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ON A SPECIAL CONCIRCULAR LIE-RECURRENCE IN A FINSLER SPACE

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Abstract: The concept of Lie-recurrence in a Finsler space was introduced by the second author [8] of the present paper in 1982.The Lie-recurrence in a Riemannian space was studied by K. L. Duggal [3] in 1992 but he used the term curvature inheriting symmetry in place of Lie-recurrence. K. L. Duggal also applied the theory to the study of fluid space time. Since then both the terms (Lie-recurrence and curvature inheriting symmetry) are in use. The present authors [11], Shivalika Saxena and P. N. Pandey [12], [13], C. K. Mishra and Gautam Lodhi [1] studied a Lie-recurrence (curvature inheriting symmetry) in a Finsler space and discussed the possibilities for contra and concurrent vector fields to generate Lie-recurrence. The present paper deals with a Lie-recurrence generated by a special concircular vector field and such Lie-recurrence is termed as a special concircular Lie-recurrence. We obtain certain results related to a special concircular Lie-recurrence in a general Finsler space as well as in birecurrent and bisymmetric Finsler spaces.

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1. Introduction

Let F_n be an n-dimensional Finsler space equipped with a metric function $F(x^i, y^i)$ satisfying the requisite conditions[2], the corresponding metric tensor $g(x^i, y^i)$ and the Berwald connection $G(x^i, y^i)$. Henceforth all the geometric objects are assumed to be functions of line-element (x^i, y^i), unless stated otherwise. Let the components of the metric tensor g and coefficients of Berwald connection G be denoted by g_{ij} and G^i_{jk} respectively. The metric function F and the components g_{ij} of the metric tensor are connected by

$$a)\quad g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2, \quad b)\quad g_{ij}y^i y^j = F^2, \tag{1}$$

where $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$.
The quantities G^i_{jkh} defined by $G^i_{jkh} = \dot{\partial}_h G^i_{jk}$, are components of a symmetric tensor and satisfy

$$G^i_{jkh}y^h = 0, \tag{2}$$

Berwald covariant derivative of a tensor field T^i_j with respect to x^k is defined by

$$\mathbf{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)G^r_{kh}y^h + T^r_j G^i_{rk} - T^i_r G^r_{jk}, \tag{3}$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$.
The commutation formula for the operators $\dot{\partial}_j$ and $\mathbf{B}_k$ is given by

$$(\dot{\partial}_j \mathbf{B}_k - \mathbf{B}_k \dot{\partial}_j)X^i = G^i_{jkh}X^h, \tag{4}$$

while the Ricci-commutation formula is given by

$$(\mathbf{B}_j \mathbf{B}_k - \mathbf{B}_k \mathbf{B}_j)X^i = H^i_{hkj}X^h - (\dot{\partial}_h X^i)H^h_{kj}, \tag{5}$$

X^i being components of an arbitrary vector field. H^i_{hkj} appearing in equation (5) are components of the Berwald curvature tensor. This tensor is skew-symmetric in last two lower indices and positively homogeneous of degree zero in y^i. This tensor and the tensor H^i_{kj} appearing in equation (5) are connected by the following:

$$a)\quad H^i_{jkh}y^j = H^i_{kh}, \quad b)\quad \dot{\partial}_j H^i_{kh} = H^i_{jkh}. \tag{6}$$

The deviation tensor H^i_h and the tensor H^i_{kh} are related by

$$a)\quad H^i_{kh} = \frac{1}{3}(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k), \quad b)\quad H^i_{kh}y^k = H^i_h. \tag{7}$$

Let a contravariant vector field $v^i(x^j)$ generates the infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i(x^j), \tag{8}$$

where ϵ is an infinitesimal constant. Let us denote the operator of Lie-differentiation with respect to this transformation by $\pounds$. The Lie-derivative of an arbitrary tensor field T^i_j with respect to above transformation [4] is given by

$$\pounds T^i_j = v^r \mathbf{B}_r T^i_j - T^r_j \mathbf{B}_r v^i + T^i_r \mathbf{B}_j v^r + (\dot{\partial}_r T^i_j) \mathbf{B}_s v^r y^s, \tag{9}$$

2. Special Concircular Lie-Recurrence

Let us consider a Finsler space admitting the infinitesimal transformation (8) generated by a special concircular vector field $v^i(x^j)$ characterized by

$$\mathbf{B}_k v^i = \rho \delta^i_k, \tag{10}$$

where ρ is not a constant.
Differentiating equation(10) partially with respect to y^j,we get

$$\dot{\partial}_j \mathbf{B}_k v^i = (\dot{\partial}_j \rho) \delta^i_k, \tag{11}$$

