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POSTULATION OF ZERO-DIMENSIONAL SCHEMES ON A SMOOTH QUADRIC SURFACE

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Abstract: Let Q be a smooth quadric surface and $Z \subset Q$ a zero-dimensional scheme. We study the postulation of a general union of Z and prescribed numbers of fat points with multiplicity 2 and 3.

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1. Introduction

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. For each $P \in Q$ and any positive integer m the m-point mP is the closed subscheme of Q with $(\mathcal{I}_P)^m$ as its ideal sheaf. Let $Z \subset Q$ be a zero-dimensional scheme. In this note we collect several results concerning the Hilbert function of $Z \cup A$, A a general union of a prescribed number of 2-points and 3-points.

Theorem 1. *Fix non-negative integers c, d, a, b such that $b \geq d+5$ and $a \geq c+2$. Set $\alpha := \sum_{i=1}^{\lfloor (b-d-2)/3 \rfloor} 2\lfloor (a+1+3-3i)/4 \rfloor$. Fix non-negative integers e, e', f, f' such that $f \leq \alpha$, $f' \leq \alpha$, $e \leq \lfloor ((a+1)(b+1)-(c+1)(d+1)-6f)/3 \rfloor$ and $e' \geq \lceil ((a+1)(b+1)-(c+1)(d+1)-6f')/3 \rceil$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such that $h^1(\mathcal{I}_Z(c,d)) = 0$ and $h^0(\mathcal{I}_{Z'}(c,d)) = 0$. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of e 2-points and f 3-points (resp. e' 2-points and f' 3-points). Then $h^1(\mathcal{I}_{Z \cup A}(a,b)) = 0$ and $h^0(\mathcal{I}_{Z' \cup A'}(a,b)) = 0$.*

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Theorem 2. *Fix non-negative integers c, d, a, b, e, e' such that $a \geq c+2$, $b \geq d+2$, $e \leq \lfloor ((a+1)(b+1)-(c+1)(d+1))/3 \rfloor$, and $e' \geq \lceil ((a+1)(b+1)-(c+1)(d+1))/3 \rceil$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such that $h^1(\mathcal{I}_Z(c,d)) = 0$ and $h^0(\mathcal{I}_{Z'}(c,d)) = 0$. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of e (resp. e') 2-points. Then $h^1(\mathcal{I}_{Z\cup A}(a,b)) = 0$ and $h^0(\mathcal{I}_{Z'\cup A'}(a,b)) = 0$.*

Theorem 3. *Fix integers c, d, a, b such that $a \geq c+2 \geq 0$ and $b \geq d \geq 0$. Let $Z \subset Q$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_Z(c,d)) = 0$ and $h^0(\mathcal{I}_Z(c,d-1)) = 0$. Set $\gamma := h^0(\mathcal{I}_Z(c,d))$. If either c is odd and $\gamma = c$ or c is even and $\gamma \in \{c-1, c\}$, then assume $(a,b) \neq (c, d+2)$. Fix an integer e such that $0 \leq 3e \leq (a+1)(b+1)-(c+1)(d+1)+\gamma$. Let $A \subset Q$ be a general union of e 2-points. Then $h^1(\mathcal{I}_{Z\cup A}(a,b)) = 0$.*

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. See Remark 2 for the restrictions on $\mathrm{char}(\mathbb{K})$ used in the proofs of some of the statements.

2. Proofs for 2-Points

Remark 1. It is easy to check that a 3-point of any smooth surface X is a flat limit of a family of disjoint unions of pairs of 2-points of X ([6]).

Lemma 1. *Fix integers $c \geq 2$ and $d \geq 0$. Let $Z \subset Q$ be a zero-dimensional scheme. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of $\lfloor 2(c+1)/3 \rfloor$ (resp. $\lceil 2(c+1)/3 \rceil$) 2-points. Then $h^1(\mathcal{I}_{Z\cup A}(c,d+2)) \leq h^1(\mathcal{I}_Z(c,d))$ and $h^0(\mathcal{I}_{Z'\cup A'}(c,d+2)) \leq h^0(\mathcal{I}_{Z'}(c,d))$.*

