-order differential equations, and several methods have been used to solve nonlinear fractional evolution equation using techniques of nonlinear analysis such as Adomian decomposition method [29], homotopy analysis method [30, 31], and homotopy perturbation method [32]. It was mentioned that the variational iteration method has been used successfully to solve different types of integer and fractional nonlinear evolution equations. Making use of the variational iteration method, this work mainly motive devoted to formulate a time-fractional Degasperis-Procesi equation and derives an approximate solitary wave solution.

This paper is organized as follows: Section 2 states some background ma-

terial from fractional calculus. Section 3 presents the principle of He's variational iteration method. Section 4 is devoted to describe the formulation of the time-fractional Degasperis-Procesi equation using the Euler-Lagrange variational technique and to derive an approximate solitary wave solution. Section 5 makes some analysis for the obtained graphs and figures and discusses the present work.

2. Preliminaries

We recall the necessary definitions for the fractional calculus (see, e.g., [34, 33]) which is used throughout the remaining sections of this paper.

Definition 1. A real multivariable function $f(x, t)$, $t > 0$ is said to be in the space C_γ , $\gamma \in \mathbb{R}$ with respect to t if there exists a real number $p(> \gamma)$, such that $f(x, t) = t^p f_1(x, t)$, where $f_1(x, t) \in C(\Omega \times T)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \leq \gamma$.

Definition 2. The left-hand side Riemann-Liouville fractional integral of a function $f \in C_\gamma$, ($\gamma \geq -1$) is defined by

$$\begin{aligned} {}_0I_t^\alpha f(x, t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x, \tau)}{(t - \tau)^{1-\alpha}} d\tau, \quad \alpha > 0, \quad t \in T, \\ {}_0I_t^0 f(x, t) &= f(x, t). \end{aligned}$$

Definition 3. The Riemann-Liouville fractional derivatives of the order $n - 1 \leq \alpha < n$ of a function $f \in C_\gamma$, ($\gamma \geq -1$) are defined as

$$\begin{aligned} {}_0D_t^\alpha f(x, t) &= \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(x, \tau)}{(t - \tau)^{\alpha+1-n}} d\tau, \\ {}_tD_{t_0}^\alpha f(x, t) &= \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial t^n} \int_t^{t_0} \frac{f(x, \tau)}{(\tau - t)^{\alpha+1-n}} d\tau, \quad t \in T. \end{aligned}$$

Lemma 4. The integration formula of Riemann-Liouville fractional derivative ($0 < \alpha < 1$)

$$\int_T f(x, t) {}_0D_t^\alpha g(x, t) dt = \int_T g(x, t) {}_tD_{t_0}^\alpha f(x, t) dt$$

is valid under the assumption that $f, g \in C(\Omega \times T)$ and that for arbitrary $x \in \Omega$, ${}_tD_{t_0}^\alpha f$, ${}_0D_t^\alpha g$ exist at every point $t \in T$ and are continuous in t .

Definition 5. The Riesz fractional integral of the order $n - 1 \leq \alpha < n$ of a function $f \in C_\gamma$, ($\gamma \geq -1$) is defined as

$$\begin{aligned} {}^R_0 I_t^\alpha f(x, t) &= \frac{1}{2} ({}_0 I_t^\alpha f(x, t) + {}_t I_{t_0}^\alpha f(x, t)) \\ &= \frac{1}{2\Gamma(\alpha)} \int_0^{t_0} |t - \tau|^{\alpha-1} f(x, \tau) d\tau, \end{aligned}$$

where ${}_0 I_t^\alpha$ and ${}_t I_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional integral operators.

Definition 6. The Riesz fractional derivative of the order $n - 1 \leq \alpha < n$ of a function $f \in C_\gamma$, ($\gamma \geq -1$) is defined by

$$\begin{aligned} {}^R_0 D_t^\alpha f(x, t) &= \frac{1}{2} ({}_0 D_t^\alpha f(x, t) + (-1)^n {}_t D_{t_0}^\alpha f(x, t)) \\ &= \frac{1}{2\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^{t_0} |t - \tau|^{n-\alpha-1} f(x, \tau) d\tau, \end{aligned}$$

where ${}_0 D_t^\alpha$ and ${}_t D_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional differential operators.

