COMMON FIXED POINT THEOREMS FOR
WEAKLY COMPATIBLE MAPPINGS IN
COMPLEX VALUED METRIC SPACES

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Abstract: In this paper, we prove a common fixed point theorem for weakly compatible mappings in complex valued metric space. Also, we prove common fixed point theorems for weakly compatible mappings with E.A. property and CLR property.

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1. Introduction

In 2011, Azam et al. [3] introduced the notion of complex valued metric space...
which is a generalization of the classical metric space and established some fixed point results for mappings satisfying a rational inequality.

A complex number \( z \in \mathbb{C} \) is an ordered pair of real numbers, whose first coordinate is called \( \text{Re}(z) \) and second coordinate is called \( \text{Im}(z) \).

Let \( \mathbb{C} \) be the set of complex numbers and \( z_1, z_2 \in \mathbb{C} \). Define a partial order \( \preceq \) on \( \mathbb{C} \) as follows:

\[
z_1 \preceq z_2 \quad \text{if and only if} \quad \text{Re}(z_1) \leq \text{Re}(z_2) \text{ and } \text{Im}(z_1) \leq \text{Im}(z_2).
\]

It follows that \( z_1 \preceq z_2 \), if one of the following holds

(i) \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \);
(ii) \( \text{Re}(z_1) < \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \);
(iii) \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) < \text{Im}(z_2) \);
(iv) \( \text{Re}(z_1) < \text{Re}(z_2) \) and \( \text{Im}(z_1) < \text{Im}(z_2) \).

In particular, we will write \( z_1 \succ z_2 \) if \( z_1 \neq z_2 \) and one of (ii), (iii) and (iv) is satisfied and we will write \( z_1 \prec z_2 \) if only (iv) is satisfied.

**Remark 1.1.** We obtained that the following statements hold:
1. If \( a, b \in \mathbb{R} \) with \( a \leq b \), then \( az \preceq bz \) for all \( z \in \mathbb{C} \).
2. If \( 0 \preceq z_1 \preceq z_2 \), then \( |z_1| < |z_2| \).
3. If \( z_1 \preceq z_2 \) and \( z_2 \prec z_3 \), then \( z_1 \prec z_3 \).

**Definition 1.2.** Let \( X \) be a nonempty set. Suppose that the mapping \( d : X \times X \to \mathbb{C} \) satisfies

(1) \( 0 \preceq d(x, y) \) for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \);
(2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(3) \( d(x, y) \preceq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a *complex valued metric* on \( X \) and \( (X, d) \) is called a *complex valued metric space*.

**Example 1.3.** Let \( X = \mathbb{C} \). Define the mapping \( d : X \times X \to \mathbb{C} \) by

\[
d(z_1, z_2) = 2i|z_1 - z_2|
\]

for all \( z_1, z_2 \in X \). Then \( (X, d) \) is a complex valued metric space.

**Definition 1.4.** Let \( (X, d) \) be a complex valued metric space and \( \{x_n\} \) be a sequence in \( X \).

(1) If for every \( c \in \mathbb{C} \) with \( 0 \prec c \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x) \prec c \) for all \( n \geq N \), then \( \{x_n\} \) is said to be *convergent* to \( x \in X \), and we denote this by \( x_n \to x \) as \( n \to \infty \) or \( \lim_{n \to \infty} x_n = x \).

(2) If for every \( c \in \mathbb{C} \) with \( 0 \prec c \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_{n+m}) \prec c \) for all \( n \geq N \), where \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be *Cauchy sequence*.

(3) If every Cauchy sequence in \( X \) is convergent, then \( (X, d) \) is said to be a *complete complex valued metric space*. 
Lemma 1.5. Let \((X, d)\) be a complex valued metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(|d(x_n, x)| \to 0\) as \(n \to \infty\).

Lemma 1.6. Let \((X, d)\) be a complex valued metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if \(|d(x_n, x_{n+m})| \to 0\) as \(n \to \infty\), where \(m \in \mathbb{N}\).


Definition 1.7. Let \(f\) and \(g\) be two self-mappings of a metric space \((X, d)\). Then a pair \((f, g)\) is said to be weakly compatible if they commute at coincidence points.

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows.

