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POSTULATION OF GENERAL UNIONS OF DECORATED LINES

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Abstract: For all integers $a \ge 0$ and $r \ge 3$ an a-decorated line $D \subset \mathbb{P}^r$ is a scheme union of a line D_{red} and a tangent vectors of $\mathbb{P}^r$ at points of D_{red}. Here we study the postulation of general disjoint unions of a-decorated lines.

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Key Words: postulation, Hilbert function, unions of lines, decorated lines

1. Introduction

Fix a line $L \subset \mathbb{P}^r$, $r \ge 2$, an integer $a \ge 0$ and a finite set $S \subset L$ such that $\sharp(S) = a$. For each $P \in S$ fix a degree two zero-dimensional scheme $v_P \subset \mathbb{P}^r$ such that $\deg(v_P) = 2$ and $\langle v_P \rangle \ne L$, where $\langle\ \ \rangle$ denote the linear span; we call v_P a tangent vector with P as its reduction and the line $\langle v_P \rangle$ as its support, or the tangent vector of $\langle v_P \rangle$ at P. Let $X \subset \mathbb{P}^r$ be the minimal closed subscheme containing L and each v_P, $P \in S$. We say that X is an a-decorated line with L as its support, S as the support of its nilradical (or of the nilradical of the sheaf $\mathcal{O}_X$) and the tangent vectors $\{v_P\}_{P \in S}$ as its nil-directions. The scheme X has dimension 1, $X_{\mathrm{red}} = L$, $h^0(\mathcal{O}_X) = a + 1$, $h^1(\mathcal{O}_X) = 1 + a$. By construction the line bundle $\mathcal{O}_X(1)$ is very ample and $x + a + 1$ is the Hilbert polynomial $p_X(x)$ of X with respect to $\mathcal{O}_X(1)$. We have $h^1(\mathcal{O}_X(x)) = 0$ for all $x \ge 0$ and hence $h^0(\mathcal{O}_X(x)) = x + 1 + a$ for all $x \ge 0$. It is elementary to check that $h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge a - 1$. For all integers $r \ge 3$, $t > 0$ and $a \ge 0$

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let $W(r,t,a)$ be the set of all disjoint unions $X \subset \mathbb{P}^r$ of t a-decorated lines of $\mathbb{P}^r$.The set of all a-decorated lines of $\mathbb{P}^r$ with a fixed line D as their support is parametrized by an irreducible variety of dimension $a+r-1$. Hence $W(r,t,a)$ is parametrized by an irreducible variety. Hence (as in [14] for the case $a=0$) it makes sense to compute the Hilbert function of a general element of $W(r,t,a)$. In this paper we prove the following result

Theorem 1. *Fix integers r,k,a,t such that $r \ge 3$, $t>0$, $k>0$ and $0 \le a \le k+1$. Let X be a general element of $W(r,t,a)$. Then either $h^0(\mathcal{I}_X(k))=0$ or $h^1(\mathcal{I}_X(k))=0$.*

We work over an algebraically closed field $\mathbb{K}$ such that either $\mathrm{char}(\mathbb{K})=0$ or $\mathrm{char}(\mathbb{K})>k$ (see Remark ++ for the positive caracteristic case).

2. Preliminaries

For all integers $r \ge 3$, $k>0$ and $a \ge 0$. Set $x_{r,k,a} := \lfloor \binom{r+k}{r}/(k+1+a) \rfloor$ and $y_{r,y,a} := \binom{r+k}{r} - (k+1+a)x_{r,k,a}$. We have $0 \le y_{r,k,a}$ and

$$(k+1+a)x_{r,k,a} + y_{r,k,a} = \binom{r+k}{r} \tag{1}$$

Remark 1. Fix integers r,k,a such that $r \ge 3$, $k>0$ and $0 \le a \le k+1$. To prove Theorem 1 for all quadruples (r,k,a,t) is sufficient to prove that $h^1(\mathcal{I}_X(k))=0$ for some $X \in W(r,x_{r,t,a},a)$ and that $h^0(\mathcal{I}_Y(k))=0$ for some $Y \in W(r,x_{r,k,a}+1,a)$. Take any $X \in W(r,x_{r,t,a},a)$. Since $h^0(\mathcal{O}_X(k)) = (k+1+a)x_{r,k,a}$, (1) shows that $h^0(\mathcal{I}_X(k)) = y_{r,k,a} + h^1(\mathcal{I}_X(k))$.