Lemma 7. Let $\alpha > 0$ and $\beta > 0$ be such that $n-1 < \alpha < n$, $m-1 < \beta < m$ and $\alpha + \beta < n$, and let $f \in L_1(\Omega \times T)$ and ${}_0 I_t^{m-\alpha} f \in AC^m(\Omega \times T)$. Then we have the following index rule:

$$\begin{aligned} {}^R_0 D_t^\alpha ({}_0^R D_t^\beta f(x, t)) &= {}^R_0 D_t^{\alpha+\beta} f(x, t) \\ &\quad - \sum_{i=1}^m {}^R_0 D_t^{\beta-i} f(x, t) \Big|_{t=0} \frac{t^{-\alpha-i}}{\Gamma(1 - \alpha - i)}. \end{aligned}$$

Remark 8. One can express the Riesz fractional differential operator ${}_0^R D_t^{\alpha-1}$ of the order $0 < \alpha < 1$ as the Riesz fractional integral operator ${}_0^R I_t^{1-\alpha}$, i.e.

$${}_0^R D_t^{\alpha-1} f(x, t) = {}_0^R I_t^{1-\alpha} f(x, t), \quad t \in T.$$

3. Variational Iteration Method

Since the variational iteration method provides an effective procedure for explicit and solitary wave solutions of a wide and general class of differential systems representing real physical problems. Moreover, the variational iteration

method can overcome the foregoing restrictions and limitations of approximate techniques so that it provides us with a possibility to analyze strongly nonlinear evolution equations. Therefore, we extend this method to solve the time-fractional Degasperis-Procesi equation. The basic features of the variational iteration method outlined as follows.

Considering a nonlinear evolution equation consists of a linear part $\mathcal{L}u(x, t)$, nonlinear part $\mathcal{N}u(x, t)$, and a free term $g(x, t)$ represented as

$$\mathcal{L}u(x, t) + \mathcal{N}u(x, t) = g(x, t). \quad (1)$$

According to the variational iteration method, the $n+1$ -th approximate solution of (1) can be read using iteration correction functional as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) (\mathcal{L}\tilde{u}(x, \tau) + \mathcal{N}\tilde{u}(x, \tau) - g(x, \tau)) d\tau, \quad (2)$$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\tilde{u}(x, t)$ is considered as a restricted variation function, i.e., $\delta\tilde{u}(x, t) = 0$. Extreming the variation of the correction functional (2) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration $u_0(x, t)$ can be used as the initial value $u(x, 0)$. As n tends to infinity, the iteration leads to the solitary wave solution of (1), i.e.

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t).$$

4. Time-Fractional Degasperis-Procesi Equation

As for any wave propagation model, the main task is that of investigating the solution which satisfies the initial condition. The Degasperis-Procesi equation in (1+1) dimensions is given as

$$\begin{aligned} u_t(x, t) - u_{xxt}(x, t) + 4u^2(x, t)u_x(x, t) - 3u_x(x, t)u_{xx}(x, t) \\ - u(x, t)u_{xxx}(x, t) = 0, \end{aligned} \quad (3)$$

where $k \neq 0$ is a constant, $u(x, t)$ is a field variable, $x \in \Omega (\Omega \subset \mathbb{R})$ is a space coordinate in the propagation direction of the field and $t \in T (= [0, T])$ is the time. Using a potential function $v(x, t)$, set $u(x, t) = v_x(x, t)$ yields the potential equation of the Degasperis-Procesi equation (3) in the form,

$$\begin{aligned} v_{xt}(x, t) - v_{xxt}(x, t) + 4v_x^2(x, t)v_{xx}(x, t) - 3v_{xx}(x, t)v_{xxx}(x, t) \\ - v_x(x, t)v_{xxxx}(x, t) = 0. \end{aligned} \quad (4)$$

The Lagrangian of this Degasperis-Procesi equation (3) can be defined using the semi-inverse method [35, 36] as follows. The functional of the potential equation (4) can be represented as

$$\begin{aligned} J(v) = \int_{\Omega} dx \int_T \big(& v(x, t)(c_1 v_{xt}(x, t) - c_2 v_{xxxt}(x, t) \\ & + 4c_3 v_x^2(x, t)v_{xx}(x, t) - 3c_4 v_{xx}(x, t)v_{xxx}(x, t) \\ & - c_5 v_x(x, t)v_{xxxx}(x, t)) \big) dt, \end{aligned} \quad (5)$$

with $c_i (i = 1, 2, \dots, 6)$ is unknown constant to be determined later. Integrating (5) by parts and taking $v_t|_{\partial\Omega} = v_x|_{\partial\Omega} = v_x|_{\partial T} = v_{xx}|_{\partial\Omega} = v_{xxt}|_{\partial\Omega} = 0$ yields

$$\begin{aligned} J(v) = \int_{\Omega} dx \int_T \big(& -c_1 v_t(x, t)v_x(x, t) + c_2 v_x(x, t)v_{xxt}(x, t) \\ & - \frac{4}{3}c_3 v_x^4(x, t) + \frac{3}{2}(c_4 - c_5)v_x(x, t)v_{xx}^2(x, t) \\ & + \frac{1}{2}c_5 v_x^2(x, t)v_{xxx}(x, t) \big) dt. \end{aligned} \quad (6)$$