Proof. Set $e := \lfloor (c+1)/3 \rfloor$. Take a line $L \subset Q$ of type $(0,1)$ such that $Z \cap L = \emptyset$. Fix $S \cup S' \subset L$ such that $S \cap S' = \emptyset$ and $\sharp(S) = \sharp(S') = e$ and general $o, o' \in L \setminus (S \cup S')$. We degenerate e of the 2-points of A and A' to the 2-points of $2S$. We apply e times the Differential Horace Lemma ([1], Lemme 1.3, [3], Lemma 5) with respect to each point of S'. If $2e = \lfloor 2(c+1)/3 \rfloor$, i.e. if $c+1 \equiv 0, 1 \pmod 3$ we get $h^1(\mathcal{I}_{Z\cup A}(c,d+2)) \leq h^1(\mathcal{I}_Z(c,d))$ and $h^0(\mathcal{I}_{Z'\cup A}(c,d+2)) \leq h^0(\mathcal{I}_{Z'}(c,d)) + 2$. Now assume $c+1 \equiv 1 \pmod 3$. Adding also $2o'$ we get $h^0(\mathcal{I}_{Z'\cup A'}(c,d)) \leq h^0(\mathcal{I}_{Z'}(c,d))$. Now assume $c \equiv 2 \pmod 3$, i.e. $c = 3e+1$. We first insert $2o$ and get $h^1(\mathcal{I}_{Z\cup A}(c,d+2)) \leq h^1(\mathcal{I}_Z(c,d))$, because at the first step in L we get a scheme $(2S \cup S' \cup 2O) \cap L$ of degree $3e+2 = c+1$ and during the second step a scheme $S \cup (2S' \cap L) \cup \{o\}$ of degree $3e+1$. To get $h^0(\mathcal{I}_{Z'\cup A'}(c,d)) \leq h^0(\mathcal{I}_{Z'}(c,d))$ we add $2o'$ at the second step. □

Lemma 2. *Fix integers c, d, e, e' such that $c \geq 2$, $d \geq 0$, $0 \leq e \leq c+1$ and $e' \geq c+1$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such* *that* $h^1(\mathcal{I}_Z(c,d)) = 0$ *and* $h^0(\mathcal{I}_{Z'}(c,d)) = 0$. *Let* $A \subset Q$ *(resp.* $A' \subset Q$*) be a general union of* e *(resp.* e'*) 2-points. Then* $h^1(\mathcal{I}_{Z\cup A}(c,d+3)) = 0$ *and* $h^0(\mathcal{I}_{Z'\cup A'}(c,d+3)) = 0$.

Proof. Since a 3-point is a flat limit of a family of disjoint unions of two 2-points (Remark 1), this lemma is a particular case of Lemma 8. □

Lemma 3. *Fix integers* $e \geq 0$, $d \geq 0$ *and* $c \geq 2$. *Let* $Z \subset Q$ *be a zero-dimensional scheme such that* $h^1(\mathcal{I}_Z(c,d)) = 0$ *and* $h^0(\mathcal{I}_Z(c,d-1)) = 0$. *Set* $\gamma := h^0(\mathcal{I}_Z(c,d))$, $u := \lfloor (c+1)/2 \rfloor$, *and* $\epsilon := c+1-2u$. *Let* $A \subset Q$ *be a general union of* e *2-points.*

(a) If $e \leq u$, *then* $h^1(\mathcal{I}_{Z\cup A}(c,d+1)) \leq \max\{0, e-\gamma\}$.

(b) If c *is even,* $e = u+1$ *and either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > c$, *then* $h^1(\mathcal{I}_{Z\cup A}(c,d+1)) \leq \max\{0, u+2-\gamma\}$.

