ON Q-FUZZY IDEALS IN ORDERED SEMIGROUPS

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Abstract: In this paper we shows that in ordered groupoids the Q-fuzzy right (resp. Q-fuzzy left) ideals are Q-fuzzy quasi-ideals, in ordered semigroups the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals, and in regular ordered semigroups the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide and show that if S is an ordered semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

AMS Subject Classification: 06F05
Key Words: ordered semigroup, regular ordered semigroup, Q-fuzzy left (right) ideal, Q-fuzzy quasi-ideal, Q-fuzzy bi-ideals

1. Introduction

A fuzzy set theory was conceptualized by Professor L. A. Zadeh at the University of California in 1965, [14] as a generalization of abstract set theory. Zadeh’s initiation is virtually a complete paradigm shift that initially gained popularity in the Far East and its successful applications has gained further ground almost round the globe. Rosenfeld [11] used the ideal of fuzzy set to introduce the no-
tions of fuzzy subgroups. The ideal of fuzzy subsemigroup was also introduced by Kuroki [7], [9]. In [8], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Xie [12] introduced the idea of extensions of fuzzy ideals in semigroups. The concept of fuzzy generalized bi-ideals of an ordered semigroup is introduced by Xie and Tang [13] and characterized regular ordered semigroups by means of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. In [10], Majumder introduced the concept of $Q$-fuzzification of ideals of $\Gamma$-semigroups and some important properties have been investigated. In this paper we shows that in ordered groupoids the $Q$-fuzzy right (resp. $Q$-fuzzy left) ideals are $Q$-fuzzy quasi-ideals, in ordered semigroups the $Q$-fuzzy quasi-ideals are $Q$-fuzzy bi-ideals, and in regular ordered semigroups the $Q$-fuzzy quasi-ideals and the $Q$-fuzzy bi-ideals coincide and show that if $S$ is an ordered semigroup, then a $Q$-fuzzy subset $f$ is a $Q$-fuzzy quasi-ideal of $S$ if and only if there exist a $Q$-fuzzy right ideal $g$ and a $Q$-fuzzy left ideal $h$ of $S$ such that $f = g \cap h$.

2. Preliminaries

Throughout this paper, unless stated otherwise, $S$ stands for an ordered semigroup. A function $f$ from $S \times Q$ to the real closed interval $[0, 1]$ is called $Q$-fuzzy subset of $S$, where $Q$ is a non-empty set. The ordered semigroup $S$ itself is a $Q$-fuzzy subset of $S$, its characteristic function, also denoted by $S$, is defined as follows:

$$S : S \times Q \rightarrow [0, 1] \mid (x, q) \mapsto S(x, q) := 1,$$

for all $x \in S$ and $q \in Q$.

Let $f$ and $g$ be two $Q$-fuzzy subsets of $S$. Then the inclusion relation $f \subseteq g$ means that

$$f(x, q) \leq g(x, q),$$

for all $x \in S$ and $q \in Q$, $f \cap g$ and $f \cup g$ are defined by

$$(f \cap g)(x, q) := \min\{f(x, q), g(x, q)\},$$

$$(f \cup g)(x, q) := \max\{f(x, q), g(x, q)\},$$

for all $x \in S$ and $q \in Q$.

Let $S$ and $Q$ be an ordered semigroup and a non-empty set, respectively. For $x \in S$, we define $A_x := \{(y, z) \in S \times S \mid x \leq yz\}$. The product $f \circ g$ of $f$
and $g$ is defined by

$$(\forall x \in S, \forall q \in Q)(f \circ g)(x, q) := \begin{cases} \bigvee_{(y,z) \in A_x} \min\{f(y,q), g(z,q)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

We denote by $f_{A \times Q}$ the characteristic function of $A \times Q$, that is, the mapping of $S \times Q$ into $[0, 1]$ defined by

$$f_{A \times Q}(x, q) := \begin{cases} 1 & \text{if } x \in A \times Q, \\ 0 & \text{if } x \not\in A \times Q, \end{cases}$$

for all $(x, q) \in A \times Q$.