The constants $c_i (i = 1, 2, \dots, 6)$ can be determined taking the variation of the functional (6) to make it optimal. By applying the variation of this functional and integrating each term by parts making use of the variation optimum condition to yield the following expression

$$\begin{aligned} & 2c_1 v_{xt}(x, t) - 2c_2 v_{xxxt}(x, t) + 16c_3 v_x^2(x, t)v_{xx}(x, t) + \left(-\frac{15}{2}c_4 \right. \\ & \left. + \frac{13}{2}c_5\right)v_{xx}(x, t)v_{xxx}(x, t) + (c_5 - 3c_4)v_x(x, t)v_{xxxx}(x, t) = 0. \end{aligned} \quad (7)$$

Notice that the above equation (7) is equivalent to (4), so we obtain the constants $c_i (i = 1, 2, \dots, 6)$ is

$$c_1 = c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{4}, \quad c_4 = \frac{7}{24}, \quad c_5 = -\frac{1}{8}.$$

In addition, the functional expression given by (6) obtains directly the Lagrangian form of the Degasperis-Procesi equation,

$$\begin{aligned} L(v_t, v_x, v_{xx}, v_{xxt}, v_{xxx}) = & -\frac{1}{2}v_t(x, t)v_x(x, t) + \frac{1}{2}v_x(x, t)v_{xxt}(x, t) \\ & - \frac{1}{3}v_x^4(x, t) + \frac{5}{8}v_x(x, t)v_{xx}^2(x, t) \end{aligned}$$

$$-\frac{1}{16}v_x^2(x,t)v_{xxx}(x,t).$$

Similarly, the Lagrangian of the time-fractional version of the time-fractional Degasperis-Procesi equation could be written as

$$\begin{aligned} F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}, v_{xxx}) = & -\frac{1}{2}{}_0D_t^\alpha v_t(x,t)v_x(x,t) - \frac{1}{3}v_x^4(x,t) \\ & + \frac{1}{2}v_x(x,t)v_{xxt}(x,t) \\ & + \frac{5}{8}v_x(x,t)v_{xx}^2(x,t) \\ & - \frac{1}{16}v_x^2(x,t)v_{xxx}(x,t), \quad \alpha \in [0, 1]. \end{aligned} \quad (8)$$

Thus, the functional of the time-fractional Degasperis-Procesi equation will take the expression

$$J(v) = \int_{\Omega} dx \int_T dt F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}, v_{xxx}), \quad (9)$$

where the time-fractional Lagrangian $F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}, v_{xxx})$ is given by (8). Following Agrawal's method [24, 25, 26], the variation of functional (9) with respect to $v(x, t)$ leads to

$$\begin{aligned} \delta J(v) = & \int_{\Omega} dx \int_T \left(\frac{\partial F}{\partial {}_0D_t^\alpha v} \delta({}_0D_t^\alpha v(x, t)) + \frac{\partial F}{\partial v_x} \delta v_x(x, t) \right. \\ & \left. + \frac{\partial F}{\partial v_{xx}} \delta v_{xx}(x, t) + \frac{\partial F}{\partial v_{xxt}} \delta v_{xxt}(x, t) + \frac{\partial F}{\partial v_{xxx}} \delta v_{xxx}(x, t) \right) dt. \end{aligned} \quad (10)$$

Upon integrating the right-hand side of (10), one has

$$\begin{aligned} \delta J(v) = & \int_{\Omega} dx \int_T \left({}_tD_T^\alpha \left(\frac{\partial F}{\partial {}_0D_t^\alpha v} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial v_{xx}} \right) \right. \\ & \left. - \frac{\partial^3}{\partial x^2 \partial t} \left(\frac{\partial F}{\partial v_{xxt}} \right) - \frac{\partial^3}{\partial x^3} \left(\frac{\partial F}{\partial v_{xxx}} \right) \right) \delta v dt, \end{aligned}$$

noting that $\delta v|_{\partial T} = \delta v|_{\partial \Omega} = \delta v_x|_{\partial \Omega} = \delta v_{xx}|_{\partial \Omega} = \delta v_{xx}|_{\partial T} = 0$.

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional Degasperis-Procesi equation in the following expression

$$\begin{aligned} & {}_tD_T^\alpha \left(\frac{\partial F}{\partial {}_0D_t^\alpha v} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial v_{xx}} \right) - \frac{\partial^3}{\partial x^2 \partial t} \left(\frac{\partial F}{\partial v_{xxt}} \right) \\ & - \frac{\partial^3}{\partial x^3} \left(\frac{\partial F}{\partial v_{xxx}} \right) = 0. \end{aligned} \quad (11)$$

Substituting the Lagrangian of the time-fractional Degasperis-Procesi equation (8) into Euler-Lagrange formula (11) obtains

$$-\frac{1}{2}{}_tD_T^\alpha v_x(x, t) + \frac{1}{2}{}_0D_t^\alpha v_x(x, t) - v_{xxxt}(x, t) + 4v_x^2(x, t)v_{xx}(x, t) + 3v_{xx}(x, t)v_{xxx}(x, t) + \frac{3}{2}v_x(x, t)v_{xxxx}(x, t) = 0.$$

