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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN COMPLEX VALUED METRIC SPACES

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Abstract: In this paper, we prove a common fixed point theorem for weakly compatible mappings in complex valued metric space. Also, we prove common fixed point theorems for weakly compatible mappings with E.A. property and CLR property.

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1. Introduction

In 2011, Azam et al. [3] introduced the notion of complex valued metric space

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which is a generalization of the classical metric space and established some fixed point results for mappings satisfying a rational inequality.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first coordinate is called $Re(z)$ and second coordinate is called $Im(z)$.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$, if one of the following holds

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied and we will write $z_1 \prec z_2$ if only (iv) is satisfied.

Remark 1.1. We obtained that the following statements hold:

1. If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \precsim bz$ for all $z \in \mathbb{C}$.
2. If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
3. If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies

(1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *complex valued metric* on X and (X, d) is called a *complex valued metric space.*

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = 2i|z_1 - z_2|$ for all $z_1, z_2 \in X$. Then (X, d) is a complex valued metric space.

Definition 1.4. Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X.

(1) If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec c$ for all $n \geq N$, then $\{x_n\}$ is said to be *convergent* to $x \in X$, and we denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$.

(2) If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec c$ for all $n \geq N$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be *Cauchy sequence.*

(3) If every Cauchy sequence in X is convergent, then (X, d) is said to be a *complete complex valued metric space.*

Lemma 1.5. *Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 1.6. *Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Jungck [4] and Vetro [7] introduced the concept of weakly compatible maps as follows.

Definition 1.7. Let f and g be two self-mappings of a metric space (X,d). Then a pair (f,g) is said to be *weakly compatible* if they commute at coincidence points.

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows.

Definition 1.8. Let f and g be two self-mappings of a metric space (X,d). Then a pair (f,g) is said to satisfy *E.A. property* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 2011, Sintunavarat and Kumam [5] introduced the notion of CLR property as follows.

Definition 1.9. Let f and g be two self-mappings of a metric space (X,d). Then a pair (f,g) is said to satisfy *CLR_f property* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some $x \in X$.

In the same way, we can introduce these notions in complex valued metric space (see [2] and [6]).

Example 1.10. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1,z_2) = 2i|z_1 - z_2|$ for all $z_1, z_2 \in X$. Then (X,d) is a complex valued metric space. Define S and $T : X \to X$ by $Sz = z + i$ and $Tz = 2z$ for all $z \in X$, respectively. Consider a sequence $\{z_n\} = \{i - \frac{1}{n}\}$ $(n \in \mathbb{N})$ in X. Then

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} (z_n + i) = 2i \quad \text{and} \quad \lim_{n\to\infty} Tz_n = \lim_{n\to\infty} 2z_n = 2i,$$

where $2i \in X$. Thus, S and T satisfy E.A. property. Also, we have

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = 2i = Si,$$

where $2i \in X$. Thus, S and T satisfy CLR_S property.

In 2013, Verma and Pathak [6] defined the 'max' function for partial order relation $\precsim$.

Definition 1.11. Define the 'max' function for the partial order relation $\precsim$ by

(1) $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \precsim z_2$.
(2) If $z_1 \precsim \max\{z_2, z_3\}$, then $z_1 \precsim z_2$ or $z_1 \precsim z_3$.
(3) $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

Using Definition 1.11, we have the following lemma.

Lemma 1.12. *Let $z_1, z_2, z_3, \cdots \in \mathbb{C}$ and the partial order relation $\precsim$ is defined on $\mathbb{C}$. Then following statements are easy to prove.*

(i) *If $z_1 \precsim \max\{z_2, z_3\}$, then $z_1 \precsim z_2$ if $z_3 \precsim z_2$;*
(ii) *If $z_1 \precsim \max\{z_2, z_3, z_4\}$, then $z_1 \precsim z_2$ if $\max\{z_3, z_4\} \precsim z_2$;*
(iii) *If $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$, then $z_1 \precsim z_2$ if $\max\{z_3, z_4, z_5\} \precsim z_2$, and so on.*

2. Fixed Point Theorems for Weakly Compatible Mappings

Now, we prove common fixed point theorems for weakly compatible mappings in complex valued metric spaces.