### 3. Main Results

In this section, we introduced the notion of $Q$-fuzzy right (resp. $Q$-fuzzy left) ideals, $Q$-fuzzy quasi-ideals, $Q$-fuzzy bi-ideals of ordered semigroups, and investigate related properties.

**Definition 3.1.** Let $S$ and $Q$ be an ordered groupoid and a non-empty set, respectively. A $Q$-fuzzy subset $f$ of $S$ is called a $Q$-fuzzy right (resp. $Q$-fuzzy left) ideal of $S$ if:

1. $x \leq y$ implies $f(x, q) \geq f(y, q)$, and
2. $f(xy, q) \geq f(x, q)$ (resp. $f(xy, q) \geq f(y, q)$),

for all $x, y \in S$ and for all $q \in Q$.

**Definition 3.2.** Let $S$ and $Q$ be an ordered groupoid and a non-empty set, respectively. A $Q$-fuzzy subset $f$ of $S$ is called a $Q$-fuzzy quasi-ideal of $S$ if:

1. $x \leq y \Rightarrow f(x, q) \geq f(y, q)$,
2. $(f \circ S) \cap (S \circ f) \subseteq f$,

for all $x, y \in S$ and for all $q \in Q$.

**Definition 3.3.** Let $S$ and $Q$ be an ordered semigroup and a non-empty set, respectively. A $Q$-fuzzy subsemigroup $f$ of $S$ is called a $Q$-fuzzy bi-ideal of $S$ if:

1. $x \leq y \Rightarrow f(x, q) \geq f(y, q)$,
2. $f(xy, q) \geq \min\{f(x, q), f(z, q)\}$,

for all $x, y, z \in S$ and for all $q \in Q$. 
**Theorem 3.4.** If $S$ is an ordered groupoid and $Q$ is a non-empty set, then the $Q$-fuzzy right (resp. left) ideals of $S$ are $Q$-fuzzy quasi-ideals of $S$.

**Proof.** Let $f$ be a $Q$-fuzzy right ideal of $S$ and $x \in S, q \in Q$. First of all,

$$(f \circ S) \cap (S \circ f)(x, q) = \min\{(f \circ S)(x, q), (S \circ f)(x, q)\}.$$  

If $A_x = \emptyset$, then we have $(f \circ S)(x, q) = 0 = (S \circ f)(x, q)$ and, since $f$ is a $Q$-fuzzy right ideal of $S$, we have $\min\{(f \circ S)(x, q), (S \circ f)(x, q)\} = 0 \leq f(x, q)$.

If $A_x \neq \emptyset$, then

$$(f \circ S)(x, q) = \bigvee_{(u,v) \in A_x} \{\min\{f(u, q), S(v, q)\}\}.$$  

On the other hand, if $(u, v) \in A_x$, then $x \leq uv$ and $f(x, q) \geq f(uv, q) \geq f(u, q) = \min\{f(u, q), S(v, q)\}$. Hence, we have

$$f(x, q) \geq \bigvee_{(u,v) \in A_x} \{\min\{f(u, q)\}\} \geq \min\{(f \circ S)(x, q), (S \circ f)(x, q)\} = (f \circ S) \cap (S \circ f)(x, q).$$

Therefore $f$ is a $Q$-fuzzy quasi-ideal of $S$. \hfill \Box

**Theorem 3.5.** If $S$ is an ordered semigroup and $Q$ is a non-empty set, then the $Q$-fuzzy quasi-ideals are $Q$-fuzzy bi-ideals of $S$.