Proof. Since $h^1(\mathcal{I}_Z(c,d)) = 0$, we have $\deg(Z) = (c+1)(d+1) - \gamma$. Since $h^0(\mathcal{I}_Z(c,d-1)) = 0$, we have $0 \leq \gamma \leq c+1$ and $h^1(\mathcal{I}_Z(c,d-1)) = c+1-\gamma$. Fix a line $L \subset Q$ of type $(0,1)$ such that $L \cap Z = \emptyset$. First assume c even and $e = u+1$. Fix $S \subset Q$ such that $\sharp(S) = u$ and $o \in L \setminus S$. Let $E \subset L$ be the 2-point of L with o as its reduction. Since $\deg(L \cap (2S \cup \{o\}) = c+1$, By the Differential Horace Lemma for double points ([1], Lemme 1.3, [3], Lemma 5) to prove the lemma it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup S\cup E}(c,d)) = \max\{0, u+2-\gamma\}$. Let $W \subset H^0(\mathcal{O}_L(c))$ be the image of the restriction map $\rho : H^0(\mathcal{I}_Z(c,d)) \to H^0(\mathcal{O}_L(c))$. Since $h^0(\mathcal{I}_Z(c,d-1)) = 0$, ρ is injective. Hence $\dim(W) = \gamma$. Hence to prove part (b) it is sufficient to prove that $S \cup E$ imposes $\min\{\gamma, \deg(S \cup E)\}$ independent conditions to the linear system W. In arbitrary characteristic S imposes independent conditions to any linear system V on L with dimension $\geq \sharp(S)$. This is also true for $S \cup E$ if either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > c$, because E is a general tangent vector of L and our assumption on $\mathrm{char}(\mathbb{K})$ implies that the rational map induced by V is separable.

The case $e \leq u$ is easier. Indeed, we do not use Differential Horace and in the residual we only have $Z \cup S$ with $\sharp(S) = e$, instead of $Z \cup S \cup E$. □

Lemma 4. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix integers* $d \geq 0$ *and* $c \geq 2$. *Let* $Z \subset Q$ *be a zero-dimensional scheme such that* $h^1(\mathcal{I}_Z(c,d)) = 0$ *and* $h^0(\mathcal{I}_Z(c,d-1)) = 0$. *Set* $\gamma := h^0(\mathcal{I}_Z(c,d))$ *and* $e' := \lfloor (2c+2+\gamma)/3 \rfloor$. *If* c *is odd and* $\gamma = c$, *then set* $e := e'-1$. *If* c *is even and* $\gamma \in \{c-1, c\}$, *then set* $e := e'-1$. *In all other cases set* $e := e'$. *Let* $U \subset Q$ *be a general union of* $e - \lceil (c+1)/2 \rceil$ *2-points. Then* $h^1(\mathcal{I}_{Z\cup U}(c,d+2)) = 0$.

Proof. Since $h^0(\mathcal{I}_Z(c,d-1))=0$, we have $\gamma \le c+1$. Moreover, $\gamma=c+1$ if and only if $h^1(\mathcal{I}_Z(c-1,d))=0$. Hence the case $\gamma=c+1$ is true by Lemma 2. From now on we assume $\gamma \le c$. Since each connected component of U contains a general tangent vector of Q at its support and U_{red} is general, it is sufficient to prove that $f:=\sharp(U_{\mathrm{red}}) \ge \gamma/2$. Set $f':=f+e'-e$. Let $g\in\{0,1,2\}$ the congruence class of $2c+2+\gamma$ modulo 3. First assume that c is odd. In this case we get $f'=(2c+2+\gamma)/3-g/3-(c+1)/2$, i.e. $2f'=c/3+2\gamma/3-2g/3+1/3$. We have $2f'\ge\gamma$ if and only if $c/3-2g/3+1/3\ge\gamma/3$. The last inequality is obviously true if $\gamma\le c-1$. Now assume that c is even. In this case we have $f'=(2c+2+\gamma)/3-g/3-c/2-1$, i.e. $2f=c/3+2\gamma/3-2g/3-1/3$. We conclude if $\gamma\le c-2$. □

Lemma 5. *Fix integers $d\ge 0$ and $c\ge 2$. Let $Z\subset Q$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_Z(c,d))=0$ and $h^0(\mathcal{I}_Z(c,d-1))=0$. Set $\gamma:=h^0(\mathcal{I}_Z(c,d))$. Take an integer e such that $0\le 3e\le 2c+2+\gamma$. Let $A\subset Q$ be a general union of e 2-points. If c is odd assume that either* $\mathrm{char}(\mathbb{K})=0$ *or* $\mathrm{char}(\mathbb{K})>c$. *Then $h^1(\mathcal{I}_{Z\cup A}(c,d+2))=0$.*