Theorem 2.1. *Let A, B, S and T be four self-mappings of a complex valued metric space (X, d) satisfying*

(C1) $$SX \subset BX \quad \text{and} \quad TX \subset AX,$$

(C2) $$d(Sx, Ty) \precsim \alpha m(x, y) + \beta M(x, y),$$

where

$$m(x, y) = d(By, Ty)\frac{1 + d(Ax, Sx)}{1 + d(Ax, By)}$$

and

$$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\}$$

for each $x, y \in X$, where α and β are non-negative real numbers with $\alpha + \beta < 1$,

(C3) *the pairs (A, S) and (B, T) are weakly compatible.*

Suppose that one of AX, BX, SX and TX is a complete subspace of X. Then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $SX \subset BX$ and $TX \subset AX$, define for each $n \geq 0$, the sequence $\{y_n\}$ in X by

$$y_{2n+1} = Sx_{2n} = Bx_{2n+1} \quad \text{and} \quad y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}.$$

Suppose that $y_{2n} = y_{2n+1}$ for some n. Then by (C2), we have $y_{2n+1} = y_{2n+2}$, and so, $y_m = y_{2n}$ for every $m > 2n$. Thus, the sequence $\{y_n\}$ is a Cauchy sequence. The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some n.

Assume that $y_n \neq y_{n+1}$ for all n. Putting $x = x_{2n}$ and $y = x_{2n-1}$ in (C2), we have

$$d(Sx_{2n}, Tx_{2n-1}) \precsim \alpha m(x_{2n}, x_{2n-1}) + \beta M(x_{2n}, x_{2n-1}),$$

where

$$\begin{aligned} m(x_{2n}, x_{2n-1}) &= d(Bx_{2n-1}, Tx_{2n-1}) \frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Ax_{2n}, Bx_{2n-1})} \\ &= d(y_{2n-1}, y_{2n}) \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})} \end{aligned}$$

and

$$\begin{aligned} &M(x_{2n}, x_{2n-1}) \\ &= \max\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n-1}, Tx_{2n-1})\} \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}. \end{aligned}$$

Now, if $M(x_{2n}, x_{2n-1}) = d(y_{2n}, y_{2n+1})$, it follows that

$$d(y_{2n-1}, y_{2n})(1 + d(y_{2n}, y_{2n+1})) \precsim d(y_{2n}, y_{2n+1})(1 + d(y_{2n}, y_{2n-1})).$$

Thus, we have

$$\begin{aligned} m(x_{2n}, x_{2n-1}) &= d(y_{2n-1}, y_{2n}) \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})} \\ &\precsim d(y_{2n}, y_{2n+1}). \end{aligned}$$

So, we obtain

$$d(y_{2n}, y_{2n+1}) \precsim \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n}, y_{2n+1}).$$

Thus, we have

$$|d(y_{2n}, y_{2n+1})| \leq (\alpha + \beta)|d(y_{2n}, y_{2n+1})|,$$

which is a contradiction to $\alpha + \beta < 1$. Consequently, we must have

$$M(x_{2n}, x_{2n-1}) = d(y_{2n-1}, y_{2n})$$

and

$$m(x_{2n}, x_{2n-1}) \leq d(y_{2n-1}, y_{2n}).$$

Thus, we have

$$d(y_{2n}, y_{2n+1}) \precsim (\alpha+\beta)d(y_{2n-1}, y_{2n}).$$

On putting $x = x_{2n-2}$ and $y = x_{2n-1}$ in (C2), we have

$$d(Sx_{2n-2}, Tx_{2n-1}) \precsim \alpha m(x_{2n-2}, x_{2n-1}) + \beta M(x_{2n-2}, x_{2n-1}),$$

where

$$\begin{aligned} m(x_{2n-2}, x_{2n-1}) &= d(Bx_{2n-1}, Tx_{2n-1})\frac{1+d(Ax_{2n-2}, Sx_{2n-2})}{1+d(Ax_{2n-2}, Bx_{2n-1})} \\ &= d(y_{2n-1}, y_{2n}) \end{aligned}$$

and

$$\begin{aligned} &M(x_{2n-2}, x_{2n-1}) \\ &= \max\{d(Ax_{2n-2}, Bx_{2n-1}), d(Ax_{2n-2}, Sx_{2n-2}), d(Bx_{2n-1}, Tx_{2n-1})\} \\ &= \max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n})\}. \end{aligned}$$