**Proof.** Let $f$ be a $Q$-fuzzy quasi-ideal of $S$ and $x, y, z \in S, q \in Q$. Then we have

$$f(xyz, q) \geq ((f \circ S) \cap (S \circ f))(xyz, q) = \min\{(f \circ S)(xyz, q), (S \circ f)(xyz, q)\}.$$  

Since $(x, yz) \in A_{xyz}$, we have

$$(f \circ S)(xyz, q) = \bigvee_{(u,v) \in A_{xyz}} \{\min\{f(u, q), S(v, q)\}\} \geq \min\{f(x, q), S(yz, q)\} = f(x, q).$$

Since $(xy, z) \in A_{xyz}$, we have

$$(S \circ f)(xyz, q) = \bigvee_{(u,v) \in A_{xyz}} \{\min\{S(u, q), f(v, q)\}\}$$
\[ \geq \min \{ S(xy, q), f(z, q) \} = f(z, q). \]

Thus we have
\[ f(xyz, q) \geq \min \{ (f \circ S)(xyz, q), (S \circ f)(xyz, q) \} \geq \min \{ f(x, q), f(z, q) \}. \]

Hence \( f \) is a \( Q \)-fuzzy bi-ideal of \( S \).

An ordered semigroup \( S \) is called regular if for any \( a \in S \) there exists \( x \in S \) such that \( a \leq axa \).

**Theorem 3.6.** If \( S \) is a regular ordered semigroup and \( Q \) is a non-empty set, then the \( Q \)-fuzzy quasi-ideals and the \( Q \)-fuzzy bi-ideals coincide.

**Proof.** Let \( f \) be a \( Q \)-fuzzy bi-ideal of \( S \) and \( x \in S, q \in Q \). We will prove that
\[ ((f \circ S) \cap (S \circ f))(x, q) \leq f(x, q). \] (1)

First of all, we have
\[ ((f \circ S) \cap (S \circ f))(x, q) = \min \{ (f \circ S)(x, q), (S \circ f)(x, q) \}. \]

If \( A_x = \emptyset \), then as we have already seen in Theorem 3.4, condition (1) is satisfied.

If \( A_x \neq \emptyset \), then
\[ (f \circ S)(x, q) = \bigvee_{(z, w) \in A_x} \{ \min \{ f(z, q), S(w, q) \} \}, \] (2)
\[ (S \circ f)(x, q) = \bigvee_{(u, v) \in A_x} \{ \min \{ S(u, q), f(v, q) \} \}. \] (3)

Let \( (f \circ S)(x, q) \leq f(x, q) \). Then, we have
\[ f(x, q) \geq (f \circ S)(x, q) \geq \min \{ (f \circ S)(x, q), (S \circ f)(x, q) \} = ((f \circ S) \cap (S \circ f))(x, q), \]

and condition (1) is satisfied.

Let \( (f \circ S)(x, q) > f(x, q) \). Then, by (2), there exists \( (z, w) \in A_x \) such that
\[ \min \{ f(z, q), S(w, q) \} > f(x, q) \] (4)
(otherwise \( f(x, q) \leq (f \circ S)(x, q) \), which is impossible). Since \((z, w) \in A_x\), we have \(z, w \in S\) and \(x \leq zw\). Similarly, from \(\min \{f(z, q), S(w, q)\} = f(z, q)\), by (4), we obtain

\[
f(z, q) > f(x, q).
\]

(5)

We will prove that \((S \circ f)(x, q) \leq f(x, q)\), then

\[
\min \{(f \circ S)(x, q), (S \circ f)(x, q)\} \leq (S \circ f)(x, q) \leq f(x, q),
\]

so that \(((f \circ S) \cap (S \circ f))(x, q) \leq f(x, q)\), and condition (1) is satisfied.