Once again, substituting for the potential function, $v_x(x, t) = u(x, t)$, yields the time-fractional Degasperis-Procesi equation for the state function $u(x, t)$ as

$$\begin{aligned} & \frac{1}{2}({}_0D_t^\alpha u(x, t) - {}_tD_T^\alpha u(x, t)) - u_{xxt}(x, t) + 4u^2(x, t)u_x(x, t) \\ & + 3u_x(x, t)u_{xx}(x, t) + \frac{3}{2}u(x, t)u_{xxx}(x, t) = 0, \end{aligned} \quad (12)$$

where the fractional derivatives ${}_0D_t^\alpha u(x, t)$ and ${}_tD_T^\alpha u(x, t)$ are respectively the left- and right-hand side Riemann-Liouville fractional derivatives.

According to the Riesz fractional derivative ${}_0^R D_t^\alpha u(x, t)$, the time-fractional Degasperis-Procesi equation represented in (12) can be rewritten as

$$\begin{aligned} & {}_0^R D_t^\alpha u(x, t) - u_{xxt}(x, t) + 4u^2(x, t)u_x(x, t) + 3u_x(x, t)u_{xx}(x, t) \\ & + \frac{3}{2}u(x, t)u_{xxx}(x, t) = 0. \end{aligned} \quad (13)$$

Acting from left-hand side by the fractional operator ${}_0^R D_t^\alpha u$ on (13) leads to us

$$\begin{aligned} & \frac{\partial}{\partial t}u(x, t) - {}_0^R D_t^{\alpha-1}u(x, t)|_{t=0} \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^R D_t^{1-\alpha} \left(-\frac{\partial^3}{\partial x^2 \partial t}u_n(x, t) \right. \\ & + 4u_n^2(x, t) \frac{\partial}{\partial x}u_n(x, t) + 3 \frac{\partial}{\partial x}u_n(x, t) \frac{\partial^2}{\partial x^2}u_n(x, t) \\ & \left. + \frac{3}{2}u_n(x, t) \frac{\partial^3}{\partial x^3}u_n(x, t) \right) = 0. \end{aligned} \quad (14)$$

From Lemma 7, the iterative correction functional of (14) becomes

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) \\ &+ \int_0^t d\tau \lambda(\tau) \left(\frac{\partial}{\partial \tau}u_n(x, \tau) - {}_0^R D_\tau^{\alpha-1}u_n(x, \tau)|_{\tau=0} \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \right. \\ &+ {}_0^R D_\tau^{1-\alpha} \left(-\frac{\partial^3}{\partial x^2 \partial t}\tilde{u}_n(x, \tau) + 4\tilde{u}_n^2(x, \tau) \frac{\partial}{\partial x}\tilde{u}_n(x, \tau) \right. \\ &\left. \left. + 3 \frac{\partial}{\partial x}\tilde{u}_n(x, \tau) \frac{\partial^2}{\partial x^2}\tilde{u}_n(x, \tau) + \frac{3}{2}\tilde{u}_n(x, \tau) \frac{\partial^3}{\partial x^3}\tilde{u}_n(x, \tau) \right) \right), \end{aligned} \quad (15)$$

where the function $\tilde{u}_n(x, t)$ is considered as a restricted variation function, i.e., $\delta\tilde{u}_n(x, t) = 0$. The extreme of the variation of (15) subject to the restricted variation function straightforwardly yields us

$$\begin{aligned}\delta u_{n+1}(x, t) &= \delta u_n(x, t) + \int_0^t \lambda(\tau) \delta \frac{\partial}{\partial \tau} u_n(x, \tau) d\tau \\ &= \delta u_n(x, t) + \lambda(\tau) \delta u_n(x, \tau)|_{\tau=t} - \int_0^t \frac{\partial}{\partial \tau} \lambda(\tau) \delta u_n(x, \tau) d\tau \\ &= 0.\end{aligned}$$

This expression yields the following stationary conditions

$$\frac{\partial}{\partial \tau} \lambda(\tau) = 0, \quad 1 + \lambda(\tau) = 0, \quad (16)$$

(16) converted to the Lagrangian multiplier at $\lambda(\tau) = -1$. Therefore, the correction functional (15) takes the following form

$$\begin{aligned}u_{n+1}(x, t) &= u_n(x, t) \\ &\quad - \int_0^t \left(\frac{\partial}{\partial \tau} u_n(x, \tau) - {}^R_0 I_\tau^{1-\alpha} u_n(x, \tau) \Big|_{\tau=0} \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \right. \\ &\quad + {}^R_0 D_\tau^{1-\alpha} \left(- \frac{\partial^3}{\partial x^2 \partial t} u_n(x, \tau) + 4u_n^2(x, \tau) \frac{\partial}{\partial x} u_n(x, \tau) \right. \\ &\quad + 3 \frac{\partial}{\partial x} u_n(x, \tau) \frac{\partial^2}{\partial x^2} u_n(x, \tau) \\ &\quad \left. \left. + \frac{3}{2} u_n(x, \tau) \frac{\partial^3}{\partial x^3} u_n(x, \tau) \right) \right) d\tau.\end{aligned} \quad (17)$$