Now, if $M(x_{2n-2}, x_{2n-1}) = d(y_{2n-1}, y_{2n})$, it follows that

$$d(y_{2n-1}, y_{2n}) \precsim \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-1}, y_{2n}).$$

Thus, we have

$$|d(y_{2n-1}, y_{2n})| \le (\alpha+\beta)|d(y_{2n-1}, y_{2n}),$$

which is a contradiction to $\alpha+\beta < 1$. Then we have $M(x_{2n-2}, x_{2n-1}) = d(y_{2n-2}, y_{2n-1})$ and hence

$$d(y_{2n-1}, y_{2n}) \precsim \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-2}, y_{2n-1}).$$

Thus, we have

$$|d(y_{2n-1}, y_{2n})| \le \frac{\beta}{1-\alpha}|d(y_{2n-2}, y_{2n-1})|.$$

Define $k = \max\left\{\alpha+\beta, \frac{\beta}{1-\alpha}\right\}$. Consequently, it can be concluded that

$$\begin{aligned} d(y_n, y_{n+1}) &\precsim kd(y_{n-1}, y_n) \\ &\precsim k^2 d(y_{n-2}, y_{n-1}) \\ &\quad \cdots \\ &\precsim k^n d(y_0, y_1). \end{aligned}$$

Now, for all $m > n$, we have

$$\begin{aligned} d(y_m, y_n) &\precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\precsim k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + \cdots + k^{m-1} d(y_0, y_1) \end{aligned}$$

Therefore, we have

$$|d(y_m, y_n)| \leq \frac{k^n}{1-k} |d(y_0, y_1)|.$$

Hence, we obtain

$$\lim_{n\to\infty} |d(y_m, y_n)| = 0.$$

Hence, $\{y_n\}$ is a Cauchy sequence.

Now, suppose that AX is complete. Note that $\{y_{2n}\}$ is contained in AX and has a limit in AX, say u, thatis, $\lim_{n\to\infty} y_{2n} = u$. Since $u \in AX$, there exists $v \in X$ such that $Av = u$.

Now, we shall prove that $Sv = u$. Let $Sv \neq u$. Putting $x = v$ and $y = x_{2n-1}$ in (C2), we have

$$d(Sv, Tx_{2n-1}) \precsim \alpha m(v, x_{2n-1}) + \beta M(v, x_{2n-1}), \tag{2.1}$$

where

$$m(v, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \frac{1 + d(u, Sv)}{1 + d(u, y_{2n-1})}$$

and

$$M(v, x_{2n-1}) = \max\{d(u, y_{2n-1}), d(u, Sv), d(y_{2n-1}, y_{2n})\}.$$

As the sequence $\{y_{2n-1}\}$ is convergent to u, therefore

$$\lim_{n\to\infty} d(u, y_{2n-1}) = \lim_{n\to\infty} d(y_{2n}, y_{2n-1}) = 0.$$

Thus, letting $n \to \infty$ in (2.1), we have

$$d(Sv, u) \precsim \beta d(Sv, u),$$

that is,

$$|d(Sv, u)| \leq \beta |d(Sv, u)|,$$

which is a contradiction to $\beta < 1$. Hence $Sv = u = Av$. Now, since $SX \subset BX$, $Sv = u \in BX$. There exists $w \in X$ such that $Bw = u$. By using the same arguments as above, one can easily verify that $Tw = u = Bw$, that is, w is the coincidence point of the pair (B, T).

The same result holds if we assume that BX is complete.

Now, if TX is complete, then by (2.1), $u \in TX \subset AX$. Similarly, if SX is complete, then $u \in SX \subset BX$.

Now, since the pairs (A, S) and (B, T) are weakly compatible, so $u = Sv = Av = Tw = Bw$ and hence $Au = ASu = SAv = Su$ and $Bu = BTw = TBw = Tu$.

Now, we claim that $Tu = u$. Let $Tu \neq u$. From (C2), we have

$$d(u, Tu) = d(Sv, Tu) \precsim \alpha m(v, u) + \beta M(v, u), \tag{2.2}$$

where

$$m(v, u) = d(Bu, Tu)\frac{1 + d(Av, Sv)}{1 + d(Av, Bu)} = 0$$

and

$$M(v, u) = \max\{d(Av, Bu), d(Av, Sv), d(Bu, Tu)\} = d(u, Tu).$$

Thus, from (2.2), we have

$$d(u, Tu) \precsim \beta d(u, Tu),$$

that is,

$$|d(u, Tu)| \leq \beta |d(u, Tu)|,$$

which is a contradiction to $\beta < 1$. Therefore, $Tu = u$.