By (3), it is enough to prove that

\[
\min \{S(u, q), f(v, q)\} \leq f(x, q), \forall (u, v) \in A_x.
\]

Let \((u, v) \in A_x\). Then \(x \leq uv\) for some \(u, v \in S\). Since \(S\) is regular, there exists \(s \in S\) such that \(x \leq xsx\). It follows that \(x \leq zwsuv\). Since \(f\) is a \(Q\)-fuzzy bi-ideal of \(S\), we have

\[
\min \{S(u, q), f(v, q)\} \leq f(x, q), \forall (u, v) \in A_x,
\]

and, we have

\[
f(x, q) \geq f(zwsuv, q) \geq \min \{f(z, q), f(v, q)\}.
\]

If \(\min \{f(z, q), f(v, q)\} = f(z, q)\), then \(f(z, q) \leq f(x, q)\) which is impossible by (5). Thus we have \(\min \{f(z, q), f(v, q)\} = f(v, q)\), then \(f(x, q) \geq f(v, q) = \min \{S(u, q), f(v, q)\}\).

\[
\square
\]

In the following, using the usual definitions of ideals mentioned above, we show that the \(Q\)-fuzzy quasi-ideals of an ordered semigroup are just intersections of \(Q\)-fuzzy right and \(Q\)-fuzzy left ideals.

**Lemma 3.7.** Let \(S\) and \(Q\) be an ordered semigroup and a non-empty set respectively. Let \(f\) be a \(Q\)-fuzzy subset of \(S\). Then we have the following:

1. \((S \circ f)(xy, q) \geq f(y, q)\) for all \(x, y \in S, q \in Q\),
2. \((S \circ f)(xy, q) \geq (S \circ f)(y, q)\) for all \(x, y \in S, q \in Q\).

**Proof.** (1) Let \(x, y \in S\) and \(q \in Q\). Since \((x, y) \in A_{xy}\), we have

\[
(S \circ f)(xy, q) = \bigvee_{(w, z) \in A_{xy}} \{\min \{S(w, q), f(z, q)\}\} \geq \min \{S(x, q), f(y, q)\} = f(y, q).
\]
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(2) Let \( x, y \in S \) and \( q \in Q \). If \( A_y = \emptyset \), then \( (S \circ f)(y, q) = 0 \). Since \((S \circ f)\) is a Q-fuzzy subset of \( S \), we have \((S \circ f)(x, q) \geq 0 = (S \circ f)(y, q)\). If \( A_x \neq \emptyset \), then

\[
(S \circ f)(y, q) = \bigvee_{(w, z) \in A_y} \{\min\{S(w, q), f(z, q)\}\}.
\]

On the other hand,

\[
(S \circ f)(xy, q) = \min\{S(w, q), f(z, q)\}, \forall (w, z) \in A_y.
\]

Indeed, let \((w, z) \in A_y\). Since \((x, y) \in A_{xy}\), we have

\[
(S \circ f)(xy, q) = \bigvee_{(s, t) \in A_{xy}} \{\min\{S(s, q), f(t, q)\}\}.
\]

Since \((w, z) \in A_y\), we have \( y \leq wz \), then \( xy \leq xwz \), and \((xw, z) \in A_{xy}\). Hence we have

\[
(S \circ f)(xy, q) = \min\{S(xw, q), f(z, q)\} = f(z, q) = \min\{S(w, q), f(z, q)\}.
\]

By (6), we have

\[
(S \circ f)(xy, q) = \bigvee_{(w, z) \in A_y} \{\min\{S(w, q), f(z, q)\}\} = (S \circ f)(y, q). \quad \square
\]

In a similar way we can prove the following lemmas:

**Lemma 3.8.** Let \( S \) and \( Q \) be an ordered semigroup and a non-empty set, respectively. Let \( f \) be a Q-fuzzy subset of \( S \). Then we have the following:

1. \((S \circ f)(xy, q) \geq f(x, q)\) for all \( x, y \in S, q \in Q \),
2. \((S \circ f)(xy, q) \geq (S \circ f)(x, q)\) for all \( x, y \in S, q \in Q \).

**Lemma 3.9.** Let \( S \) and \( Q \) be an ordered semigroup and a non-empty set, respectively. Let \( f \) be a Q-fuzzy subset of \( S \) and \( x \leq y \). Then we have \((S \circ f)(x, q) \geq (S \circ f)(y, q)\), for all \( q \in Q \).