Definition 1.8. Let \(f\) and \(g\) be two self-mappings of a metric space \((X, d)\). Then a pair \((f, g)\) is said to satisfy E.A. property if there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t\) for some \(t \in X\).

In 2011, Sintunavarat and Kumam [5] introduced the notion of CLR property as follows.

Definition 1.9. Let \(f\) and \(g\) be two self-mappings of a metric space \((X, d)\). Then a pair \((f, g)\) is said to satisfy CLR\(_f\) property if there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = fx\) for some \(x \in X\).

In the same way, we can introduce these notions in complex valued metric space (see [2] and [6]).

Example 1.10. Let \(X = \mathbb{C}\). Define the mapping \(d : X \times X \to \mathbb{C}\) by \(d(z_1, z_2) = 2i|z_1 - z_2|\) for all \(z_1, z_2 \in X\). Then \((X, d)\) is a complex valued metric space. Define \(S\) and \(T : X \to X\) by \(Sz = z + i\) and \(Tz = 2z\) for all \(z \in X\), respectively. Consider a sequence \(\{z_n\}\) = \(\{i - \frac{1}{n}\}\) \((n \in \mathbb{N})\) in \(X\). Then
\[
\lim_{n \to \infty} Sz_n = \lim_{n \to \infty} (z_n + i) = 2i \quad \text{and} \quad \lim_{n \to \infty} Tz_n = \lim_{n \to \infty} 2z_n = 2i,
\]
where \(2i \in X\). Thus, \(S\) and \(T\) satisfy E.A. property. Also, we have
\[
\lim_{n \to \infty} Sz_n = \lim_{n \to \infty} Tz_n = 2i = Si,
\]
where \(2i \in X\). Thus, \(S\) and \(T\) satisfy CLR\(_S\) property.

In 2013, Verma and Pathak [6] defined the ‘max’ function for partial order relation \(\preceq\).
**Definition 1.11.** Define the ‘max’ function for the partial order relation \( \preceq \) by

1. \( \max\{z_1, z_2\} = z_2 \) if and only if \( z_1 \preceq z_2 \).
2. If \( z_1 \preceq \max\{z_2, z_3\} \), then \( z_1 \preceq z_2 \) or \( z_1 \preceq z_3 \).
3. \( \max\{z_1, z_2\} = z_2 \) if and only if \( z_1 \preceq z_2 \) or \( |z_1| \leq |z_2| \).

Using Definition 1.11, we have the following lemma.

**Lemma 1.12.** Let \( z_1, z_2, z_3, \cdots \in \mathbb{C} \) and the partial order relation \( \preceq \) is defined on \( \mathbb{C} \). Then following statements are easy to prove.

(i) If \( z_1 \preceq \max\{z_2, z_3\} \), then \( z_1 \preceq z_2 \) if \( z_3 \preceq z_2 \);

(ii) If \( z_1 \preceq \max\{z_2, z_3, z_4\} \), then \( z_1 \preceq z_2 \) if \( \max\{z_3, z_4\} \preceq z_2 \);

(iii) If \( z_1 \preceq \max\{z_2, z_3, z_4, z_5\} \), then \( z_1 \preceq z_2 \) if \( \max\{z_3, z_4, z_5\} \preceq z_2 \), and so on.

2. **Fixed Point Theorems for Weakly Compatible Mappings**

Now, we prove common fixed point theorems for weakly compatible mappings in complex valued metric spaces.

**Theorem 2.1.** Let \( A, B, S \) and \( T \) be four self-mappings of a complex valued metric space \((X, d)\) satisfying

(C1) \( SX \subset BX \) and \( TX \subset AX \),

(C2) \( d(Sx, Ty) \preceq \alpha m(x, y) + \beta M(x, y) \),

where

\[
m(x, y) = d(By, Ty) \frac{1 + d(Ax, Sx)}{1 + d(Ax, By)}
\]

and

\[
M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\}
\]

for each \( x, y \in X \), where \( \alpha \) and \( \beta \) are non-negative real numbers with \( \alpha + \beta < 1 \),

(C3) the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

Suppose that one of \( AX \), \( BX \), \( SX \) and \( TX \) is a complete subspace of \( X \). Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \). Since \( SX \subset BX \) and \( TX \subset AX \), define for each \( n \geq 0 \), the sequence \( \{y_n\} \) in \( X \) by

\[
y_{2n+1} = Sx_{2n} = Bx_{2n+1} \quad \text{and} \quad y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}.
\]
Suppose that $y_{2n} = y_{2n+1}$ for some $n$. Then by (C2), we have $y_{2n+1} = y_{2n+2}$, and so, $y_m = y_{2n}$ for every $m > 2n$. Thus, the sequence $\{y_n\}$ is a Cauchy sequence. The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some $n$.