Similarly, one can prove that $Su = u$ and hence u is a common fixed point of A, B, S and T.

Finally, for the uniqueness, let z $(z \neq u)$ be another common fixed point of A, B, S and T. From (C2), we have

$$d(u, z) = d(Su, Tz) \precsim \alpha m(u, z) + \beta M(u, z), \tag{2.3}$$

where

$$m(u, z) = d(Bz, Tz)\frac{1 + d(Au, Su)}{1 + d(Au, Bz)} = 0$$

and

$$M(u, z) = \max\{d(Au, Bz), d(Au, Su), d(Bz, Tz)\} = d(u, z).$$

Thus, from (2.3), we have

$$|d(u, z)| \leq \beta |d(u, z)|,$$

which is a contradiction to $\beta < 1$. Hence $u = z$. Therefore A, B, S and T have a unique common fixed point. This completes the proof. $\square$

From Theorem 2.1, if $A = B$ and $S = T$, we get the following.

Corollary 2.2. *Let A and S be two self-mappings of a complex valued metric space (X, d) satisfying*

(C4) $$SX \subset AX,$$

(C5) $$d(Sx, Sy) \precsim \alpha m(x, y) + \beta M(x, y),$$

where

$$m(x, y) = d(Ay, Sy)\frac{1 + d(Ax, Sx)}{1 + d(Ax, Ay)}$$

and

$$M(x, y) = \max\{d(Ax, Ay), d(Ax, Sx), d(Ay, Sy)\}$$

for each x, y in X,where α and β are non-negative real numbers with $\alpha + \beta < 1$,

(C6) *the pair (A, S) is weakly compatible.*

Suppose that one of AX or SX is complete subspace of X. Then A and S have a unique common fixed point.

3. Fixed Point Theorems for Weakly Compatible Mappings with E.A. Property

Now, we shall prove common fixed point theorems for weakly compatible mappings with E.A. property.

Theorem 3.1. *Let A, B, S and T be four self-mappings of a complex valued metric space (X, d) satisfying* (C1)-(C3) *and*

(C7) *the pairs (A, S) or (B, T) satisfy the E.A. property.*

Suppose that any one of AX, BX, SX and TX is a closed subspace of X. Then A, B, S and T have a unique common fixed point.

Proof. Suppose that (A, S) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $Ax_n = Sx_n = z$ for some $z \in X$. Since $SX \subset BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = z$.

We shall show that $\lim_{n\to\infty} Ty_n = z$. Let $\lim_{n\to\infty} Ty_n = t \neq z$. From (C2), we have

$$d(Sx_n, Ty_n) \precsim \alpha m(x_n, y_n) + \beta M(x_n, y_n).$$

Letting $n \to \infty$, we have

$$d(z,t) \precsim \alpha \lim_{n\to\infty} m(x_n, y_n) + \beta \lim_{n\to\infty} M(x_n, y_n), \tag{3.1}$$

where

$$\lim_{n\to\infty} m(x_n, y_n) = \lim_{n\to\infty} d(By_n, Ty_n)\frac{1 + d(x_n, Sx_n)}{1 + d(Ax_n, By_n)} = d(z,t)$$

and

$$\begin{aligned}\lim_{n\to\infty} M(x_n, y_n) &= \lim_{n\to\infty} \max\{d(Ax_n, By_n), d(Ax_n, Sx_n), d(By_n, Ty_n)\} \\ &= d(z,t).\end{aligned}$$

Thus, from (3.1), we have

$$d(z,t) \precsim \alpha d(z,t) + \beta d(z,t),$$

that is,

$$|d(z,t)| \le (\alpha + \beta)|d(z,t)|,$$

which is a contradiction to $\alpha+\beta < 1$. Therefore, $t = z$, that is, $\lim_{n\to\infty} Ty_n = z$. Suppose that BX is a closed space of X. Then there exists $u \in X$ such that $z = Bu$. Subsequently, we have