Fix $P \in \mathbb{P}^r$, $r \ge 2$, and a line $D \subset \mathbb{P}^r$. The 2-point $2P$ is the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_P)^2$ as its ideal sheaf. The 2-line $2D$ is the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_D)^2$ as its ideal sheaf.

Lemma 1. *Let $V \subset H^0(\mathcal{O}_{\mathbb{P}^r}(2))$, $r \ge 2$, be a linear system containing a line L in its base locus. Fix a general $P \in L$ and let v be a general tangent vector of $\mathbb{P}^r$ with $v_{\mathrm{red}} = \{P\}$. Set $V' := \{f \in V : f|v \equiv 0\}$. We have $\dim(V') = \dim(V)$ if a general $T \in |V|$ is not a quadric cone with L in its vertex and $V' = V$ if every $\{g=0\}$, $g \in V \setminus \{0\}$, is a quadric cone with vertex containing L.*

Proof. A quadric hypersurface contains every (or a general) degree 2 connected scheme v with $v_{\mathrm{red}} = \{P\}$ if and only if it is a quadric cone with vertex containing P. A quadric cone has vertex containing L if and only if its vertex contains a general $P \in L$. $\square$

Remark 2. Fix positive integers r, k, t. Fix any linear subspace of $H^0(\mathcal{O}_{\mathbb{P}^r}(k))$. Let $Z \subset \mathbb{P}^r$ be a general union of t tangent vectors. By [8, Lemma 1.4] or [12] (in characteristic zero) or if $\mathrm{char}(\mathbb{K}) > k$ ([7], Z imposes $\min\{\dim(V), 2t\}$ independent conditions to V.

3. Proof in $\mathbb{P}^3$

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface.

Remark 3. No cubic surface $T \subset \mathbb{P}^3$ has a singular locus containing 3 disjoint lines (e.g. by Bezout and the fact that 3 disjoint lines of $\mathbb{P}^3$ are contained in a smooth quadric surface, but no plane contains two disjoint lines).