Assume that $y_n \neq y_{n+1}$ for all $n$. Putting $x = x_{2n}$ and $y = x_{2n-1}$ in (C2), we have

$$d(Sx_{2n}, Tx_{2n-1}) \prec \alpha m(x_{2n}, x_{2n-1}) + \beta M(x_{2n}, x_{2n-1}),$$

where

$$m(x_{2n}, x_{2n-1}) = d(Bx_{2n-1}, Tx_{2n-1}) \frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Ax_{2n}, Bx_{2n-1})} = d(y_{2n-1}, y_{2n}) \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})}$$

and

$$M(x_{2n}, x_{2n-1}) = \max\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n-1}, Tx_{2n-1})\} = \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}.$$ 

Now, if $M(x_{2n}, x_{2n-1}) = d(y_{2n}, y_{2n+1})$, it follows that

$$d(y_{2n-1}, y_{2n})(1 + d(y_{2n}, y_{2n+1})) \prec d(y_{2n}, y_{2n+1})(1 + d(y_{2n}, y_{2n-1})).$$

Thus, we have

$$m(x_{2n}, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})} \prec d(y_{2n}, y_{2n+1}).$$

So, we obtain

$$d(y_{2n}, y_{2n+1}) \prec \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n}, y_{2n+1}).$$

Thus, we have

$$|d(y_{2n}, y_{2n+1})| \leq (\alpha + \beta)|d(y_{2n}, y_{2n+1})|,$$

which is a contradiction to $\alpha + \beta < 1$. Consequently, we must have

$$M(x_{2n}, x_{2n-1}) = d(y_{2n-1}, y_{2n})$$

and

$$m(x_{2n}, x_{2n-1}) \leq d(y_{2n-1}, y_{2n}).$$
Thus, we have

\[ d(y_{2n}, y_{2n+1}) \preceq (\alpha + \beta) d(y_{2n-1}, y_{2n}). \]

On putting \( x = x_{2n-2} \) and \( y = x_{2n-1} \) in (C2), we have

\[ d(Sx_{2n-2}, Tx_{2n-1}) \preceq \alpha m(x_{2n-2}, x_{2n-1}) + \beta M(x_{2n-2}, x_{2n-1}), \]

where

\[ m(x_{2n-2}, x_{2n-1}) = d(Bx_{2n-1}, Tx_{2n-1}) \frac{1 + d(Ax_{2n-2}, Sx_{2n-2})}{1 + d(Ax_{2n-2}, Bx_{2n-1})} = d(y_{2n-1}, y_{2n}) \]

and

\[ M(x_{2n-2}, x_{2n-1}) \]

\[ = \max\{d(Ax_{2n-2}, Bx_{2n-1}), d(Ax_{2n-2}, Sx_{2n-2}), d(Bx_{2n-1}, Tx_{2n-1})\} \]

\[ = \max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n})\}. \]

Now, if \( M(x_{2n-2}, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \), it follows that

\[ d(y_{2n-1}, y_{2n}) \preceq \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-1}, y_{2n}). \]

Thus, we have

\[ |d(y_{2n-1}, y_{2n})| \leq (\alpha + \beta)|d(y_{2n-1}, y_{2n}), \]

which is a contradiction to \( \alpha + \beta < 1 \). Then we have \( M(x_{2n-2}, x_{2n-1}) = d(y_{2n-2}, y_{2n-1}) \) and hence

\[ d(y_{2n-1}, y_{2n}) \preceq \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-2}, y_{2n-1}). \]

Thus, we have

\[ |d(y_{2n-1}, y_{2n})| \leq \frac{\beta}{1 - \alpha} |d(y_{2n-2}, y_{2n-1})|. \]