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z = Bu.$$

Now, we shall show that $Tu = Bu$. Let $Tu \neq Bu$. From (C2), we have

$$d(Sx_n, Tu) \precsim \alpha m(x_n, u) + \beta M(x_n, u).$$

Letting $n \to \infty$, we have

$$d(z, Tu) \precsim \alpha \lim_{n\to\infty} m(x_n, u) + \beta \lim_{n\to\infty} M(x_n, u), \tag{3.2}$$

where

$$\lim_{n\to\infty} m(x_n, u) = \lim_{n\to\infty} d(Bu, Tu)\frac{1 + d(Ax_n, Sx_n)}{1 + d(Ax_n, Bu)} = d(z, Tu)$$

and

$$\begin{aligned}\lim_{n\to\infty} M(x_n, u) &= \lim_{n\to\infty} \max\{d(Ax_n, Bu), d(Ax_n, Sx_n), d(Bu, Tu)\} \\ &= \max\{0, 0, d(Bu, Tu)\} = d(z, Tu).\end{aligned}$$

Thus, from (3.2), we have

$$d(z, Tu) \precsim \alpha d(z, Tu) + \beta d(z, Tu),$$

that is,

$$|d(z, Tu)| \leq (\alpha + \beta)|d(z, Tu)|,$$

which is a contradiction to $\alpha + \beta < 1$. Therefore, $Tu = z = Bu$. Since B and T are weakly compatible, we have $BTu = TBu$ and hence $TTu = TBu = BTu = BBu$. Since $TX \subset AX$, there exists $v \in X$ such that $Tu = Av$.

Now, we claim that $Av = Sv$. Let $Av \neq Sv$. From (C2), we have

$$d(Sv, Tu) \precsim \alpha m(v, u) + \beta M(v, u), \tag{3.3}$$

where

$$m(v, u) = d(Bu, Tu)\frac{1 + d(Av, Sv)}{1 + d(Av, Bu)} = 0$$

and

$$\begin{aligned} M(v, u) &= \max\{d(Av, Bu), d(Av, Sv), d(Bu, Tu)\} \\ &= \max\{0, d(Av, Sv), 0)\} = d(Tu, Sv). \end{aligned}$$

Thus, from (3.3), we have

$$d(Sv, Tu) \precsim \beta d(Sv, Tu),$$

that is,

$$|d(Sv, Tu)| \leq \beta |d(Sv, Tu)|,$$

which is a contradiction to $\beta < 1$. Therefore, $Sv = Tu = Av$. Thus, we have $Tu = Bu = Sv = Av$. The weak compatibility of A and S implies that $ASv = SAv = SSv = AAv$.

Now, we claim that Tu is the common fixed point of A, B, S and T. Suppose that $TTu \neq Tu$. From (C2), we have

$$d(Tu, TTu) = d(Sv, TTu) \precsim \alpha m(v, Tu) + \beta M(v, Tu), \tag{3.4}$$

where

$$m(v, Tu) = d(BTu, TTu)\frac{1 + d(Av, Sv)}{1 + d(Av, BTu)} = 0$$

and

$$\begin{aligned} M(v, Tu) &= \max\{d(Av, BTu), d(Av, Sv), d(BTu, TTu)\} \\ &= d(Av, BTu) = d(Tu, TTu). \end{aligned}$$

Thus, from (3.4), we have

$$|d(Tu, TTu)| \leq \beta|d(Tu, TTu)|,$$

which is a contradiction to $\beta < 1$. Therefore, $Tu = TTu = BTu$. Hence Tu is the common fixed point of B and T. Similarly, we prove that Sv is the common fixed point of A and S. Since $Tu = Sv$, Tu is the common fixed point of A, B, S and T.

The proof is similar when AX is assumed to be a closed subspace of X.

The cases in which TX or SX is a complete subspace of X are similar to the cases in whichAX or BX, respectively is complete subspace of X since $TX \subset AX$ and $SX \subset BX$.