As $\alpha - 1 < 0$, the operator ${}^R_0 D_\tau^{\alpha-1}$ reduced to integral one. It means that, one can express the fractional operator as the Riesz fractional integral ${}^R_0 I_\tau^{1-\alpha}$.

In view of the right-hand side Riemann-Liouville fractional derivative is interpreted as a future state of the process in physics. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development. Therefore, the right-derivative is used equal to zero in the following calculations. The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x, t) = u(x, 0) = - \frac{\frac{15}{4}r}{a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r},$$

where $a > b, r = \pm\sqrt{a^2 - b^2}$ and ξ_0 are constants.

Substituting this zero order approximate solitary wave solution into (17) and using the definition of the Riesz fractional derivative leads to the first order approximate solitary wave solution as

$$\begin{aligned}
 u_1(x, t) = & -\frac{\frac{15}{4}r}{a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r} \\
 & - \frac{675r^2t^\alpha}{4\Gamma(\alpha + 1)} \left(\frac{\frac{5r}{4}(a \sinh(x + \xi_0) + b \cosh(x + \xi_0))}{(a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r)^4} \right. \\
 & - \frac{\frac{5}{4}(a \sinh(x + \xi_0) + b \cosh(x + \xi_0))^3}{(a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r)^5} \\
 & - \frac{\frac{1}{8}(a \sinh(x + \xi_0) + b \cosh(x + \xi_0))}{(a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r)^3} \\
 & + \frac{(a \sinh(x + \xi_0) + b \cosh(x + \xi_0))}{(a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r)^4} \\
 & \left. \times (a \cosh(x + \xi_0) + b \sinh(x + \xi_0)) \right).
 \end{aligned}$$

Substituting first order approximate solitary wave solution into (17), using the Riesz fractional derivative and lead to the second order approximate solitary wave solution $u_2(x, t)$,