Define \( k = \max\{\alpha + \beta, \frac{\beta}{1 - \alpha}\} \). Consequently, it can be concluded that

\[ d(y_n, y_{n+1}) \preceq kd(y_{n-1}, y_n) \]

\[ \preceq k^2 d(y_{n-2}, y_{n-1}) \]

\[ \cdots \]

\[ \preceq k^n d(y_0, y_1). \]
Now, for all $m > n$, we have
\[
d(y_m, y_n) \preceq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\
\preceq k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + \cdots + k^{m-1} d(y_0, y_1)
\]
Therefore, we have
\[
|d(y_m, y_n)| \leq \frac{k^n}{1 - k} |d(y_0, y_1)|.
\]
Hence, we obtain
\[
\lim_{n \to \infty} |d(y_m, y_n)| = 0.
\]
Hence, $\{y_n\}$ is a Cauchy sequence.

Now, suppose that $AX$ is complete. Note that $\{y_{2n}\}$ is contained in $AX$ and has a limit in $AX$, say $u$, that is, $\lim_{n \to \infty} y_{2n} = u$. Since $u \in AX$, there exists $v \in X$ such that $Av = u$.

Now, we shall prove that $Sv = u$. Let $Sv \neq u$. Putting $x = v$ and $y = x_{2n-1}$ in (C2), we have
\[
d(Sv, Tx_{2n-1}) \preceq \alpha m(v, x_{2n-1}) + \beta M(v, x_{2n-1}), \quad (2.1)
\]
where
\[
m(v, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \frac{1 + d(u, Sv)}{1 + d(u, y_{2n-1})}
\]
and
\[
M(v, x_{2n-1}) = \max\{d(u, y_{2n-1}), d(u, Sv), d(y_{2n-1}, y_{2n})\}.
\]
As the sequence $\{y_{2n-1}\}$ is convergent to $u$, therefore
\[
\lim_{n \to \infty} d(u, y_{2n-1}) = \lim_{n \to \infty} d(y_{2n}, y_{2n-1}) = 0.
\]
Thus, letting $n \to \infty$ in (2.1), we have
\[
d(Sv, u) \preceq \beta d(Sv, u),
\]
that is,
\[
|d(Sv, u)| \leq \beta |d(Sv, u)|,
\]
which is a contradiction to $\beta < 1$. Hence $Sv = u = Av$. Now, since $SX \subset BX$, $Sv = u \in BX$. There exists $w \in X$ such that $Bw = u$. By using the same arguments as above, one can easily verify that $Tw = u = Bw$, that is, $w$ is the coincidence point of the pair $(B, T)$.

The same result holds if we assume that $BX$ is complete.
Now, if $TX$ is complete, then by (2.1), $u \in TX \subset AX$. Similarly, if $SX$ is complete, then $u \in SX \subset BX$.

Now, since the pairs $(A, S)$ and $(B, T)$ are weakly compatible, so $u = Sv = Av = Tw = Bw$ and hence $Au = ASu = SAv = Su$ and $Bu = BTw = TBw = Tu$.

Now, we claim that $Tu = u$. Let $Tu \neq u$. From (C2), we have

$$d(u, Tu) = d(Sv, Tu) \preceq \alpha m(v, u) + \beta M(v, u),$$

(2.2)

where

$$m(v, u) = d(Bu, Tu)^{1 + d(Av, Sv)} + d(Av, Bu) = 0$$

and

$$M(v, u) = \max\{d(Av, Bu), d(Av, Sv), d(Bu, Tu)\} = d(u, Tu).$$

Thus, from (2.2), we have

$$d(u, Tu) \preceq \beta d(u, Tu),$$

that is,

$$|d(u, Tu)| \leq \beta |d(u, Tu)|,$$

which is a contradiction to $\beta < 1$. Therefore, $Tu = u$.

Similarly, one can prove that $Su = u$ and hence $u$ is a common fixed point of $A, B, S$ and $T$.

Finally, for the uniqueness, let $z (z \neq u)$ be another common fixed point of $A, B, S$ and $T$. From (C2), we have

$$d(u, z) = d(Su, Tz) \preceq \alpha m(u, z) + \beta M(u, z),$$

(2.3)

where

$$m(u, z) = d(Bz, Tz)^{1 + d(Au, Su)} + d(Au, Bz) = 0$$

and

$$M(u, z) = \max\{d(Au, Bz), d(Au, Su), d(Bz, Tz)\} = d(u, z).$$

Thus, from (2.3), we have

$$|d(u, z)| \leq \beta |d(u, z)|,$$

which is a contradiction to $\beta < 1$. Hence $u = z$. Therefore $A, B, S$ and $T$ have a unique common fixed point. This completes the proof. \hfill \Box
From Theorem 2.1, if \( A = B \) and \( S = T \), we get the following.

