SOME PROPERTIES OF WEAK-$\bigoplus$-SUPPLEMENTED MODULE

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Abstract: In this paper we give explicit necessary and sufficient conditions for weak-$\bigoplus$-supplemented module. If $M$ is lifting module then it is weak-$\bigoplus$-supplemented module. Moreover if we have an $R$-module $M$ such that is generalized lifting module then $M$ is weak-$\bigoplus$-supplemented module. We prove that if $M$ is supplemented and projective then $M$ is weak-$\bigoplus$-supplemented.

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1. Introduction and Preliminaries

Throughout, all rings are associative rings with identity, and all modules are unital left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If $N$ is a submodule (respectively proper submodule) of $M$ we write $N \leq M$ (respectively $N$ less than $M$). Let $N$ be submodule of $M$ then $N$ is small in $M$ ($N \ll M$) if there is no proper submodule $L$ of $M$ such that $N+L=M$. Now $N$ is called supplement of $L$ in $M$, if $N+L=M$ and $N$ minimal.

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with respect to this property, therefore a module $M$ is called supplemented if every submodule of $M$ has a supplement which is a direct summand. A module $M$ is called $\oplus$-supplemented if for any submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $M=N+K$ and $N \cap K \ll K$, namely every submodule $N$ of $M$ has a direct summand supplement in $M$. A module $M$ is called $\bigoplus$-supplemented if for any submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K \ll K$, namely every submodule $N$ of $M$ has a direct summand supplement in $M$. A module $M$ is called lifting module (or satisfies $(D_1)$) if for every submodule $A$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq A$ and $(A/K) \ll (M/K)$. Clearly, every lifting module is $\oplus$-supplemented, but the converse is not true (see,[7]). A module $M$ is called amply supplemented if $B$ contains a supplement of $A$ whenever $M = A + B$. Recall that a left $R$-module $M$ is said to be semisimple if it is the direct sum of simple submodules and hence any module $M$ is called a weak lifting module provided, for each semisimple submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $(N/K) \ll (M/K)$, or there exists a decomposition $M = M_1 \bigoplus M_2$, such that $M_1 \leq N$ and $M \cap N \ll M_2$.

In this article we generalize $\bigoplus$-supplemented module in order to we obtain weak-$\bigoplus$-supplemented module. We use weak lifting, $\oplus$-supplemented and strongly $\bigoplus$-supplemented modules to study the generalization of $\bigoplus$-supplemented module and the relation between them.

2. Main Results

Let $M$ be an $R$-module, then $M$ is called weak-$\bigoplus$-supplemented module if for each semisimple submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $M = N + K$ and $(N \cap K) \ll K$. Since there is a relation between weak lifting and weak-$\bigoplus$-supplemented module therefore we can depend on a concept lifting module in order to get some properties of weak-$\bigoplus$-supplemented module. Any lifting module is amply supplemented and every amply supplemented module is weak lifting but the converse is not true, for example the ring of integers $Z$ is weak lifting but not amply supplemented, also lifting module is weak lifting and so weak-$\bigoplus$-supplemented. Every submodule of a module $M$ lies above a direct summand of $M$. Let $M$ be an $R$-module. If $M$ is lifting then $M$ is $\bigoplus$-supplemented and so it is weak-$\bigoplus$-supplemented.

**Lemma 2.1** (5, Proposition 2.3). *If $M$ is weak lifting then $M$ is weak-$\bigoplus$-supplemented module.*

**Theorem 2.2.** Let $M$ be an $R$-module. If $M$ is lifting then $M$ is weak-$\bigoplus$-supplemented module.

**Proof.** Since $M$ is a lifting module then $M$ is amply supplemented and any
supplement submodule of $M$ is direct summand of $M$. Therefore if for any two submodules $L$ and $K$ of $M \ni L+K=M$ and $K$ contains a supplement of $L$ in $M$. Since every submodule of $M$ lies above a direct summand of $M$ then $M$ is $(D_1)$-module therefore by [4] $M$ is a lifting module and hence $M$ is a weak lifting. Thus $M$ is weak-$\bigoplus$-supplemented module.

**Theorem 2.3.** Let $M$ be an $R$-module. If every amply supplemented module with every submodule of a module $M$ lies above a direct summand, then $M$ is weak-$\bigoplus$-supplemented module.

**Proof.** Suppose $M$ is amply supplemented module, then there exists $N_1$ and $N_2$ are submodules of $M$ with $N_1+N_2=M$, $N_2 \supseteq A_i$ such that $A_i$ are supplemented of $N_1$. Since every submodule of $M$ lies above a direct summand of $M$, then $M$ is $(D_1)$- module and this means if for every submodule $A$ of $M$, there exists a direct summand $K$ of $M$ such that $N \subseteq A$ and $(A/N) \ll (M/N)$, and so $M$ is lifting module. Hence by [Theorem 2.2] $M$ is weak-$\bigoplus$-supplemented module.