Proof. Since $h^1(\mathcal{I}_Z(c,d))=0$, we have $\deg(Z)=(c+1)(d+1)-\gamma$. Since $h^0(\mathcal{I}_Z(c,d-1))=0$, we have $0\le\gamma\le c+1$ and $h^1(\mathcal{I}_Z(c,d-1))=c+1-\gamma$. Since the case $\gamma=c+1$ is true by Lemma 2 applied to the integers $(c,d-1)$, we may assume $\gamma\le c$. Hence $e\le c$. It is sufficient to do the case $e=\lfloor(2c+2+\gamma)/3\rfloor$. In particular we assume $e\ge\lceil(c+1)(2\rceil$. Fix a line $L\subset Q$ of type $(0,1)$ such that $L\cap Z=\emptyset$. First assume that c is odd. Let $A'\subset Q$ be a general union of $e-(c+1)/2$ 2-points. Take $S\subset L$ such that $\sharp(S)=(c+1)/2$. It is sufficient to prove that $h^1(\mathcal{I}_{Z\cup A'\cup 2S}(c,d+2))=0$. Since $\deg(2S\cap L)=c+1$, it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup A'\cup S}(c,d+1))=0$. Since $3e\le 2c+2+\gamma$, we have $\deg(A')\le c+1+\gamma$. Lemma 3 gives $h^1(\mathcal{I}_{Z\cup A'}(c,d+1))=0$. Hence $h^0(\mathcal{I}_{Z\cup A'}(c,d+1))=(c+1)(d+2)-\deg(Z)-\deg(A')=\gamma+c+1-3e+3(c+1)/2\ge(c+1)/2$ and to prove the lemma for c odd it is sufficient to prove that S imposes $\sharp(S)$ independent conditions to $H^0(\mathcal{I}_{Z\cup A'}(c,d+1))$. Let $W\subset H^0(\mathcal{O}_L(c))$ be the image of the restriction map $\rho: H^0(\mathcal{I}_{Z\cup A'}(c,d+1))\to H^0(\mathcal{O}_L(c))$. Since $h^0(\mathcal{I}_{Z\cup A'}(c,d))=0$ (Lemma 4), ρ is injective. Hence $\dim(W)\ge(c+1)/2=\sharp(S)$. Hence it is sufficient to use that S is general in L.

Now assume that c is even. Let $A''\subset Q$ be a general union of $e-c/2-1$ 2-points of Q. Take a general $S'\cup\{o\}\subset L$ such that $\sharp(S')=c/2$ and $o\notin S'$. Let $E\subset L$ be the 2-point of L with o as its support. Since $\deg(2S\cap L)+\deg(\{o\})=c+1$, the Differential Horace Lemma for 2-points shows that to prove that a general union Y of $Z\cup 2S\cup A''$ and a 2-point satisfies $h^1(\mathcal{I}_Y(c,d+2))=0$ (and hence to prove the lemma in the case c even), it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup A''\cup S\cup E}(c,d+1)) = 0$. Lemma 3 gives $h^1(\mathcal{I}_{Z\cup A''}(c,d+1)) = 0$. Let $V \subset H^0(\mathcal{O}_L(c))$ be the image of the restriction map $\rho' : H^0(\mathcal{I}_{Z\cup A''}(c,d+1)) \to H^0(\mathcal{O}_L(c))$. Since $h^0(\mathcal{I}_{Z\cup A''}(c,d)) = 0$ (Lemma 4), ρ is injective. Hence $\dim(V) \geq \sharp(S) + \deg(E)$. Hence it is sufficient to use that $S \cup \{o\}$ is general in L and hence that E is a general tangent vector of $L \setminus S$. □

Lemma 6. *Fix integers $c \geq 2$ and $d \geq 0$. Let $Z \subset Q$ be a zero-dimensional scheme. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of $\lfloor 4(c+1)/3 \rfloor$ (resp. $\lceil 4(c+1)/3 \rceil$) 2-points. Then $h^1(\mathcal{I}_{Z\cup A}(c,d+4)) \leq h^1(\mathcal{I}_Z(c,d))$ and $h^0(\mathcal{I}_{Z'\cup A'}(c,d+4)) \leq h^0(\mathcal{I}_{Z'}(c,d))$.*

Proof. Adapt either the proof of Lemma 1 or the one of Lemma 5 quoting Lemmas 2 and 1 for the residual with respect to L instead of Lemmas 4 and 3. □