**Proof.** Let \( x, y \in S \) and \( q \in Q \). Then, if \( A_y = \emptyset \), then \((S \circ f)(y, q) = 0\). Since \( S \circ f \) is a Q-fuzzy subset of \( S \), we have \((S \circ f)(x, q) \geq 0\), then \((S \circ f)(x, q) \geq (S \circ f)(y, q)\). If \( A_y \neq \emptyset \), then

\[
(S \circ f)(y, q) = \bigvee_{(w, z) \in A_y} \{\min\{S(w, q), f(z, q)\}\} = \bigvee_{(w, z) \in A_y} \{f(z, q)\}.
\]
On the other hand, 
\[(S \circ f)(x, q) \geq f(z, q), \forall (w, z) \in A_y. \quad (7)\]
Indeed, let \((w, z) \in A_y\). Since \(x \leq y \leq wz\), we have \((w, z) \in A_x\). Then
\[(S \circ f)(xy, q) = \bigvee_{(s, t) \in A_{xy}} \{\min\{S(s, q), f(t, q)\}\} \geq \min\{S(w, q), f(z, q)\} = f(z, q).\]
Thus, by \((7)\), we have
\[(S \circ f)(x, q) \geq \bigvee_{(w, z) \in A_y} \{f(z, q)\} = (S \circ f)(y, q).\]
The proof is completed. 

**Lemma 3.10.** Let \(S\) and \(Q\) be an ordered semigroup and a non-empty set, respectively. Let \(f\) be a \(Q\)-fuzzy subset of \(S\) and \(x \leq y\). Then we have \((f \circ S)(x, q) \geq (f \circ S)(y, q)\), for all \(q \in Q\).

**Lemma 3.11.** Let \(S\) and \(Q\) be an ordered semigroup and a non-empty set, respectively. Let \(f\) be a \(Q\)-fuzzy subset of \(S\) such that \(x \leq y\), we have \(f(x, q) \geq f(y, q)\) for all \(x, y \in S, q \in Q\). Then the \(Q\)-fuzzy subset \(f \cup (S \circ f)\) is a \(Q\)-fuzzy left ideal of \(S\).

**Proof.** Let \(x, y \in S\) and \(q \in Q\). By, Theorem 3.9, we have \((f \cup (S \circ f))(xy, q) \geq (f \cup (S \circ f))(y, q)\). Let now \(x \leq y\). Then \((f \cup (S \circ f))(x, q) \geq (f \cup (S \circ f))(y, q)\). Indeed: Since \(f\) is a \(Q\)-fuzzy subset of \(S\) and \(x \leq y\), by Lemma 3.7, we get \((S \circ f)(x, q) \geq (S \circ f)(y, q)\) and, by hypothesis, \(f(x, q) \geq f(y, q)\). Then
\[
(f \cup (S \circ f))(x, q) = \max\{f(x, q), (S \circ f)(x, q)\} \\
\geq \max\{f(y, q), (S \circ f)(y, q)\} \\
= (f \cup (S \circ f))(y, q). \quad \Box
\]

In a similar way we can prove the following:

**Lemma 3.12.** Let \(S\) and \(Q\) be an ordered semigroup and a non-empty set, respectively. Let \(f\) be a \(Q\)-fuzzy subset of \(S\) such that \(x \leq y\), we have \(f(x, q) \geq f(y, q)\) for all \(x, y \in S, q \in Q\). Then the \(Q\)-fuzzy subset \(f \cup (f \circ S)\) is a \(Q\)-fuzzy right ideal of \(S\).
Lemma 3.13. Let $S$ and $f, g, h$ be an ordered semigroup and $Q$-fuzzy subsets of $S$, respectively. Then

$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h).$$

Proof. Let $x \in S$ and $q \in Q$. Then we have

$$(f \cap (g \cup h))(x, q) = \min\{f(x, q), (g \cup h)(x, q)\}$$

$$= \min\{f(x, q), \max\{g(x, q), h(x, q)\}\}$$

$$= \max\{\min\{f(x, q), g(x, q)\}, \min\{f(x, q), h(x, q)\}\}$$

$$= \max\{min\{(f \cap g)(x, q), (f \cap h)(x, q)\}\}$$

$$= (f \cap g) \cup (f \cap h))(x, q).$$

Corollary 3.14. Let $S$ and $Q$ be an ordered semigroup. Then the set of all $Q$-fuzzy subsets of $S$ is a distributive lattice.