**Lemma 2.4** (10,Theorem 3.6). Let $M$ be a Noetherian $R$-module. If $M$ is finitely lifting, then $M$ is lifting.

**Theorem 2.5.** Let $M$ be an $R$-module. If $M$ satisfying the following conditions:

1. $M$ is Noetherian $R$-module,
2. $M$ is finitely lifting module.

Then $M$ is weak-$\bigoplus$-supplemented module.

**Proof.** Since $M$ is Noetherian module then every submodule $N$ of $M$ is finitely generated and $M$ is finitely lifting then it is lifting module and by [Theorem 2.2] $M$ is weak-$\bigoplus$-supplemented module.

**Proposition 2.6.** Let $M$ be an $R$-module. If $M$ satisfying the following conditions:

1. $M$ is hollow-lifting module,
2. $M$ has finite hollow dimension,
3. $M$ is amply supplemented module.

Then $M$ is weak-$\bigoplus$-supplemented module.
Proof. Let $M$ be an $R$-module. Suppose that $M$ satisfying hollow-lifting conditions. If $N$ coclosed submodule of $M$, then $(M/N)$ has finite hollow dimension, therefore we must prove that $N$ is a direct summand of $M$. Now we use induction on hollow dimension of $(M/N)$. Let hollow dimension of $(M/N)$ is $n$. If $n=1$, this means $N$ is a direct summand of $M$ because $M$ is hollow-lifting. Suppose that hollow dimension of $(M/N)$ is $n_1$ and for every coclosed submodule $F$ of $M$ such that $(M/N)$ has hollow dimension less then $n_1$, $F$ is a direct summand of $M$. Let $(G/N)$ be coclosed in $(M/N)$ such that $(M/N)/(G/N)$ is hollow. By [1], $G$ is coclosed in $M$. Hence $M=G+G_1$ for some submodule $G_1$ of $M$. If $M$ is hollow-lifting. Then $n_1=G \cap (N \oplus G_1)$ and $(M/N)=(G/N)\oplus((N \oplus G_1)/N)$. Thus $(N \oplus G_1)/N$ is coclosed in $(M/N)$. Again by [1], $(N \oplus G_1)$ is coclosed in $M$. By induction, $(N \oplus G_1)$ is a direct summand of $M$ and so $N$ is a direct summand of $M$. Then $M$ is lifting. So is weak-$\oplus$-supplemented.

Theorem 2.7. Let $\bigoplus M_i \ (i=1,...,n)$ be a finite direct sum of $M$ such that is weak-$\oplus$-supplemented modules then $\bigoplus M_i$, also weak-$\oplus$-supplemented.

Proof. Let $M=M_1 \oplus ... \oplus M_n$. For $i=1,...,n$. Let $p_i: M \to M_i$ be the projection map and let $L$ be a semisimple submodule of $M$. We have $0$ is a supplement of $(L+M_1)+M_2+...+M_n$ in $M$ then $(L+M_1)\cap M_i$, $i=2,...,n$ has a direct summand supplement $N$ in $M_2$, $M_3,...,M_n$ because $(L+M_1)\cap M_i=p_2(L)$ is semisimple. Now by [2], $N$ is a supplement of $(L+M_1)$ in $M$. Since

$$(L+N) \cap M_1 \cap (L+M_i) \cap M_1 = p_1(L)$$

is semisimple such that $i=2,...,n$ implies $(L+N)\cap M_1$ has a direct summand supplement $K$ in $M_1$. Again by [2], $(N+K)$ is a supplement of $L$ in $M$ implies $(N \oplus K)$ is a direct summand of $M$. Hence $M$ is weak-$\oplus$-supplemented.

A module $M$ is called a generalized lifting module if the following condition satisfied:

(If $M=M_1 \oplus M_2$ and $A \leq M$, then there exist $C_i \leq \bigoplus M_i \ (i=1,2)$ such that $C_1 \oplus C_2$ is a supplement of $A$ in $M$).

Let $M$ be an $R$-module, then any lifting module is a generalization lifting module and so is $\oplus$-supplemented. Also every generalization lifting module is $\oplus$-Supplemented module and the converse is true if we put some conditions on submodules of $M$. See the following theorem:
**Theorem 2.8.** If $M$ is a $\bigoplus$-supplemented, and satisfies in this condition that for every two direct summands $N_1$ and $N_2$ of $M$ such that $(N_1 \cap N_2)$ is coclosed in $M$, implies that $(N_1 \cap N_2)$ is a direct summand of $M$. Then $M$ is a generalization lifting module.

Recall that lifting module is generalization lifting module, and so $\bigoplus$-supplemented but the GL-module is not lifting module. See the following example:

**Example 2.9.** Let $p$ be any prime integer. $\mathbb{Z}$-Module $(\mathbb{Z}/p\mathbb{Z}) \bigoplus (\mathbb{Z}/p^3 \mathbb{Z})$ is generalization lifting module but not lifting module [8].