Remark 4. We have

$$
\begin{aligned}
&(x_{3,3,0}, y_{3,3,0}) = (5,0), && (x_{3,3,1}, y_{3,3,1}) = (4,0), && (x_{3,3,2}, y_{3,3,2}) = (3,2),\\
&(x_{3,3,3}, y_{3,3,3}) = (2,6), && (x_{3,3,4}, y_{3,3,4}) = (2,4), && (x_{3,4,0}, y_{3,4,0}) = (7,0),\\
&(x_{3,4,1}, y_{3,4,1}) = (5,5), && (x_{3,4,2}, y_{3,4,2}) = (5,0), && (x_{3,4,3}, y_{3,4,3}) = (4,3),\\
&(x_{3,4,4}, y_{3,4,4}) = (3,8), && (x_{3,4,5}, y_{3,4,5}) = (3,5), && (x_{3,5,0}, y_{3,5,0}) = (9,2),\\
&(x_{3,5,1}, y_{3,5,1}) = (8,0), && (x_{3,5,2}, y_{3,5,2}) = (7,0), && (x_{3,5,3}, y_{3,5,3}) = (6,2),\\
&(x_{3,5,4}, y_{3,5,4}) = (5,6), && (x_{3,5,5}, y_{3,5,5}) = (5,1), && (x_{3,5,6}, y_{3,5,6}) = (4,7),\\
&(x_{3,6,0}, y_{3,6,0}) = (12,0), && (x_{3,6,1}, y_{3,6,1}) = (10,4), && (x_{3,6,2}, y_{3,6,2}) = (9,3),\\
&(x_{3,6,3}, y_{3,6,3}) = (8,4), && (x_{3,6,4}, y_{3,6,4}) = (7,7), && (x_{3,6,5}, y_{3,6,5}) = (7,0),\\
&(x_{3,6,6}, y_{3,6,6}) = (6,6), && (x_{3,6,7}, y_{3,6,7}) = (6,0), && (x_{3,7,0}, y_{3,7,0}) = (15,0),\\
&(x_{3,7,1}, y_{3,7,1}) = (13,3), && (x_{3,7,2}, y_{3,7,2}) = (12,0), && (x_{3,7,3}, y_{3,7,3}) = (10,9),\\
&(x_{3,7,4}, y_{3,7,4}) = (10,0), && (x_{3,7,5}, y_{3,7,5}) = (9,3), && (x_{3,7,6}, y_{3,7,6}) = (8,8),\\
&(x_{3,7,7}, y_{3,7,7}) = (8,0), && (x_{3,7,8}, y_{3,7,8}) = (7,8), && (x_{3,8,0}, y_{3,8,0}) = (18,3),\\
&(x_{3,8,1}, y_{3,8,1}) = (16,5), && (x_{3,8,2}, y_{3,8,2}) = (15,0), && (x_{3,8,3}, y_{3,8,3}) = (13,9),\\
&(x_{3,8,4}, y_{3,8,4}) = (12,9), && (x_{3,8,5}, y_{3,8,5}) = (11,11), && (x_{3,8,6}, y_{3,8,6}) = (11,0),\\
&(x_{3,8,7}, y_{3,8,7}) = (10,5), && (x_{3,8,8}, y_{3,8,8}) = (9,12), && (x_{3,8,9}, y_{3,8,9}) = (9,3),\\
&(x_{3,9,0}, y_{3,9,0}) = (22,0), && (x_{3,9,1}, y_{3,9,1}) = (20,0), && (x_{3,9,2}, y_{3,9,2}) = (18,4),\\
&(x_{3,9,3}, y_{3,9,3}) = (16,12), && (x_{3,9,4}, y_{3,9,4}) = (15,10), && (x_{3,9,5}, y_{3,9,5}) = (14,10),
\end{aligned}
$$

$(x_{3,9,6}, y_{3,9,6}) = (13,12)$, $(x_{3,9,7}, y_{3,9,7}) = (12,16)$, $(x_{3,9,8}, y_{3,9,8}) = (12,4)$, $(x_{3,9,9}, y_{3,9,9}) = (11,11)$, $(x_{3,9,10}, y_{3,9,10}) = (11,0)$.

Proof of Theorem 1 in $\mathbb{P}^3$. By Lemma 1 we may assume $k \ge 3$. Taking the difference between (1) with $(r',k',a') = (3,k,a)$ and $(r',k',a') = (3,k-2,a-2)$ we get

$$4x_{3,k-2,a-2} + (k+1+a)(x_{3,k,a} - x_{3,k-2,a-2}) + y_{3,k,a} - y_{3,k-2,a-2} = (k+1)^2. \quad (2)$$

If $a \ge 3$ (resp. $a \ge 4$), then we also have the following relations:

$$5x_{3,k-2,a-3} + (k+1+a)(x_{3,k,a} - x_{3,k-2,a-3}) + y_{3,k,a} - y_{3,k-2,a-3} = (k+1)^2, \quad (3)$$

$$6x_{3,k-2,a-4} + (k+1+a)(x_{3,k,a} - x_{3,k-2,a-4}) + y_{3,k,a} - y_{3,k-2,a-4} = (k+1)^2 \quad (4)$$

By [14] and [5] the theorem is true if $a \le 1$. Fix an integer a with $2 \le a \le k+1$. Let $Y \subset \mathbb{P}^3$ be a general union of $x_{3,k-2,a-2}$ $(a-2)$-decorated lines. Since $a-2 \le k-1$, the inductive assumption gives $h^1(\mathcal{I}_Y(k-2)) = 0$ and $h^0(\mathcal{I}_Y(k-2)) = y_{3,k-2,a-2}$. Set $w := x_{3,k,a} - x_{3,k-2,a-2}$. By (2) we have $w \le k$ for all $k \ge 3$. We assume $k \ge 5$ and use Lemma 1 and Remark 3 for the integer $k-2$ (plus the explicit values of the integer $x_{3,k-2,a-2}$) to handle the cases $k = 3,4$. For each $P \in Y \cap Q$ let v_P be a general tangent vector of Q with P as its reduction. Set $Z := \cup_{P \in Y \cap Q} v_P$. For general Y the scheme Z is a general union of $x_{3,k-2,a-2}$ tangent vectors of Q.