$$\begin{aligned}
 u_2(x, t) = & -\frac{15r}{4h} \\
 & - \frac{675r^2t^\alpha}{16\Gamma(\alpha + 1)} \left(\frac{5rv}{h^4} + \frac{vwh - 2v^3}{h^5} - \frac{6w^3 - 6vwh + vh^2}{2h^5} \right) \\
 & + \frac{675r^2t^{2\alpha}}{4\Gamma(2\alpha)} \left(\frac{5rvh^2 - 60rvw + 100rv^3}{4h^6} \right. \\
 & - \frac{30vw^2h^2 - 175v^3wh + 15v^3h^2 + 150v^5}{4h^7} - \frac{12vw^2 + 8w^3}{h^5} \\
 & + \frac{4vw}{h^4} + \frac{20v^3w}{h^6} - \frac{vh^2 - 9vwh + 12v^3}{8h^5} \left. \right) \\
 & + \frac{151875r^4t^{2\alpha}}{16h^2\Gamma(2\alpha + 1)\Gamma(\alpha + 1)} \left(\frac{5rwh - 20rv^2}{4h^5} - \frac{15v^2wh - 25v^4}{4h^6} \right. \\
 & + \frac{v^2 + w^2}{h^4} - \frac{4v^2w}{h^5} - \frac{wh - 3v^2}{8h^4} \left. \right) \\
 & + \frac{151875r^4vt^{2\alpha}}{8h^3\Gamma(2\alpha + 1)\Gamma(\alpha + 1)} \left(\frac{5rv + 4vw}{4h^4} - \frac{5v^3}{4h^5} - \frac{v}{8h^3} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{30375r^3vt^{2\alpha}}{16h^2\Gamma(2\alpha+1)} \left(\frac{5rwh^2 - 60rvwh + 100rv^3}{4h^6} \right. \\
& - \frac{30vw^2h^2 - 175v^3wh + 15v^3h^2 + 150v^5}{4h^7} \\
& + \frac{4vw}{h^4} - \frac{12vw^2 + 8v^3}{h^5} + \frac{20v^3w}{h^6} - \frac{vh^2 - 9vw + 12v^3}{8h^5} \Bigg) \\
& + \frac{2025r^3(15wh - 30v^2)t^{2\alpha}}{16h^3\Gamma(2\alpha+1)} \left(\frac{5rwh - 20rv^2}{4h^5} - \frac{15v^2wh - 25v^4}{4h^6} \right. \\
& + \frac{v^2 + w^2}{h^4} - \frac{4v^2w}{h^5} - \frac{wh - 3v^2}{8h^4} \Bigg) \\
& + \frac{2025r^3(90w^3 - 90vwh + 15vh^2)t^{2\alpha}}{32h^4\Gamma(2\alpha+1)} \left(\frac{5rv}{4h^4} - \frac{5v^3}{4h^5} + \frac{vw}{h^4} - \frac{v}{8h^3} \right) \\
& - \frac{30375r^3t^{2\alpha}}{32h\Gamma(2\alpha+1)} \left(\frac{5rwh^3 - 80rv^2h^2 + 60rv^2wh - 60rw^2h^2 - 60rv^4}{4h^7} \right. \\
& - \frac{30w^3 + 105v^2w}{4h^5} + \frac{775v^2w^2 + 250v^4}{4h^6} - \frac{450v^4w}{h^7} + \frac{525v^6}{2h^8} \\
& + \frac{4v^2 + 4w^2}{h^4} - \frac{40v^2w + 24v^2w + 12w^2}{h^5} + \frac{120v^2w^2 + 60v^4}{h^6} \\
& - \frac{120v^4w}{h^7} - \frac{w}{8h^3} + \frac{12v^2 + 9w^2}{8h^4} - \frac{9v^2w}{h^5} + \frac{15v^4}{2h^6} \Bigg) \\
& - \frac{1366875\Gamma(2\alpha+1)r^4t^{3\alpha}}{32\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} \left(\frac{5rv}{4h^4} - \frac{5v^3}{4h^5} + \frac{vw}{h^4} - \frac{v}{8h^3} \right) \\
& \times \left(\frac{5rwh^3 - 80rv^2h^2 + 60rv^2wh - 60rw^2h^2 - 60rv^4}{4h^7} \right. \\
& - \frac{30w^3 + 105v^2w}{4h^5} + \frac{775v^2w^2 + 250v^4}{4h^6} - \frac{450v^4w}{h^7} + \frac{525v^6}{2h^8} \\
& + \frac{4v^2 + 4w^2}{h^4} - \frac{40v^2w + 24v^2w + 12w^2}{h^5} + \frac{120v^2w^2 + 60v^4}{h^6} \\
& - \frac{120v^4w}{h^7} - \frac{w}{8h^3} + \frac{12v^2 + 9w^2}{8h^4} - \frac{9v^2w}{h^5} + \frac{15v^4}{2h^6} \Bigg) \\
& - \frac{1366875\Gamma(2\alpha+1)r^4t^{3\alpha}}{16\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} \left(\frac{5rwh - 20rv^2}{4h^5} - \frac{15v^2wh - 25v^4}{4h^6} \right. \\
& + \frac{v^2 + w^2}{h^4} - \frac{4v^2w}{h^5} - \frac{wh - 3v^2}{8h^4} \Bigg) \left(\frac{5rvh^2 - 60rvw + 100rv^3}{4h^6} \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{30vw^2h^2 - 175v^3wh + 15v^3h^2 + 150v^5}{4h^7} - \frac{12vw^2 + 8w^3}{h^5} \\
& + \frac{4vw}{h^4} + \frac{20v^3w}{h^6} - \frac{vh^2 - 9vwh + 12v^3}{8h^5} \Bigg) \\
& - \frac{6834375\Gamma(2\alpha + 1)r^5vt^{3\alpha}}{16h^2\Gamma(3\alpha + 1)\Gamma^2(\alpha + 1)} \left(\frac{5rv + 4vw}{4h^4} - \frac{5v^3}{4h^5} - \frac{v}{8h^3} \right)^2 \\
& - \frac{6834375\Gamma(2\alpha + 1)r^5t^{3\alpha}}{8h\Gamma(3\alpha + 1)\Gamma^2(\alpha + 1)} \left(\frac{5rv + 4vw}{4h^4} - \frac{5v^3}{4h^5} - \frac{v}{8h^3} \right) \\
& \times \left(\frac{5rwh - 20rv^2}{4h^5} - \frac{15v^2wh - 25v^4}{4h^6} + \frac{v^2 + w^2}{h^4} - \frac{4v^2w}{h^5} \right. \\
& \left. - \frac{wh - 3v^2}{8h^4} \right) \\
& + \frac{307546875\Gamma(3\alpha + 1)r^6t^{4\alpha}}{16\Gamma(4\alpha + 1)\Gamma^3(\alpha + 1)} \left(\frac{5rv + 4vw}{4h^4} - \frac{5v^3}{4h^5} - \frac{v}{8h^3} \right)^2 \\
& \times \left(\frac{5rwh - 20rv^2}{4h^5} - \frac{15v^2wh - 25v^4}{4h^6} + \frac{v^2 + w^2}{h^4} - \frac{4v^2w}{h^5} \right. \\
& \left. - \frac{wh - 3v^2}{8h^4} \right),
\end{aligned}$$