Lemma 7. *Fix integers $c \geq 2$ and $d \geq 2$. Let $Z \subset Q$ be a zero-dimensional scheme. Set $\gamma := h^0(\mathcal{I}_Z(c,d))$. Assume $\gamma \leq c$, $h^0(\mathcal{I}_Z(c-1,d-2)) = 0$ and $h^1(\mathcal{I}_Z(c,d)) = 0$. Let $U \subset Q$ be a general union of $\lfloor (2c+d+4+\gamma)/3 \rfloor$ 2-points of Q. Then $h^1(\mathcal{I}_{Z\cup U}(c+1,d+2)) = 0$.*

Proof. Let $C \subset Q$ be a smooth curve of type $(1,2)$ such that $C \cap Z = \emptyset$. We have $(c+2)(d+3)-(c+1)(d+1) = 2c+d+4$. Since $\gamma \leq c$, $c \geq 2$ and $d \geq 2$, we have $\lfloor (2c+d+4+\gamma)/3 \rfloor \leq \lfloor ((2c+d+1)/2 \rfloor$. Take a general $S \subset C$ such that $\sharp(S) = \lfloor (2c+d+4+\gamma)/3 \rfloor$. It is sufficient to prove that $h^1(\mathcal{I}_{Z\cup 2S}(c+1,d+2)) = 0$. Since $\deg(2S \cap C) \leq 2c+d+1$ and $Z \cap C = \emptyset$, it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup S}(c,d)) = 0$. Let $V \subset H^0(\mathcal{O}_C(c))$ be the image of the restriction map $\rho' : H^0(\mathcal{I}_Z(c,d)) \to H^0(\mathcal{O}_C(c,d))$. Since $h^0(\mathcal{I}_Z(c-1,d-2)) = 0$ and $Z \cap C = \emptyset$, ρ is injective. Since $\dim(V) = \gamma$ and S is general in C, S imposes γ independent conditions to V. Since ρ is injective, we get $h^0(\mathcal{I}_{Z\cup S}(c,d)) = 0$. □

Proof of Theorem 3. If $a = c$ and $b = d+2$, then we quote Lemma 5. Now assume $a = c$ and $b \geq d+3$. In this case we copy the proof of Lemma 5 and induction on d. If $a = c+1$ and $b = d+2$, then we quote Lemma 7. Then for a fixed a we get all cases with $b > d+2$ using induction on b and the Horace Differential lemma for double points with respect to a general $L \in |\mathcal{O}_Q(1,0)|$. □

Proof of Theorem 2. Taking a union of Z and $\gamma := h^0(\mathcal{I}_Z(a,b))$ general points of Q we reduce to the case $\gamma = 0$. Then we apply Theorem 3. □

Proposition 1. *Fix integers $m \geq 3$, $e > 0$, $a > m$, $b > m$ such that $(a,b) \neq (m+1, m+1)$. Let $A \subset Q$ be a general union of two m-points and e 2-points. Then either $h^1(\mathcal{I}_A(a,b)) = 0$ or $h^0(\mathcal{I}_A(a,b)) = 0$.*

Proof. Just using Bezout theorem we get

$$h^i(\mathcal{O}_{mP \cup mO}(m, m-1)) = h^i(\mathcal{O}_{mP \cup mO}(m-1, m)),$$

$i = 0, 1$. ApplyTheorem 2. □

3. The Proofs for 3-Points

Lemma 8. *Fix non-negative integers c, d, e, f, e', f' such that $c \geq 2$, $d \geq 0$, $f \leq 2\lfloor (c+1)/4 \rfloor$, $f' \leq 2\lfloor (c+1)/4 \rfloor$, $e \leq c+1-2f$ and $e' \geq c+1-2f'$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such that $h^1(\mathcal{I}_Z(c,d)) = 0$ and $h^0(\mathcal{I}_{Z'}(c,d)) = 0$. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of e 2-points and f 3-points (resp. e' 2-points and f' 3-points). Then $h^1(\mathcal{I}_{Z \cup A}(c, d+3)) = 0$ and $h^0(\mathcal{I}_{Z' \cup A'}(c, d+3)) = 0$.*

Proof. Set $x := \lfloor (c+1)/4 \rfloor$. Since any 3-point of Q is a flat limit of a family of disjoint unions of pairs of 2-points of Q (Remark 1), the semicontinuity theorem for cohomology shows that it is sufficient to do the case $f = f' = 2x$. Fix a line $L \subset Q$ of type $(0,1)$ such that $L \cap Z = L \cap Z' = \emptyset$. Fix a general $S \cup S' \subset L$ such that $S \cap S' = \emptyset$ and $\sharp(S) = \sharp(S') = x$. Fix general $o, o', o'' \in L \setminus S \cup S'$. Fix $o'_1 \in S'$ and set $S'_1 := S' \setminus \{o'_1\}$.