Claim 1. *if* $2 \le a \le k+1$ *and either* $k \ge 20$ *or* $a \ge (k+2)/2$, *then* $w \le (k+1)/2$.

Proof of Claim 1. Assume $w \ge (k+2)/2$. Since $y_{3,k,a} \ge 0$, $y_{3,k,a-2} \le k-3+a$ and $4(k-3+a)x_{3,k-2,a-2} + 4y_{3,k-2,a-2} = 4\binom{k+1}{3}$, (2) gives $4\binom{k+1}{3} - (k+1+a)(k-3+a) + (k+1+a)(k-3+a)(k+2)/2 < (k+1+a)(k+1)^2$. Hence $4\binom{k+1}{3} < (k+1+a)(k^2 - ak + 5k - 4)/2$. Since the right hand side of the last inequality is a decreasing function of a when $a > 0$, we get a contradiction for all $k \ge 20$ and, if $a \ge (k+2)/2$, for all $k \ge 10$.

(a) In this step we assume $y_{3,k,a} \geq y_{3,k-2,a-2}$. Let $E \subset Q$ be a general union of w lines of type $(0,1)$. For each line $L \subseteq E$ fix a general $S_L \subset L$ such that $\sharp(S_L) = a$. For each line $L \subseteq E$ and each $P \in S_L$ let v_P be a tangent vector of Q with $(v_P)_{\mathrm{red}} = \{P\}$ and not tangent to L. Set $S := \cup_{L\subseteq E} S_L$ and $X := Y \cup Z \cup E \cup \bigcup_{P\in S} v_P$. Since X is a disjoint union of $x_{3,k,a}$ a-decorated lines, to prove the theorem for the integers k and a it is sufficient to prove that $h^1(\mathcal{I}_X(k)) = 0$. Since $\mathrm{Res}_Q(X) = Y$, it is sufficient to prove that $h^1(Q, \mathcal{I}_{X\cap Q}(k)) = 0$. Since $\mathrm{Res}_E(X \cap Q) = S \cup Z$, it is sufficient to prove that $h^1(Q, \mathcal{I}_{S\cup Z}(k, k-w)) = 0$. Since $\deg(S \cup Z) = (k+1)(k+1-w) + y_{3,k,a} - y_{3,k-2,a-2}$, it is sufficient to prove that $h^1(Q, \mathcal{I}_S(k, k-w)) = 0$ (Remark 3). For each line $L \subseteq E$ fix $S'_L \subset L$ such that $\sharp(S'_L) = k+1$ and $S'_L \supset S_L$. It is sufficient to prove that $h^1(Q, \mathcal{I}_{\cup_{L\subseteq E} S'_L}(k, k-w) = 0$. Assume for the moment that either $k \geq 20$ or $k \geq 10$ and $a \geq (k+2)/2$. Since $\sharp(S'_L) = k+1$ for all L, we have $h^1(Q, \mathcal{I}_{\cup_{L\subseteq E} S'_L}(k, k-w) = h^1(Q, \mathcal{O}_Q(k, k-2w) = 0$, the latter equality coming from the inequality $2w \leq k+1$ (Claim 1). Now assume $a \leq (k+1)/2$. In this case we only need the inequality $w + \lceil w/2 \rceil \leq k+1$, instead of the inequality $2w \leq k+1$; this weaker inequality is true for all $k \geq 5$. The explicit values in Remark 4 show that we alway have $w \leq (k+1)/2$.