Using the commutation formula exhibited by equation (4), we have

$$v^r G^i_{jkr} = (\dot{\partial}_j \rho) \delta^i_k, \tag{12}$$

Transvecting equation (12) by y^k and using (2), we get $y^i(\dot{\partial}_j \rho) = 0$, which implies $\dot{\partial}_j \rho = 0$, This shows that ρ is a function of positional co-ordinates x^i only.The second author [7] proved that a special concircular vector field can not generate an affine motion in a Finsler space. This means $\pounds G^i_{jk} \neq 0$ for every special concircular transformation. Therfore, there is a possibility of existence of a special concircular transformation with respect to which $\pounds H^i_{jkh} \neq 0$.

Suppose the special concircular transformation (8) is a Lie-recurrence (curvature inheriting symmetry) in a Finsler space F_n, i.e.

$$\pounds H^i_{jkh} = \Phi H^i_{jkh}, \tag{13}$$

where Φ is a non-zero scalar. The Second author [8] proved that the scalar function Φ is independent of y^i. In view of equation (9), equation (13) may be written as

$v^r \mathcal{B}_r H^i_{hjk} + (\dot{\partial}_r H^i_{hjk}) \mathcal{B}_s v^r y^s - H^r_{hjk} \mathcal{B}_r v^i + H^i_{rjk} \mathcal{B}_h v^r + H^i_{hrk} \mathcal{B}_j v^r + H^i_{hjr} \mathcal{B}_k v^r = \Phi H^i_{hjk}$,

Using equation (10) and the fact that the curvature tensor H^i_{hjk} is positively homogeneous of degree zero in y^i, we get

$$v^r \mathcal{B}_r H^i_{hjk} = (\phi - 2\rho) H^i_{hjk}. \tag{14}$$

Differentiating equation (10) covariantly with respect to x^j ,we get

$$\mathcal{B}_j \mathcal{B}_k v^i = \rho_j \delta^i_k, \tag{15}$$

where $\rho_j = \mathcal{B}_j \rho$.
Taking skew- symmetric part of equation (15), we have

$$\mathcal{B}_j \mathcal{B}_k v^i - \mathcal{B}_k \mathcal{B}_j v^i = \rho_j \delta^i_k - \rho_k \delta^i_j.$$

Using the commutation formula exhibited by equation (5), we get

$$v^r H^i_{rkj} = \rho_j \delta^i_k - \rho_k \delta^i_j. \tag{16}$$

Contracting the indices i and j in equation (16), we have

$$v^r H_{rk} = -(n-1)\rho_k, \tag{17}$$

where H_{rk} is the Ricci tensor defined by $H_{rk} = H^s_{rks}$.
Using equation (17) in equation (16), we find

$$\left((n-1) H^i_{rkj} + H_{rj}\delta^i_k - H_{rk}\delta^i_j\right) v^r = 0. \tag{18}$$

Now,

$$\pounds \rho_k = v^m \mathcal{B}_m \rho_k + \left(\dot{\partial}_r \rho_k\right) \mathcal{B}_s v^r y^s + \rho_r \mathcal{B}_k v^r,$$

Using equation (10) and the fact that ρ_k is homogenous of degree zero in y^i, we get

$$\pounds \rho_k = v^m \mathcal{B}_m \rho_k + \rho \rho_k. \tag{19}$$

Transvecting equation (14) by v^h, we have

$$v^h v^r \mathcal{B}_r H^i_{hjk} = (\phi - 2\rho) H^i_{hjk} v^h. \tag{20}$$

Differentiating equation (16) covariantly with respect to x^m, we have

$$\mathcal{B}_m v^r H^i_{rkj} + v^r \mathcal{B}_m H^i_{rkj} = \mathcal{B}_m \rho_j \delta^i_k - \mathcal{B}_m \rho_k \delta^i_j,$$

Transvecting by v^m and using equation (10), we have

$$v^r \rho H^i_{rkj} + v^m v^r \mathcal{B}_m H^i_{rkj} = v^m \left(\mathcal{B}_m \rho_j \delta^i_k - \mathcal{B}_m \rho_k \delta^i_j \right), \tag{21}$$

In view of equation (20), equation (21) implies

$$v^r \rho H^i_{rkj} + (\phi - 2\rho) H^i_{rkj} v^r = v^m \mathcal{B}_m \left(\rho_j \delta^i_k - \rho_k \delta^i_j \right), \tag{22}$$

Using equation (16) in equation (22), we have

$$\rho \left(\rho_j \delta^i_k - \rho_k \delta^i_j \right) + (\phi - 2\rho) \left(\rho_j \delta^i_k - \rho_k \delta^i_j \right) = v^m \mathcal{B}_m \left(\rho_j \delta^i_k - \rho_k \delta^i_j \right), \tag{23}$$