First assume $c \equiv 3 \pmod 4$. In this case we apply 3 times the Horace Differential Method for 3-points to each point of S' with respect to the integers $3 > 2$ so that $(1,2,3)$ are the degrees of the intersection with L of the first, second and third trace ([5], [7], [8], [9]), while we specialize x of the 3-points of A and A' to 3-points with a point of S as their support. At each step in L we have a scheme of degree $c+1$.

Now assume $c \equiv 1 \pmod 4$. In the first step we also add $2o$. In this step we have a scheme whose intersection with L has degree $c+1$, while in the second step the scheme on L has only degree c, because $\mathrm{Res}_L(2o) = \{o\}$. Hence at the second step we also use the Horace Differential lemma for double points with respect to o', so that in L we have a degree $c+1$ scheme, while at the third step we have a scheme sitting of degree $c+1$ inside L, with no connected component with o as its reduction and with a degree 2 component with o' as its reduction.

Now assume $c \equiv 0 \pmod 4$. We take $3S$ and apply the Differential Horace Method for 3-points with respect to the integers $3 > 2$, i.e. with traces of degrees $(1, 2, 3)$, for each point of S'_1 and with respect to the sequence 2 (i.e. with traces of degrees $(2, 1, 3)$) with respect to o'_1; hence the intersection of L with this virtual scheme has degree $3x + (x - 1) + 2 = c + 1$. (x components of degree 3, $x - 1$ components of degree 1 and one of degree 2). At the second step we also add the scheme $2o$. The new virtual scheme intersects L in a scheme with degree $c + 1$ ($2x + 1$ of its connected components have degree 2, the one supported by o'_1 has degree 1). In the last step, after taking the virtual residual scheme, we get a scheme of degree $c + 1$ ($x + 1$ components of degree 1, x components of degree 3).

Now assume $c \equiv 2 \pmod 4$. We first add $3S$, apply the Differential Horace lemma for 3-points with respect to the integers $3 > 2$, i.e. with traces of degrees $(1, 2, 3)$, at each point of S'_1, the Differential Horace points for 3-points with intersections with L of degree $(3, 1, 2)$ (it is the example done in the introduction of [5]) at o'_1 and the differential Horace lemma for double points with respect to o'. The intersection of the virtual residual scheme with L has only degree $4x + 1 = c - 1$ ($2x$ connected components of degree 2 and one of degree 1). Therefore in the second step we also apply the Differential Horace lemma for 2-points at o' and o''; in this way the restriction of this scheme to L has degree $c + 1$ ($2x$ components of degree 2 and 3 components of degree 1). The virtual residue scheme with respect to L is contained in L and it has degree $c + 1$ (it has x components of degree 1 with S as the union of their reductions, $x - 1$ components of degree $x - 1$ with S'_1 as the union of their reductions and 3 components of degree 2 with o'_1, o' and o'' as their reduction). □

Proof of Theorem 1. We have $h^1(\mathcal{I}_Z(a, x)) = 0$ for all $x \geq d$. We apply $\lfloor (d - b - 2)/ \rfloor$ times Lemma 8, first with respect to the pair $(c, d) := (a, b)$, then with respect to the pairs $(c, d) := (a, b - 3)$, and so on. Then we apply the lemmas for 2-points with respect to the integers $(a, b) := (a, b - 3\lfloor (d - b - 2)/3 \rfloor)$. □

Remark 2. The quoted parts of [8] and [9] and the Differential Horace lemma for double points ([3]) are characteristic free. We only use the condition $\mathrm{char}(\mathbb{K})$ to say that a general tangent vector gives two conditions to any linear system of dimension > 1 ([4], [2], Lemma 1.4). In a few cases we state the minimal assumptions for our proofs. In all theorem it is sufficient to assume $\mathrm{char}(\mathbb{K}) > \max\{a, b, 2c + d\}$ (the only problem is to check Lemma 4 when $\mathrm{char}(\mathbb{K}) > 2c + d$).

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