(b) In this step we assume $y_{3,k,a} < y_{3,k-2,a-2}$. Set $e := y_{3,k-2,a-2} - y_{3,k,a}$. We modify the construction just given in step (a) in the following way. Fix $S' \subset Y \cap Q$ such that $\sharp(S') = e$ and each line of Y_{red} contains at most one point of S'; this is possible because $x_{3,k-2,a-2} = \lfloor \binom{k+1}{3}/(k-3+a) \rfloor \geq k-4+a \geq y_{3,k-2,a-2} \geq e$. Call $R_1, \dots, R_e$ the lines of Y_{red} containing a point of S'. For each $P \in S'$ with, say $P \in R_i \cap Q$ let v_P be a general tangent vector of $\mathbb{P}^3$ with as v_{red} a general point of R_i. Set $Y_1 := Y \cup \bigcup_{P\in S} v_P$. Then take $E \subset Q$ and $v_P \subset Q$, $P \in S_L \subset L$, $L \subseteq E$ as in step (a). Set $X := Y_1 \cup E \cup \bigcup_{P\in S_L, L\subseteq E} v_P$. It is sufficient to prove that $h^1(\mathcal{I}_X(k)) = 0$. As in step (a) we have $h^1(Q, \mathcal{I}_{X\cap Q}(k)) = 0$. Hence it is sufficient to prove that $h^1(\mathcal{I}_{\mathrm{Res}_Q(X)}(k-2)) = 0$. We have $\mathrm{Res}_Q(X) = Y_1$. Assume $f := h^1(\mathcal{I}_{Y_1}(k-2)) > 0$. Since $Y \subset Y_1$, $h^1(\mathcal{I}_Y(k-2)) = 0$ and the ideal sheaf of Y in Y_1 is a skyscraper sheaf with S as its support and length 1 at each $P \in S$, we have $e \leq \sharp(S') = e$ and we may order the lines $L_1, \dots, L_{x_{3,k-2,a-2}}$ of Y_{red} so that $S' \subset (L_1 \cup \cdots \cup L_{y_{3,k-2,a-2} - y_{3,k,a}}) \cap Q$ and, calling $Y[i]$, $1 \leq i \leq y_{3,k-2,a-2} - y_{3,k,a}$, the union of Y and the schemes v_P, $P \in Q \cap (L_1 \cup \cdots \cup L_i)$ (with $Y[0] := Y$), we have $h^1(\mathcal{I}_{Y[i]}(k-2)) = 0$ for all $i \leq \alpha := w - f$ and $h^0(\mathcal{I}_{Y[\alpha]}(k-2)) = h^0(\mathcal{I}_{Y[\alpha+1}(k-2))$. Call $D_1, \dots, D_\alpha$ the lines of Y_{red} supporting the ideal sheaf of Y in $Y[\alpha]$ and $D_{\alpha+1}, \dots D_{y_{3,k-2,a-2}}$ all the other lines of Y_{red}. Moving $D_{\alpha+1}, \dots D_{y_{3,k-2,a-2}}$ and the vectors v_P at all points of the lines R_i not on some $D_1, \dots, D_\alpha$, we get a monodromy group which is a full symmetric group. Hence every surface $T \in |\mathcal{I}_{Y[\alpha]}(k-2)|$ contains the double line D_i. Call W the union of these double lines and of the connected components of Y_1 with $D_1,\dots,D_\alpha$ as its support. We have $h^0(\mathcal{I}_{Y_1}(k-2)) = h^0(\mathcal{I}_W(k-2)) = y_{3,k-2,a-2} - \alpha$ and $W \cap Q$ is a general union of α tangent vectors of Q, α points of Q and $2(x_{3,k-2,a-2}-\alpha)$ 2-points of Q. Let $E_1 \subset Q$ be a general union of $w1$ lines of type $(0,1)$. For each $L \subseteq E_1$ take subset $S_L \subset L$ with $\sharp(S_L) = k+1$. For any line $L \subseteq E_1$ and any $P \in S_L$ let v_P be any tangent vector of Q with P as its reduction and not contained in L. Let $L \subset Q$ be a general line of type $(1,0)$. Set $X_1 := Y_1 \cup L \cup E_1 \cup \bigcup_{P\in S_L, L\subseteq E_1} v_P$. Since X_1 is a disjoint union of $x_{3,k,a}-1$ a-decorated lines and $y_{3,k-2,a-2}-\alpha < 2k+1+y_{3,k,a}$, to get a contradiction it is sufficient to prove that $h^0(\mathcal{I}_{X_1}(k)) \le y_{3,k-2,a-2}-\alpha$. Since $h^0(\mathcal{I}_{Y_1}(k-2)) = h^0(\mathcal{I}_W(k-2)) = y_{3,k-2,a-2}-\alpha$, it is sufficient to prove that $h^0(Q, \mathcal{I}_{Q\cap W}(k, k+1-w)) = 0$. This is true by (2) and [15], because $\deg(W\cap Q) - \deg(Y(Y\cup Z)\cap Q)) > k+1+e$.