Finally, for uniqueness, let p and q $(p \neq q)$ be two common fixed points of A, B, S and T. From (C2), we have

$$d(p, q) = d(Sp, Tq) \precsim \alpha m(p, q) + \beta M(p, q), \tag{3.5}$$

where

$$m(p, q) = d(Bq, Tq)\frac{1 + d(Ap, Sp)}{1 + d(Ap, Bq)} = 0$$

and

$$M(p, q) = \max\{d(Ap, Bq), d(Ap, Sp), d(Bq, Tq)\} = d(p, q).$$

Thus, from (3.5), we have

$$|d(p, q)| \leq \beta|d(p, q)|,$$

which is a contradiction to $\beta < 1$. Therefore, $p = q$. Hence A, B, S and T have a unique common fixed point. This completes the proof. □

From Theorem 3.1, if $A = B$ and $S = T$ in Theorem 2.1, we get the following.

Corollary 3.2. *Let A and S be two self-mappings of a complex valued metric space (X, d) satisfying* (C4)-(C6).

(C8) *the pair (A, S) satisfies the E.A. property.*

If one of AX or SX is closed subspace of X, then A and S have a unique common fixed point.

4. Fixed Point Theorems for Weakly Compatible Mappingss with CLR Property

Now, we prove common fixed point theorems for weakly compatible mappings with CLR property.

Theorem 4.1. *Let A, B, S and T be four self-mappings of a complex valued metric space (X, d) satisfying* (C2), (C3) *and*

(C9) *$SX \subset BX$ and the pair (A, S) satisfies CLR$_A$ property or $TX \subset AX$ and the pair (B, T) satisfies CLR$_B$ property.*
Then A, B, S and T have a unique common fixed point.

Proof. Without loss of generality, assume that $SX \subset BX$ and the pair (A, S) satisfies CLR$_A$ property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Ax$ for some $x \in X$. Since $SX \subset BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = Ax$.

We shall show that $\lim_{n\to\infty} Ty_n = Ax$. Let $\lim_{n\to\infty} Ty_n = z \neq Ax$. From (C2), we have

$$d(Sx_n, Ty_n) \precsim \alpha m(x_n, y_n) + \beta M(x_n, y_n).$$

Letting $n \to \infty$, we have

$$d(Ax, z) \precsim \alpha \lim_{n\to\infty} m(x_n, y_n) + \beta \lim_{n\to\infty} M(x_n, y_n), \tag{4.1}$$

where

$$\lim_{n\to\infty} m(x_n, y_n) = \lim_{n\to\infty} d(By_n, Ty_n) \frac{1 + d(Ax_n, Sx_n)}{1 + d(Ax_n, By_n)} = d(Ax, z)$$

and

$$\begin{aligned}\lim_{n\to\infty} M(x_n, y_n) &= \lim_{n\to\infty} \max\{d(Ax_n, By_n), d(Ax_n, Sx_n), d(By_n, Ty_n)\} \\ &= d(Ax, z).\end{aligned}$$

Thus, from (4.1), we have

$$d(Ax, z) \precsim \alpha d(Ax, z) + \beta d(Ax, z),$$

that is,

$$|d(Ax, z)| \leq (\alpha + \beta)|d(Ax, z)|,$$

which is a contradiction to $\alpha+\beta < 1$. Therefore, $Ax = z$ and hence $\lim_{n\to\infty} Ty_n = Ax$. Subsequently, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Ax = z.$$

Now, we shall show that $Sx = z$. Let $Sx \neq z$. From (C2), we have

$$d(Sx, Ty_n) \precsim \alpha m(x, y_n) + \beta M(x, y_n).$$

Letting $n \to \infty$, we have

$$d(Sx, z) \precsim \alpha \lim_{n\to\infty} m(x, y_n) + \beta \lim_{n\to\infty} M(x, y_n), \tag{4.2}$$

where

$$\lim_{n\to\infty} m(x, y_n) = \lim_{n\to\infty} d(By_n, Ty_n)\frac{1 + d(Ax, Sx)}{1 + d(Ax, By_n)} = 0$$

and

$$\begin{aligned}\lim_{n\to\infty} M(x, y_n) &= \lim_{n\to\infty} \max\{d(Ax, By_n), d(Ax, Sx), d(By_n, Ty_n)\}\\ &= d(z, Sx).\end{aligned}$$

Thus, from (4.2), we have

$$d(Sx, z) \precsim \beta d(Sx, z),$$

that is,

$$|d(Sx, z)| \leq \beta |d(Sx, z)|,$$

which is a contradiction to $\beta < 1$. Therefore, $Sx = z = Ax$. Since the pair (A, S) is weakly compatible, it follows that $Az = Sz$. Also, since $SX \subset BX$, there exists $y \in X$ such that $z = Sx = By$.