**Corollary 2.2.** Let \( A \) and \( S \) be two self-mappings of a complex valued metric space \((X,d)\) satisfying

\[
\begin{align}
& (C4) \quad SX \subset AX, \\
& (C5) \quad d(Sx, Sy) \preceq \alpha m(x, y) + \beta M(x, y),
\end{align}
\]

where
\[
m(x, y) = d(Ay, Sy) \frac{1 + d(Ax, Sx)}{1 + d(Ax, Ay)}
\]

and
\[
M(x, y) = \max\{d(Ax, Ay), d(Ax, Sx), d(Ay, Sy)\}
\]

for each \( x, y \) in \( X \), where \( \alpha \) and \( \beta \) are non-negative real numbers with \( \alpha + \beta < 1 \), \( (C6) \) the pair \((A, S)\) is weakly compatible.

Suppose that one of \( AX \) or \( SX \) is complete subspace of \( X \). Then \( A \) and \( S \) have a unique common fixed point.

**3. Fixed Point Theorems for Weakly Compatible Mappings with E.A. Property**

Now, we shall prove common fixed point theorems for weakly compatible mappings with E.A. property.

**Theorem 3.1.** Let \( A, B, S \) and \( T \) be four self-mappings of a complex valued metric space \((X,d)\) satisfying \((C1)-(C3)\) and

\[
\text{with \( (C7) \) the pairs \((A, S)\) or \((B, T)\) satisfy the E.A. property.}
\]

Suppose that any one of \( AX, BX, SX \) and \( TX \) is a closed subspace of \( X \). Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Suppose that \((A, S)\) satisfies the E.A. property. Then there exists a sequence \( \{x_n\} \) in \( X \) such that \( Ax_n = Sx_n = z \) for some \( z \in X \). Since \( SX \subset BX \), there exists a sequence \( \{y_n\} \) in \( X \) such that \( Sx_n = By_n \). Hence \( \lim_{n \to \infty} By_n = z \).

We shall show that \( \lim_{n \to \infty} Ty_n = z \). Let \( \lim_{n \to \infty} Ty_n = t \neq z \). From \((C2)\), we have
\[
d(Sx_n, Ty_n) \preceq \alpha m(x_n, y_n) + \beta M(x_n, y_n).
\]
Letting $n \to \infty$, we have

\[ d(z, t) \preceq \alpha \lim_{n \to \infty} m(x_n, y_n) + \beta \lim_{n \to \infty} M(x_n, y_n), \tag{3.1} \]

where

\[ \lim_{n \to \infty} m(x_n, y_n) = \lim_{n \to \infty} d(By_n, T y_n) \frac{1 + d(x_n, S x_n)}{1 + d(A x_n, B y_n)} = d(z, t) \]

and

\[ \lim_{n \to \infty} M(x_n, y_n) = \lim_{n \to \infty} \max \{d(A x_n, B y_n), d(A x_n, S x_n), d(B y_n, T y_n)\} = d(z, t). \]

Thus, from (3.1), we have

\[ d(z, t) \preceq \alpha d(z, t) + \beta d(z, t), \]

that is,

\[ |d(z, t)| \leq (\alpha + \beta)|d(z, t)|, \]

which is a contradiction to $\alpha + \beta < 1$. Therefore, $t = z$, that is, $\lim_{n \to \infty} T y_n = z$. Suppose that $B X$ is a closed space of $X$. Then there exists $u \in X$ such that $z = Bu$. Subsequently, we have

\[ \lim_{n \to \infty} T y_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} A x_n = \lim_{n \to \infty} B y_n = z = Bu. \]

Now, we shall show that $T u = Bu$. Let $T u \neq Bu$. From (C2), we have

\[ d(S x_n, T u) \preceq \alpha m(x_n, u) + \beta M(x_n, u). \]