(c) To check that $h^0(\mathcal{I}_U(k)) = 0$ for some $U \in W(r, x_{3,k,a}+1, a)$ modify step (b) taking $e := y_{3,k-2,a-2}$ and adding one more a-line in Q. □

4. Proof in $\mathbb{P}^r$, $r \geq 4$

We fix an integer $r \ge 4$ and we assume that Theorem 1 is true in $\mathbb{P}^{r-1}$. By [5] to prove Theorem 1 in $\mathbb{P}^r$ we may assume $a \ge 2$. We have $a \le k+1$ by assumption. Since Theorem 1 is true for the integer $k=2$ (Lemma 1) we may assume $k \ge 3$ and that Theorem 1 is true for the triple $(r', k', a') = (r, k-1, a-1)$. Let $H \subset \mathbb{P}^r$ be a hyperplane. Taking the difference between (1) and the same equation for the integers r, $a' := a-1$ and $k' := k-1$ we get

$$2x_{r,k-1,a-1} + (k+1+a)(x_{r,k,a} - x_{r,k-1,a-1}) + y_{r,k,a} - y_{r,k-1,a-1} = \binom{r+k-1}{r-1}. \quad (5)$$

Set $w := x_{r,k,a} - x_{r,k-1,a-1}$.

Claim 1. *We have $w \ge 0$.*

Proof of Claim 1. Assume $w \le -1$. Since $x_{r,k-1,a-1} = \lfloor \binom{r+k-1}{r}/(k+a-1) \rfloor$, $y_{r,k,a} \le k+a$ and $y_{r,k-1,a-1} \ge 0$, (5) gives the inequality $2\binom{r+k-1}{r} - 2(k+a-1) \ge (k+a-1)\binom{r+k-1}{r-1}$. Since $a > 0$ and $k\binom{r+k-1}{r-1} - 2\binom{r+k-1}{r} = \binom{r+k-1}{r}[r-2] > 0$, we get a contradiction.

Let $Y \subset \mathbb{P}^r$ be a general union of $x_{r,k-1,a-1}$ $(a-1)$-decorated lines. By the inductive assumption we have $h^0(\mathcal{I}_Y(k-1)) = 0$ and $h^0(\mathcal{I}_Y(k-1)) = y_{r,k-1,a-1}$.

Since Y is general $Y \cap H$ is a general subset of H with cardinality $x_{r,k-1,a-1}$. For each $P \in H \cap Y$ let v_P be a general tangent of H with $\{P\} = (v_P)_{\mathrm{red}}$. The scheme $Y \cup Z$ is a disjoint union of $x_{r,k-1,a-1}$ $(a-1)$-decorated lines, $\mathrm{Res}_H(Y \cup Z) = Y$ and $(Y \cup Z) \cap H = Z$ (scheme-theoretic intersection). We have $\deg(Z) = 2x_{r,k-1,a-1}$.

(a) In this step we assume $y_{r,k,a} \geq y_{r,k,a-1}$. Let $E \subset H$ be a general union of w a-decorated lines of H. Set $X := Y \cup Z \cup E$. Since $\mathrm{Res}_H(X) = Y$, $X \cap H = Z \cup E$ and $h^1(\mathcal{I}_Y(k-1)) = 0$, a Castelnuovo's sequence shows that it is sufficient to prove that $h^1(H, \mathcal{I}_{Z \cup E}(k)) = 0$. By (5) we have $h^0(\mathcal{O}_{X \cup E}(k)) = \binom{r+k-1}{r-1} + y_{r,k-1,a-1} - y_{r,k,a} \leq \binom{r+k-1}{r-1}$. By Remark 3 it is sufficient to prove that $h^1(H, \mathcal{I}_E(k)) = 0$. This vanishing is true, because $H \cong \mathbb{P}^{r-1}$.