Contracting the indices i and j in equation (23), we get

$$v^m \mathcal{B}_m \rho_k = (\phi - \rho) \rho_k, \tag{24}$$

Using equation (24) in equation (19), we have

$$\pounds \rho_k = \phi \rho_k. \tag{25}$$

This leads to:

Theorem 1. *If a Finsler space admits a special concircular Lie-recurrence characterized by equation* (10) *and equation* (13) , *the covariant derivative of scalar* ρ *appearing in equation* (10) *is Lie-recurrent with respect to the Lie-recurrence.*

3. Special Concircular Lie-Recurrence in a Birecurrent Finsler Space

Let us consider a birecurrent Finsler space F_n characterized [7] by

$$\mathcal{B}_l \mathcal{B}_m H^i_{jkh} = a_{lm} H^i_{jkh}, \tag{26}$$

where a_{lm} are components of a non-zero covariant tensor of type $(0, 2)$ and $H^i_{jkh} \neq 0$.
suppose that this space admits a special concircular Lie-recurrence. Then we have equation (14).
Differentiating equation (14) covariantly with respect to x^m , we get

$$\mathcal{B}_m v^r \mathcal{B}_r H^i_{hjk} + v^r \mathcal{B}_m \mathcal{B}_r H^i_{hjk} = (\phi_m - 2\rho_m) H^i_{hjk} + (\phi - 2\rho) \mathcal{B}_m H^i_{hjk}, \tag{27}$$

where $\phi_m = \mathbf{B}_m \phi$.
Using equation (10) and equation (26) in equation (27), we have

$$(v^r a_{mr} - \phi_m + 2\rho_m)\, H^i_{hjk} = (\phi - 3\rho)\, \mathbf{B}_m H^i_{hjk}. \tag{28}$$

A birecurrent Finsler space is non-flat i.e. $H^i_{jkh} \neq 0$. Also, it is not symmetric i.e. $\mathbf{B}_m H^i_{hjk} \neq 0$, for $\mathbf{B}_m H^i_{hjk} = 0$ implies $\mathbf{B}_l \mathbf{B}_m H^i_{hjk} = 0$, in view of (26), gives $a_{lm} = 0$, a contradiction.
Therefore, equation (28) implies either of the following conditions
(i) $\phi - 3\rho = 0, \quad v^r a_{mr} - \phi_m + 2\rho_m = 0$,
(ii) $\phi - 3\rho \neq 0, \quad v^r a_{mr} - \phi_m + 2\rho_m \neq 0$.
we can write the condition (i) as $\phi = 3\rho,\ v^r a_{mr} = \rho_m$.
Let us consider the condition (ii). In this case, equation (28) may be written as

$$\mathbf{B}_m H^i_{hjk} = \frac{(v^r a_{mr} - \phi_m + 2\rho_m)}{\phi - 3\rho} H^i_{hjk}, \tag{29}$$

which shows that the space is recurrent, The second author [6] proved that a recurrent Finsler space does not admit a special concircular vector field, which implies that a recurrent Finsler space does not admit a special concircular Lie-recurrence. Therefore, the pair of conditions (ii) is not possible. Hence, we may conclude:

Theorem 2. *A birecurrent Finsler space F_n admitting a special concircular Lie-recurrence necessarily satisfies the conditions $\phi = 3\rho$ and $v^r a_{mr} = \rho_m$.*

Taking skew-symmetric part of equation (26) and using the Ricci-commutation formula exhibited by (5), we have

$$H^r_{jkh} H^i_{rml} - H^i_{rkh} H^r_{jml} - H^i_{jrh} H^r_{kml} - H^i_{jkr} H^r_{hml} - (\dot\partial_r H^i_{jkh}) H^r_{ml} = A_{lm} H^i_{jkh}, \tag{30}$$

where $A_{lm} = a_{lm} - a_{ml}$. Operating both sides of equation (30) by the operator $\pounds$ and using equation (13), we get

$$\pounds A_{lm} = \phi A_{lm}.$$

This leads to:

Theorem 3. *The skew-symmetric part of the recurrence tensor a_{lm} of a birecurrent Finsler space admitting a special concircular Lie-recurrence is Lie-recurrent with respect to the Lie-recurrence.*