Letting $n \to \infty$, we have

\[ d(z, T u) \preceq \alpha \lim_{n \to \infty} m(x_n, u) + \beta \lim_{n \to \infty} M(x_n, u), \tag{3.2} \]

where

\[ \lim_{n \to \infty} m(x_n, u) = \lim_{n \to \infty} d(B u, T u) \frac{1 + d(A x_n, S x_n)}{1 + d(A x_n, B u)} = d(z, T u) \]

and

\[ \lim_{n \to \infty} M(x_n, u) = \lim_{n \to \infty} \max \{d(A x_n, B u), d(A x_n, S x_n), d(B u, T u)\} = \max \{0, 0, d(B u, T u)\} = d(z, T u). \]
Thus, from (3.2), we have
\[ d(z, Tu) \preceq \alpha d(z, Tu) + \beta d(z, Tu), \]
that is,
\[ |d(z, Tu)| \leq (\alpha + \beta)|d(z, Tu)|, \]
which is a contradiction to \( \alpha + \beta < 1 \). Therefore, \( Tu = z = Bu \). Since \( B \) and \( T \) are weakly compatible, we have \( BTu = TBu \) and hence \( TTu = TBu = BTu = BBu \). Since \( TX \subset AX \), there exists \( v \in X \) such that \( Tu = Av \).

Now, we claim that \( Av = Sv \). Let \( Av \neq Sv \). From (C2), we have
\[ d(Sv, Tu) \preceq \alpha m(v, u) + \beta M(v, u), \]
where
\[ m(v, u) = d(Bu, Tu) \frac{1 + d(Av, Sv)}{1 + d(Av, Bu)} = 0 \]
and
\[ M(v, u) = \max\{d(Av, Bu), d(Av, Sv), d(Bu, Tu)\} = \max\{0, d(Av, Sv), 0\} = d(Tu, Sv). \]
Thus, from (3.3), we have
\[ d(Sv, Tu) \preceq \beta d(Sv, Tu), \]
that is,
\[ |d(Sv, Tu)| \leq \beta|d(Sv, Tu)|, \]
which is a contradiction to \( \beta < 1 \). Therefore, \( Sv = Tu = Av \). Thus, we have \( Tu = Bu = Sv = Av \). The weak compatibility of \( A \) and \( S \) implies that \( ASv = SAv = SSv = AAv \).

Now, we claim that \( Tu \) is the common fixed point of \( A, B, S \) and \( T \). Suppose that \( TTu \neq Tu \). From (C2), we have
\[ d(Tu, TTu) = d(Sv, TTu) \preceq \alpha m(v, Tu) + \beta M(v, Tu), \]
where
\[ m(v, Tu) = d(BTu, TTu) \frac{1 + d(Av, Sv)}{1 + d(Av, BTu)} = 0 \]
and
\[ M(v, Tu) = \max\{d(Av, BTu), d(Av, Sv), d(BTu, TTu)\} = d(Av, BTu) = d(Tu, TTu). \]
Thus, from (3.4), we have

\[ |d(Tu, TTu)| \leq \beta |d(Tu, TTu)|, \]

which is a contradiction to \( \beta < 1 \). Therefore, \( Tu = TTu = BTu \). Hence \( Tu \) is the common fixed point of \( B \) and \( T \). Similarly, we prove that \( Sv \) is the common fixed point of \( A \) and \( S \). Since \( Tu = Sv, Tu \) is the common fixed point of \( A, B, S \) and \( T \).

The proof is similar when \( AX \) is assumed to be a closed subspace of \( X \).

The cases in which \( TX \) or \( SX \) is a complete subspace of \( X \) are similar to the cases in which \( AX \) or \( BX \), respectively is complete subspace of \( X \) since \( TX \subset AX \) and \( SX \subset BX \).

Finally, for uniqueness, let \( p \) and \( q \) \((p \neq q)\) be two common fixed points of \( A, B, S \) and \( T \). From (C2), we have

\[ d(p, q) = d(Sp, Tq) \leq \alpha m(p, q) + \beta M(p, q), \quad (3.5) \]

where

\[ m(p, q) = d(Bq, Tq) \frac{1 + d(Ap, Sp)}{1 + d(Ap, Bq)} = 0 \]

and

\[ M(p, q) = \max\{d(Ap, Bq), d(Ap, Sp), d(Bq, Tq)\} = d(p, q). \]

Thus, from (3.5), we have

\[ |d(p, q)| \leq \beta |d(p, q)|, \]

which is a contradiction to \( \beta < 1 \). Therefore, \( p = q \). Hence \( A, B, S \) and \( T \) have a unique common fixed point. This completes the proof.