(b) In this step we assume $y_{r,k,a} < y_{r,k,a-1}$ and $w \geq y := y_{r,k,a-1} - y_{r,k,a}$. Set $a' := \min\{a-1, k-1\}$ and let $G \subseteq Y$ be any union of (a')-decorated lines. Since $h^0(\mathcal{I}_Y(k-1)) = y_{r,k-1,a-1} \leq k-2-a < \binom{r+k-1}{r} - \binom{r+k-2}{r}$, the inductive assumption gives $h^0(\mathcal{I}_G(k-2)) = 0$. Hence $h^0(\mathcal{I}_Y(k-1)) = 0$ (this vanishing is easily proved, without using induction on k). Let $F \subset H$ be a general union of w lines. Write $F = A \sqcup B$ with B union of y lines and A union of $w-y$ lines. Let $A' \subset H$ be a general union of a-decorated lines of H with $A'_{\mathrm{red}} = A'$. Let $B'' \subset \mathbb{P}^r$ be a general union of a-decorated lines with the following restriction; for each line $D \subseteq B$ let E_D be the connected component of B'' with D as its support; we assume that $E_D \cap H$ is an $(a-1)$-decorated lines and hence the scheme $\mathrm{Res}_H(E_D)$ is a point of D. Set $X := Y \cup \bigcup Z \cup A' \cup B''$ and $S' := \mathrm{Res}_H(B'')$. We have $\mathrm{Res}_H(X) = Y \cup S'$. Since B is a general union of y lines of H and each point $\mathrm{Res}_H(E_D)$, D a connected component of B, is a general point of D, S' is a general union of y points of H. Since $\mathrm{Res}_H(Y) = Y$, S' is general in H and $h^0(\mathcal{I}_Y(k-2)) = 0$, we have $h^0(\mathcal{I}_{Y \cup S'}(k-1)) = y_{r,k,a}$ and $h^1(\mathcal{I}_{Y \cup S'}(k-1)) = 0$ (see, e.g., [6], Lemma 4). A Castelnuovo's sequence gives $h^1(\mathcal{I}_X(k)) = 0$.

(c) In this step we assume $y_{r,k,a} < y_{r,k,a-1}$ and $w < y := y_{r,k,a-1} - y_{r,k,a}$. We have $x_{r,k-1,a-1} = \lfloor \binom{r+k-1}{r}/(k-1-a) \rfloor > 2y$

Claim 2. *We have* $x_{r,k-1,a-1} - y > \binom{t+k-1}{r-1}/r$.