Theorem 3.15. Let $S$ and $Q$ be an ordered semigroup and a non-empty set, respectively. Then a $Q$-fuzzy subset $f$ of $S$ is a $Q$-fuzzy quasi-ideal of $S$ if and only if there exist a $Q$-fuzzy right ideal $g$ and a $Q$-fuzzy left ideal $h$ of $S$ such that $f = g \cap h$.

Proof. ($\Rightarrow$). By Lemma 3.11 and Lemma 3.12, $f \cup (S \circ f)$ is a $Q$-fuzzy left ideal and $f \cup (f \circ S)$ is a $Q$-fuzzy right ideal of $S$. Moreover, we have

$$f = (f \cup (S \circ f)) \cap (f \cup (f \circ S)).$$

In fact, by Corollary 3.14, we have

$$(f \cup (S \circ f)) \cap (f \cup (f \circ S)) = ((f \cup (S \circ f)) \cap f) \cup ((f \cup (S \circ f)) \cap (f \circ S))$$

$$= (f \cap f) \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f)$$

$$\cap (f \circ S))$$

$$= f \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f)$$

$$\cap (f \circ S)).$$

Since $f$ is a $Q$-fuzzy quasi-ideal of $S$, we have $(f \circ S) \cap (S \circ f) \subseteq f$. Besides, $(S \circ f) \cap f \subseteq f$ and $f \cap (f \circ S) \subseteq f$. Hence

$$(f \cup (S \circ f)) \cap (f \cup (f \circ S)) = f.$$

($\Leftarrow$). Let $x \in S$ and $q \in Q$. Then

$$(f \circ S) \cap (S \circ f))(x, q) \leq f(x, q)$$

(8)
In fact, \((f \circ S) \cap (S \circ f))(x, q) = \min\{(f \circ S)(x, q), (S \circ f)(x, q)\}\). If \(A_x = \emptyset\), then \((f \circ S)(x, q) = 0 = (S \circ f)(x, q)\). Thus, in this case condition (8) is satisfied. If \(A_x \neq \emptyset\), then

\[
(f \circ S)(x, q) = \bigvee_{(y,z) \in A_x} \{\min\{f(y, q), S(z, q)\}\} = \bigvee_{(y,z) \in A_x} \{f(y, q)\}. \tag{9}
\]

We have

\[
f(y, q) \leq h(x, q), \forall (y, z) \in A_x. \tag{10}
\]

Indeed, for \((y, z) \in A_x\), we have \(x \leq yz\) and \(h(x, q) \geq h(yz, q) \geq h(y, q)\) because \(h\) is a \(Q\)-fuzzy left ideal of \(S\). Thus, applying (10) to (9), we obtain

\[
(f \circ S)(x, q) = \bigvee_{(y,z) \in A_x} \{f(y, q)\} \leq h(x, q).
\]

In a similar way, we get \((S \circ f)(x, q) \leq g(x, q)\). Hence

\[
((f \circ S) \cap (S \circ f))(x, q) = \min\{(f \circ S)(x, q), (S \circ f)(x, q)\} \\
\leq \min\{h(x, q), g(xq)\} \\
= (h \cap g)(x, q) \\
= f(x, q),
\]

which completes the proof of (8).

\[\square\]

**Acknowledgments**

This research is supported by Thailand Research Fund under Grant: MRG5580042.

**References**


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