From Theorem 3.1, if \( A = B \) and \( S = T \) in Theorem 2.1, we get the following.

**Corollary 3.2.** Let \( A \) and \( S \) be two self-mappings of a complex valued metric space \((X, d)\) satisfying (C4)-(C6).

(C8) the pair \((A, S)\) satisfies the E.A. property.

If one of \( AX \) or \( SX \) is closed subspace of \( X \), then \( A \) and \( S \) have a unique common fixed point.
4. Fixed Point Theorems for Weakly Compatible Mappings with CLR Property

Now, we prove common fixed point theorems for weakly compatible mappings with CLR property.

**Theorem 4.1.** Let $A, B, S$ and $T$ be four self-mappings of a complex valued metric space $(X, d)$ satisfying (C2), (C3) and

(C9) $SX \subset BX$ and the pair $(A, S)$ satisfies CLR$_A$ property or $TX \subset AX$ and the pair $(B, T)$ satisfies CLR$_B$ property.

Then $A, B, S$ and $T$ have a unique common fixed point.

**Proof.** Without loss of generality, assume that $SX \subset BX$ and the pair $(A, S)$ satisfies CLR$_A$ property. Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = Ax$ for some $x \in X$. Since $SX \subset BX$, there exists a sequence $\{y_n\}$ in $X$ such that $Sx_n = By_n$. Hence $\lim_{n \to \infty} By_n = Ax$.

We shall show that $\lim_{n \to \infty} Ty_n = Ax$. Let $\lim_{n \to \infty} Ty_n = z \neq Ax$. From (C2), we have

$$d(Sx_n, Ty_n) \preceq \alpha m(x_n, y_n) + \beta M(x_n, y_n).$$

Letting $n \to \infty$, we have

$$d(Ax, z) \preceq \alpha \lim_{n \to \infty} m(x_n, y_n) + \beta \lim_{n \to \infty} M(x_n, y_n),$$

where

$$\lim_{n \to \infty} m(x_n, y_n) = \lim_{n \to \infty} d(By_n, Ty_n) \frac{1 + d(Ax_n, Sx_n)}{1 + d(Ax_n, By_n)} = d(Ax, z)$$

and

$$\lim_{n \to \infty} M(x_n, y_n) = \lim_{n \to \infty} \max\{d(Ax_n, By_n), d(Ax_n, Sx_n), d(By_n, Ty_n)\} = d(Ax, z).$$

Thus, from (4.1), we have

$$d(Ax, z) \preceq \alpha d(Ax, z) + \beta d(Ax, z),$$

that is,

$$|d(Ax, z)| \leq (\alpha + \beta)|d(Ax, z)|,$$

which is a contradiction to $\alpha + \beta < 1$. Therefore, $Ax = z$ and hence $\lim_{n \to \infty} Ty_n = Ax$. Subsequently, we have

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Ax = z.$$
Now, we shall show that $Sx = z$. Let $Sx \neq z$. From (C2), we have
\[ d(Sx, Ty) \preceq \alpha m(x, y) + \beta M(x, y). \]
Letting $n \to \infty$, we have
\[ d(Sx, z) \preceq \alpha \lim_{n \to \infty} m(x, y_n) + \beta \lim_{n \to \infty} M(x, y_n), \quad (4.2) \]
where
\[ \lim_{n \to \infty} m(x, y_n) = \lim_{n \to \infty} d(By_n, Ty_n) \frac{1 + d(Ax, Sx)}{1 + d(Ax, By_n)} = 0 \]
and
\[ \lim_{n \to \infty} M(x, y_n) = \lim_{n \to \infty} \max\{d(Ax, By_n), d(Ax, Sx), d(By_n, Ty_n)\} \]
\[ = d(z, Sx). \]
Thus, from (4.2), we have
\[ d(Sx, z) \preceq \beta d(Sx, z), \]
that is,
\[ |d(Sx, z)| \leq \beta |d(Sx, z)|, \]
which is a contradiction to $\beta < 1$. Therefore, $Sx = z = Ax$. Since the pair $(A, S)$ is weakly compatible, it follows that $Az = Sz$. Also, since $SX \subset BX$, there exists $y \in X$ such that $z = Sx = By$.