**Theorem 2.10.** Let $M$ be an $\mathbb{R}$-module. If $M$ is generalization lifting module then it is weak-$\bigoplus$-supplemented module

A module $M$ is called a strongly $\bigoplus$-supplemented module if every supplement submodule of $M$ is a direct summand of $M$. All strongly $\bigoplus$-supplemented modules are $\bigoplus$-supplemented. And so every $\bigoplus$-supplemented is weak-$\bigoplus$-supplemented module, but the converse is not true in general. See the following example:

**Example 2.11.** Let $R$ be a local Artinian ring with radical $W$ such that $W^2=0$, $Q=R=W$ is commutative, $\dim(QW)=2$ and $\dim(WQ)=1$. Then the indecomposable injective right $R$-module $U=([R \bigoplus R]/D)R$ with $D=(ur,-vr)$—$r \in R$ in [2] is a weak-$\bigoplus$-supplemented module, but is not $\bigoplus$-supplemented.

Given a right $R$-module $M$, the socle of $M$ is defined as the sum of all the simple submodules of $M_R$. Now we introduce another example to show $M$ is weak-$\bigoplus$-supplemented module. See the following example:

**Example 2.12.** Let $N$ be a nonzero semisimple submodule of $M$. Then $N=\text{Soc}(M)$. Since $\text{Soc}(M)$ is simple and $N \ll M$ then $\text{Soc}(M) \ll M$. Hence $M$ is weak-$\bigoplus$-supplemented module.

**Theorem 2.13.** Let $M$ be an $\mathbb{R}$-module. If $M$ supplemented module and every supplement submodule of $M$ lies above a direct summand then $M$ is weak-$\bigoplus$-supplemented module.

**Proof.** Suppose $V$ be supplement submodule of $M$ and supplement of $U$ in $M$. But we have every supplement submodule of $M$ lies above a direct summand then there exist $M_1$ submodule of $M$ and $M_2$ submodule of $M$ such that $M=M_1 \bigoplus M_2$, $M_1$ submodule of $V$ and $(V \cap M_2)$ small in $M_2$. Therefore $V=V \cap M=M_1 \bigoplus V \cap M_2$ and since $V \cap M_2$ small in $M$, then

$$M = U + V = U + V \cap M_2 + M_1 = U + M_1.$$
Also, since $V$ is a supplement of $U$ and $V = M_1$. Thus $M = V \oplus M_2$ and $V$ is a direct summand of $M$. Hence $M$ is strongly $\oplus$-supplemented. But every strongly $\oplus$-supplemented module is $\oplus$-supplemented and then $M$ is weak-$\oplus$-supplemented module.

**Proposition 2.14.** Let $M$ be an $R$-module. If $M$ is supplemented and projective then $M$ is weak-$\oplus$-supplemented.

**Proof.** Suppose $M$ is projective $R$-module such that $M = U + V$. We must prove that the $\beta : U \oplus V \to M$ is splits where $\beta$ is epimorphism. Since $M$ projective module then there exists a mapping $\delta : M \oplus U \to V$ and then $\beta \circ \delta = 1_M$ is the identity mapping implies $\beta : U \oplus V \to M$ is splits. Hence $\beta$ splits and so $\pi$-projective. Now let $M = N + K$ and $A$ be a supplement of $N$ in $M$. Also, let $g \in \text{End}(M) \supseteq \text{Img}(g) \subseteq K$ and $\text{Img}(1-g) \subseteq N$, since we have $g(N) \subseteq N$, $M = N + g(B)$ and $g(N \cap B) = N \cap g(B)$ implies $b \cap g(A) = (1-g)(b) \in N)$. Since $N \cap B \ll A$, $N \cap g(A)) \ll g(A)$,

and then $g(A)$ is a supplement of $N \ni g(A) \subseteq K$. Hence $M$ is amply supplemented module. Let $N$ be a semisimple submodule of $M$, there exists a submodule $K$ of $M \ni M = N + K$ and $N \cap B \ll K$. Now there exists a submodule $T$ of $M \ni M = T + K$, $T \cap K \ll N$. Since $T$ is semisimple ($T \cap K = 0$) and hence $M = T \oplus K$. Then $M$ is weak lifting [5], but a weak lifting is weak-$\oplus$-supplemented module.

**Proposition 2.15.** For a Prufer ring, any finitely generated torsion free supplemented $R$-module is weak-$\oplus$-supplemented module.

**Proof.** Suppose $R$ is a Prufer ring, then every finitely generated torsion free $R$-module is projective (See [6]). Since every projective module is $\pi$-projective, then every finitely generated torsion free supplemented $R$-module is strongly $\oplus$-supplemented and this implies $M$ is $\oplus$-supplemented. Hence $M$ is weak-$\oplus$-supplemented module.

**Corollary 2.16.** Every $(D_1)$ module is weak-$\oplus$-supplemented module.

**Corollary 2.17.** Every amply supplemented module is weak-$\oplus$-supplemented.

**Corollary 2.18.** Every strongly-$\oplus$-supplemented module is a weak-$\oplus$-supplemented.
The following implications are now clear for a module $M$:

Lifting module $\Rightarrow$ Generalization lifting module $\Rightarrow$ $\oplus$-Supplemented module

$\Downarrow$

Strongly $\oplus$-supplemented $\Rightarrow$ Weak-$\oplus$-Supplemented module.

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