Now, we show that $Ty = z$. Let $Ty \neq z$. From (C2), we have

$$d(Sx_n, Ty) \precsim \alpha m(x_n, y) + \beta M(x_n, y).$$

Letting $n \to \infty$, we have

$$d(Ax, z) \precsim \alpha \lim_{n\to\infty} m(x_n, y_n) + \beta \lim_{n\to\infty} M(x_n, y_n), \tag{4.3}$$

where

$$\lim_{n\to\infty} m(x_n, y) = \lim_{n\to\infty} d(By, Ty)\frac{1 + d(Ax_n, Sx_n)}{1 + d(Ax_n, By)} = d(z, Ty)$$

and

$$\begin{aligned}\lim_{n\to\infty} M(x_n, y) &= \lim_{n\to\infty} \max\{d(Ax_n, By), d(Ax_n, Sx_n), d(By, Ty)\}\\ &= d(z, Ty).\end{aligned}$$

Thus, from (4.3), we have

$$d(z, Ty) \precsim \alpha d(z, Ty) + \beta d(z, Ty),$$

that is,

$$|d(z, Ty)| \leq (\alpha + \beta)|d(z, Ty)|,$$

which is a contradiction to $\alpha+\beta < 1$. Thus, $z = Ty = By$. Since the pair (B, T) is weakly compatible, it follows that $Tz = Bz$. Now, we claim that $Sz = Tz$. Let $Sz \neq Tz$. From (C2), we have

$$d(Sz, Tz) \precsim \alpha m(z, z) + \beta M(z, z), \tag{4.4}$$

where

$$m(z, z) = d(Bz, Tz)\frac{1 + d(Az, Sz)}{1 + d(Az, Bz)} = 0$$

and

$$M(z, z) = \max\{d(Az, Bz), d(Az, Sz), d(Bz, Tz)\} = d(Sz, Tz).$$

Thus, from (4.4), we have

$$d(Sz, Tz) \precsim \beta d(Sz, Tz),$$

that is,

$$|d(Sz, Tz)| \leq \beta|d(Sz, Tz)|,$$

which is a contradiction to $\beta < 1$. Therefore, $Sz = Tz$, that is, $Az = Sz = Tz = Bz$.

Now, we shall show that $z = Tz$. Let $z \neq Tz$. From (C2), we have

$$d(z, Tz) = d(Sx, Tz) \precsim \alpha m(x, z) + \beta M(x, z), \tag{4.5}$$

where

$$m(x, z) = d(Bz, Tz)\frac{1 + d(Ax, Sx)}{1 + d(Ax, Bz)} = 0$$

and

$$M(x, z) = \max\{d(Ax, Bz), d(Ax, Sx), d(Bz, Tz)\} = d(z, Tz).$$

Thus, from (4.5), we have

$$|d(z, Tz)| \leq \beta|d(z, Tz)|,$$

which is a contradiction to $\beta < 1$. Therefore, $z = Tz = Bz = Az = Sz$. Hence z is the common fixed point of A, B, S and T.

Finally, for uniqueness, let u $(u \neq z)$ be another common fixed point of A, B, S and T.

$$d(u,z) = d(Su,Tz) \precsim \alpha m(u,z) + \beta M(u,z), \tag{4.6}$$

where

$$m(u,z) = d(Bz,Tz)\frac{1 + d(Au,Su)}{1 + d(Au,Bz)} = 0$$

and

$$M(u,z) = \max\{d(Au,Bz), d(Au,Su), d(Bz,Tz)\} = d(u,z).$$

Thus, from (4.6), we have

$$|d(u,z)| \leq \beta |d(u,z)|,$$

which is a contradiction to $\beta < 1$. Therefore, $u = z$. Hence A, B, S and T have a unique common fixed point. This completes the proof. □

From Theorem 4.1, if $A = B$ and $S = T$, we get the following.

Corollary 4.2. *Let A and S be two self-mappings of a complex valued metric space (X, d) satisfying* (C4)-(C6).

(C10) *the pair (A, S) satisfies CLR_A property.*

Then A and S have a unique common fixed point.

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