Now, we show that $Ty = z$. Let $Ty \neq z$. From (C2), we have
\[ d(Sx, Ty) \preceq \alpha m(x, y) + \beta M(x, y). \]
Letting $n \to \infty$, we have
\[ d(Ax, z) \preceq \alpha \lim_{n \to \infty} m(x, y_n) + \beta \lim_{n \to \infty} M(x, y_n), \quad (4.3) \]
where
\[ \lim_{n \to \infty} m(x, y_n) = \lim_{n \to \infty} d(By, Ty) \frac{1 + d(Ax_n, Sx_n)}{1 + d(Ax_n, By)} = d(z, Ty) \]
and
\[ \lim_{n \to \infty} M(x, y_n) = \lim_{n \to \infty} \max\{d(Ax_n, By), d(Ax_n, Sx_n), d(By_n, Ty_n)\} \]
\[ = d(z, Ty). \]
Thus, from (4.3), we have
\[ d(z, Ty) \preceq \alpha d(z, Ty) + \beta d(z, Ty), \]
that is,
\[ |d(z, Ty)| \leq (\alpha + \beta)|d(z, Ty)|, \]
which is a contradiction to \( \alpha + \beta < 1 \). Thus, \( z = Ty = By \). Since the pair \((B, T)\) is weakly compatible, it follows that \( Tz = Bz \). Now, we claim that \( Sz = Tz \).

Let \( Sz \neq Tz \). From (C2), we have
\[ d(Sz, Tz) \preceq \alpha m(z, z) + \beta M(z, z), \] (4.4)
where
\[ m(z, z) = d(Bz, Tz) \frac{1 + d(Az, Sz)}{1 + d(Az, Bz)} = 0 \]
and
\[ M(z, z) = \max\{d(Az, Bz), d(Az, Sz), d(Bz, Tz)\} = d(Sz, Tz). \]
Thus, from (4.4), we have
\[ d(Sz, Tz) \preceq \beta d(Sz, Tz), \]
that is,
\[ |d(Sz, Tz)| \leq \beta|d(Sz, Tz)|, \]
which is a contradiction to \( \beta < 1 \). Therefore, \( Sz = Tz \), that is, \( Az = Sz = Tz = Bz \).

Now, we shall show that \( z = Tz \). Let \( z \neq Tz \). From (C2), we have
\[ d(z, Tz) = d(Sz, Tz) \preceq \alpha m(z, z) + \beta M(z, z), \] (4.5)
where
\[ m(z, z) = d(Bz, Tz) \frac{1 + d(Az, Sz)}{1 + d(Az, Bz)} = 0 \]
and
\[ M(z, z) = \max\{d(Az, Bz), d(Az, Sz), d(Bz, Tz)\} = d(z, Tz). \]
Thus, from (4.5), we have
\[ |d(z, Tz)| \leq \beta|d(z, Tz)|, \]
which is a contradiction to \( \beta < 1 \). Therefore, \( z = Tz = Bz = Az = Sz \). Hence \( z \) is the common fixed point of \( A, B, S \) and \( T \).
Finally, for uniqueness, let $u (u \neq z)$ be another common fixed point of $A, B, S$ and $T$.

$$d(u, z) = d(Su, Tz) \preceq \alpha m(u, z) + \beta M(u, z), \quad (4.6)$$

where

$$m(u, z) = d(Bz, Tz) \frac{1 + d(Au, Su)}{1 + d(Au, Bz)} = 0$$

and

$$M(u, z) = \max\{d(Au, Bz), d(Au, Su), d(Bz, Tz)\} = d(u, z).$$

Thus, from (4.6), we have

$$|d(u, z)| \leq \beta |d(u, z)|,$$

which is a contradiction to $\beta < 1$. Therefore, $u = z$. Hence $A, B, S$ and $T$ have a unique common fixed point. This completes the proof.

From Theorem 4.1, if $A = B$ and $S = T$, we get the following.

**Corollary 4.2.** Let $A$ and $S$ be two self-mappings of a complex valued metric space $(X, d)$ satisfying (C4)-(C6).

(C10) the pair $(A, S)$ satisfies CLR$_A$ property.

Then $A$ and $S$ have a unique common fixed point.

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**References**