4. Special Concircular Lie-Recurrence in a Bisymmetric Finsler Space

Let us consider a bisymmetric Finsler space F_n characterized by

$$\mathbf{B}_m \mathbf{B}_l H^i_{hjk} = 0. \tag{31}$$

Differentiating equation (14), covariantly with respect to x^m , we have

$$\mathbf{B}_m v^r \mathbf{B}_r H^i_{hjk} + v^r \mathbf{B}_m \mathbf{B}_r H^i_{hjk} = (\phi_m - 2\rho_m) H^i_{hjk} + (\phi - 2\rho) \mathbf{B}_m H^i_{hjk} \tag{32}$$

Using equations (10) and (31) in (32),we get

$$(\phi - 3\rho) \mathbf{B}_m H^i_{hjk} = (2\rho_m - \phi_m) H^i_{hjk} \tag{33}$$

If $\phi = 3\rho$, equation (33) reduces to $\rho_m H^i_{hjk} = 0$, which implies $H^i_{hjk} = 0$ for $\rho_m \neq 0$. Thus we conclude:

Theorem 4. *A bisymmetric Finsler space F_n admitting a special concircular Lie-recurrence with condition $\phi = 3\rho$ is flat.*

If $\phi = 2\rho$ then $\phi_m - 2\rho_m = 0$. Therefore equation (33) may be written as

$$\mathbf{B}_m H^i_{hjk} = 0. \tag{34}$$

This shows that the space is symmetric. Thus, we see that a bisymmetric Finsler space admitting a special concircular Lie-recurrence with $\phi = 2\rho$ is a symmetric space admitting a special concircular Lie-recurrence.
In view of a result due to the second author [10], a symmetric Finsler space $F_n (n > 2)$ admitting a special concircular transformation is a Riemannian space of constant Riemannian curvature.
Therefore, we may conclude:

Theorem 5. *A bisymmetric Finsler space $F_n (n > 2)$ admitting a special concircular Lie-recurrence with $\phi = 2\rho$ is a Riemannian space with constant Riemannian curvature.*

If $\phi \neq 2\rho$ and $\phi \neq 3\rho$, then equation (33) may be written as

$$\mathbf{B}_m H^i_{hjk} = \frac{(2\rho_m - \phi_m)}{\phi - 3\rho} H^i_{hjk}. \tag{35}$$

This shows that the space is recurrent, but a recurrent space admitting a special concircular Lie-recurrence does not exist.
Hence we may conclude:

Theorem 6. *A bisymmetric Finsler space $F_n(n > 2)$can not admit a special concircular Lie-recurrence if ϕ is neither 2ρ nor 3ρ.*

From Theorems 4, 5 and 6, we may conclude:

Theorem 7. *A bisymmetric Finsler space $F_n(n > 2)$ admitting a special concircular Lie-recurrence is either flat or a Riemannian space of constant Riemannian curvature.*

References

[1] C.K.Mishra, Gautam Lodhi, On Curvature Inheriting Symmetry in Finsler space, *Acta Universitatis Apulensis*, **30** (2012), 39-48.

[2] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin (1959).

[3] K.L.Duggal, Curvature inheriting symmetry in Riemannian space with application of fluid space time, *J. Math. Phys.*, **33(9)** (1992),2989-2996.

[4] K. Yano, *The Theory of Lie derivatives and its applications*, North-Holland Publishing co. Amsterdam (1957).

[5] P. N. Pandey, On bisymmetric Finsler manifold, *Math education* , **11 (4)** (1977),77-80.

[6] P. N. Pandey, A recurrent Finsler manifold with a concircular vector field, *Acta. Math. Acad. Sci. (Hungar.)*, **35 (3-4)** (1980), 465-466.

[7] P. N. Pandey, Certain types of affine motion in a Finsler manifold. I, *Colloq. math.*, **49** (1985),243-252.

[8] P. N. Pandey, On Lie-recurrent Finsler manifold, *Indian Journal of Mathematics.*, **24** (1982), 135-143.

[9] P. N. Pandey, On some Finsler spaces of scalar curvature, *Prog. of Maths.*, **18 (1)** (1984), 41-48.

[10] P. N. Pandey, A symmetric Finsler manifold with a concircular vector field, *Proc. Nat. Acad. Sci. (India)*, **54** (1984), 271-273.

[11] P. N. Pandey, Vaishali Pandey, On $\widetilde{K}$-curvature inheritance in a Finsler space, *Int. Acad. Phy. Sci.***13(4)** (2009), 395-400.

[12] P. N. Pandey, S. Saxena, On Projective N-curvature inheritance, *Acta. Math. Hungar.*, **136(1-2)** (2012), 30-38.

[13] Shivalika Saxena, P. N. Pandey, On Lie-recurrence in Finsler spaces, *Diff. Geo. Dyn. Sys.*, **13** (2011), 201-207.