Proof of Claim 2. By (5) and the assumption $w < y := y_{r,k,a-1} - y_{r,k,a}$ we have $2x_{r,k-1,a-1} - 2y + (k+1+a)y - k - 1 - a \geq \binom{r+k-1}{r-1} + y$. Since $r \geq 4$ and $x_{r,k-1,a-1} = \lfloor \binom{r+k-1}{r}/(k-1-a) \rfloor > 2y$, we get $r(x_{r,k-1,a-1} - y) > \binom{t+k-1}{r-1}$. For each $P \in Y \cap H$ let D_P be the line of Y_{red} containing P. Fix $S \subset Y \cap H$ such that $\sharp(S) = y$. Set $Z' := \cup_{P \in Y \cap H \setminus S} v_P$. Let Y' be the union of the connected components of Y with D_P, $P \in Y \cap H \setminus S$ as their support. Y' is a general union of $x_{r,k-1,a-1} - y_{r,k,a-1} + y_{r,k,a}$ $(a-1)$-decorated lines. For each $P \in S$ fix a general $O_P \in D_P$ and let w_P be a general tangent vector of $\mathbb{P}^r$ with O_P as its reduction. The scheme $Y_1 := Y \cup \bigcup_{P \in S} w_P$ is a general union of $x_{r,k-1,a-1} - y$ $(a-1)$-decorated lines and y a-decorated lines. Set $Y_2 := Y_1 \cup Z'$. Y_2 is a disjoint union of $x_{r,k-1,a-1}$ a-decorated lines, $Y \cap Y_2 = Z' \cup S$ and $\mathrm{Res}_H(Y_2) = Y_1$. We have $h^0(\mathcal{I}_{Y_1}(k-1)) = y_{r,k,a} + h^1(\mathcal{I}_{Y_1}(k-1))$. First assume $h^1(\mathcal{I}_{Y_1}(k-1)) = 0$. In this case to prove Theorem 1 for the triple (r,k,a) it is sufficient to repeat the proof of step (a) adding to Y_2 a general union of w a-decorated lines of H. Now assume $h^1(\mathcal{I}_{Y_1}(k-1)) > 0$. Since $h^1(\mathcal{I}_Y(k-1)) = 0$, we may order the points of S, say $S = \{P_1, \dots, P_y\}$, so that the $h^1(\mathcal{I}_{Y \cup \bigcup_{1 \le i \le e} w_P}(k-1)) = 0$ and $h^1(\mathcal{I}_{Y \cup \bigcup_{1 \le i \le e+1} w_{P_i}}(k-1)) > 0$. Since $w_{P_{e+1}}$ is a general tangent vector of $\mathbb{P}^r$ supported by a general point of $D_{P_{e+1}}$, we get that every $T \in |\mathcal{I}_{Y \cup \bigcup_{1 \le i \le e+1} w_{P_i}}(k-1)|$ contains the double line $2D_{P_{e+1}}$ of $\mathbb{P}^r$. The monodromy group of the symmetric product of $x_{r,k-1,a-1} - e$ copies of H is the full symmetric group and for each $P \in H$ the set of all $(a-1)$-decorated lines of $\mathbb{P}^r$ containing P, but intersecting H only at P is irreducible. Hence a monodromy argument gives that every $T \in |\mathcal{I}_{Y \cup \bigcup_{1 \le i \le e} w_{P_i}}(k-1)|$ contains the 2-line $2D_P$ for all $P \in Y \cap H \setminus S$. Fix a general hyperplane $M \subset \mathbb{P}^r$. For general Y the scheme $T \cap M$ contains $x_{r,k-1,a-1} - y$ general 2-points; call $W \subset M$ the union of these 2-points. Assume for the moment either $k \ge 4$ and $r \ge 6$ or $r = 4, 5$ and $k \ge 5$. By Claim 2 and a theorem of Alexander-Hirschowitz ([1], [2], [3], [4], [13]), we have $h^0(M, \mathcal{I}_W(k-1)) = 0$. Hence $M \subset T$. Varying M we get a contradiction (it is even easily proved that no $T' \in |\mathcal{I}_Y(k-1)|$ contains a hyperplane). Now assume $k = 3$. In this case we get $h^0(\mathcal{I}_{Y \cup \bigcup_{1 \le i \le e} w_P}(2)) = 0$, because Y_{red} spans $\mathbb{P}^r$ and the singular locus of a quadric hypersurface is a linear subspace. In the case $k = 4$ and $r = 4$ use Remark 2. Now assume $k = 4$ and $r = 5$. We have $r_{5,3,a-1} \ge \lfloor \binom{8}{3}/8 \rfloor = 7$ and no cubic hypersurface of $\mathbb{P}^4$ has a singular locus containing 7 general lines (e.g., use a Castelnuovo's sequence).

(d) A small adaptation of steps (b) and (c) gives that$h^0(\mathcal{I}_A(k)) = 0$ for some $A \in W(r, x_{r,k,a} + 1, a)$. □

Remark 5. To prove an equivalent of Theorem 1 for larger values of a, say $k + 2 \le a \le 2k$, one should use (3) and 4). Fix $Y \in W(3, x_{3,k-2,a-2}, a-2)$ as in the proof of the case $r = 3$ of Theorem 1 as in the proof and a line $D \subset Y_{\mathrm{red}}$. Notice that the 2-point of Q with P as its support is the flat limit of a family of tangent vectors of $\mathbb{P}^3$ supported by general points of D. The same observation may be used in $\mathbb{P}^r$, using both Alexander-Hirschowitz theorem and [9].

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