where $v = v(x) = a \sinh(x + \xi_0) + b \cosh(x + \xi_0)$, $w = w(x) = a \cosh(x + \xi_0) + b \sinh(x + \xi_0)$ and $h = h(x) = a \cosh(x + \xi_0) + b \sinh(x + \xi_0) + r$. Substituting $n - 1$ order approximate solitary wave solution into (17) leads to the n order approximate solitary wave solution in the following form, making use of the Maple package, we obtain $u_n(x, t)$, $u_{n+1}(x, t)$, ..., the iteration leads to the solitary wave solution of the time-fractional Degasperis-Procesi equation is

$$\begin{aligned}
u(x, t) &= \lim_{n \rightarrow \infty} u_n(x, t) \\
&= - \frac{\frac{15}{4}r}{a \cosh(x - \frac{5}{2}t + \xi_0) + b \sinh(x - \frac{5}{2}t + \xi_0) + r}.
\end{aligned}$$

5. Discussion

We employ the variational iteration method to solve the time-fractional Degasperis-Procesi equation that is obtained using the Euler-Lagrange variational technique. In fact, we continue our calculations until the three-order iteration of the variational iteration method. Therefore, our approximate calculations are carried out concerning the solution of the time-fractional Degasperis-Procesi

Figure 1: The distribution function u as a 3-dimensions graph for the order α

equation taking into account the values of the coefficients $a = 1$, $b = 0$, $r = 1$ and $\xi_0 = 0$. Because our main objective of the present paper is to explore

Figure 2: The function u as a function of space x at time $t = 1$ for the order α : (b1) 3-dimensions graph, (b2) 2-dimensions graph

Figure 3: The amplitude of the function u as a function of time t at space $x = 1$ of the fractional order α : (c1) 3-dimensions graph, (c2) 2-dimensions graph

the effect of the fractional order derivative, therefore, our solution is calculated for some interesting values namely, $\alpha = 1, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, \frac{1}{4}$ and $\frac{1}{8}$. In addition, 3-dimensional representation of the solution of the time-fractional Degasperis-Procesi equation with space x and time t for different values of the fractional order α is presented in Fig. 1, the solution $u(x, t)$ is still a single soliton wave solution for all values of the fractional parameter. This means that the balanc-

ing scenario between nonlinearity and dispersion is still valid. Fig. 2 gives a good impression about the change of amplitude and width of the soliton due to the variation of the fractional power order, both 2- and 3-dimensional graphs depicted the behavior of the solution $u(x, t)$ against the space x at time $t = 1$ corresponding to different values of the fractional order α . The behavior shows that the increasing of the fractional parameter α increasing both the height and the width of the solitary wave solution. This means that, the fractional parameter can be used to modify the shape of the solitary wave without change of the nonlinearity and the dispersion effects in the medium. Fig. 3 devoted to study the expression between the amplitude of the soliton and the fractional order at different time values. These figures show that at the same time, the increasing of the fractional α , the amplitude of the solitary wave to some value of α is invariant.

References

- [1] J. He, A new approach to nonlinear partial differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **2(4)** (1997), 230-235. DOI: 10.1016/S1007-5704(97)90007-1.
- [2] J. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Comput. Methods Appl. Mech. Eng.*, **167** (1998), 57-68. DOI: 10.1016/S0045-7825(98)00108-X.
- [3] R.Y. Molliq, M.S.M. Noorani, I. Hashim, Variational iteration method for fractional heat- and wave-like equations, *Nonlinear Anal., Real World Appl.*, **10** (2009), 1854-1869. DOI: 10.1016/j.nonrwa.2008.02.026.
- [4] S. Momani, Z. Odibat, A. Alawnah, Variational iteration method for solving the space- and time-fractional KdV equation, *Numer. Methods Part. Differ. Equ.*, **24(1)** (2008), 261-271. DOI: 10.1002/num.20247.
- [5] A. Degasperis, M. Procesi, *Asymptotic Integrability, in Symmetry and Perturbation Theory*, World Scientific Publishing (1999).
- [6] R.S. Johnson, The classical problem of water waves: a reservoir of integrable and nearly-integrable equations, *J. Nonlin. Math. Phys.*, **10** (2003), 72-92. DOI: 10.2991/jnmp.2003.10.s1.6.
- [7] H.R. Dullin, G.A. Gottwald, D.D. Holm, On asymptotically equivalent shallow water wave equations, *Phys. D*, **190** (2004), 1-14. DOI: 10.1016/j.physd.2003.11.004.

- [8] A. Fokas, B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Phys. D*, **4** (1981), 47-66. DOI: 10.1016/0167-2789(81)90004-X.
- [9] J. Lenells, Conservation laws of the Camassa-Holm equation, *J. Phys. A*, **38** (2005), 869-880. DOI: 10.1088/0305-4470/38/4/007.
- [10] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, *Phys Rev Lett*, **71** (1993), 1661-1664. DOI: 10.1103/PhysRevLett.71.1661.
- [11] S. El-Wakil, E. Abulwafa, M. Zahran, A. Mahmoud, Time-fractional KdV equation: formulation and solution using variational methods, *Nonlinear Dyn.*, **65** (2011), 55-63. DOI: 10.1007/s11071-010-9873-5.
- [12] R. Gorenflo, F. Mainardi, E. Scalas, M. Raberto, Fractional calculus and continuous-time finance. III. The diffusion limit, *Mathematical finance (Konstanz, 2000)*, *Trends Math.*, (2001), 171-180. DOI: 10.1007/978-3-0348-8291-0-17.
- [13] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).
- [14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Studies. Elsevier, Amsterdam, The Netherlands (2006).
- [15] B. Lundstrom, M. Higgs, W. Spain, A. Fairhall, Fractional differentiation by neocortical pyramidal neurons, *Nature Neuroscience*, **11** (2008), 1335-1342. DOI: 10.1038/nn.2212.
- [16] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A*, **37** (2004), R161-R208. DOI: 10.1088/0305-4470/37/31/R01.
- [17] Y.A. Rossikhin, M.V. Shitikova, Application of fractional derivatives to the analysis of damped vibrations of viscoelastic single mass systems, *Acta Mechanica*, **120** (1997), 109-125. DOI: 10.1007/BF01174319.
- [18] L. Sabatelli, S. Keating, J. Dudley, P. Richmond, Waiting time distributions in financial markets, *Eur. Phys. J. B*, **27** (2002), 273-275. DOI: 10.1140/epjb/e20020151.

- [19] R. Schumer, D.A. Benson, M.M. Meerschaert, B. Baeumer, Multiscaling fractional advection-dispersion equations and their solutions, *Water Resour. Res.*, **39** (2003), 1022-1032. DOI: 10.1029/2001WR001229, 2003.
- [20] M.S. Tavazoei, M. Haeri, Describing function based methods for predicting chaos in a class of fractional order differential equations, *Nonlinear Dyn.*, **57(3)** (2009), 363-373. DOI: 10.1007/s11071-008-9447-y.
- [21] H. Bateman, On dissipative systems and related variational principles, *Phys. Rev.*, **38(4)** (1931), 815-819. DOI: 10.1103/PhysRev.38.815.
- [22] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, *Phys. Rev. E*, **53(2)** (1996), 1890-1899. DOI: 10.1103/PhysRevE.53.1890.
- [23] F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E*, **55(3)** (1997), 3581-3592. DOI: 10.1103/PhysRevE.55.3581.
- [24] O.P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, *J. Math. Anal. Appl.*, **272(1)** (2002), 368-379. DOI: 10.1016/S0022-247X(02)00180-4.
- [25] O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dyn.*, **38(4)** (2004), 323-337. DOI: 10.1007/s11071-004-3764-6.
- [26] O.P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives, *J. Phys. A, Math. Theor.*, **40** (2007), 6287-6303. DOI: 10.1088/1751-8113/40/24/003.
- [27] D. Baleanu, S.I. Muslih, Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives, *Phys Scr.*, **72** (2005), 119-121. DOI: 10.1238/Physica.Regular.072a00119.
- [28] M. Inokuti, H. Sekine, T. Mura, *General use of the Lagrange multiplier in non-linear mathematical physics. In: Nemat-Nasser, S. (ed.) Variational Method in the Mechanics of Solids*, Pergamon Press, Oxford (1978).
- [29] S. Saha Ray, R. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method, *Appl. Math. Comput.*, **167** (2005), 561-571. DOI: 10.1016/j.amc.2004.07.020.
- [30] J. Cang, Y. Tan, H. Xu, S. Liao, Series solutions of non-linear Riccati differential equations with fractional order, *Chaos Solitons Fractals*, **40(1)** (2009), 1-9. DOI: 10.1016/j.chaos.2007.04.018.

- [31] S. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, *Ph.D. Thesis*. Shanghai Jiao Tong University (1992).
- [32] N.H. Sweilam, M.M. Khader, R.F. Al-Bar, Numerical studies for a multi-order fractional differential equation, *Phys. Lett. A*, **371** (2007), 26-33. DOI: 10.1016/j.physleta.2007.06.016.
- [33] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1999).
- [34] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland (1993).
- [35] J. He, Semi-inverse method and generalized variational principles with multi-variables in elasticity, *Appl. Math. Mech.*, **21(7)** (2000), 797-808. DOI: 10.1007/BF02428378.
- [36] J. He, Variational principles for some nonlinear partial differential equations with variable coefficients, *Chaos Solitons Fractals*, **19** (2004), 847-851. DOI: 10.1016/S0960-0779(03)00265-0.