POSTULATION OF ZERO-DIMENSIONAL SCHEMES
ON A SMOOTH QUADRIC SURFACE

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Abstract: Let $Q$ be a smooth quadric surface and $Z \subset Q$ a zero-dimensional scheme. We study the postulation of a general union of $Z$ and prescribed numbers of fat points with multiplicity 2 and 3.

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1. Introduction

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. For each $P \in Q$ and any positive integer $m$ the $m$-point $mP$ is the closed subscheme of $Q$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. Let $Z \subset Q$ be a zero-dimensional scheme. In this note we collect several results concerning the Hilbert function of $Z \cup A$, $A$ a general union of a prescribed number of 2-points and 3-points.

Theorem 1. Fix non-negative integers $c, d, a, b$ such that $b \geq d + 5$ and $a \geq c + 2$. Set $\alpha := \sum_{i=1}^{\lfloor (b-d-2)/3 \rfloor} 2\lfloor (a+1+3-3i)/4 \rfloor$. Fix non-negative integers $e, e', f, f'$ such that $f \leq \alpha$, $f' \leq \alpha$, $e \leq \lfloor ((a+1)(b+1)-(c+1)(d+1)-6f)/3 \rfloor$ and $e' \geq \lfloor ((a+1)(b+1)-(c+1)(d+1)-6f')/3 \rfloor$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such that $h^1(\mathcal{I}_Z(c, d)) = 0$ and $h^0(\mathcal{I}_{Z'}(c, d)) = 0$. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of $e$ 2-points and $f$ 3-points (resp. $e'$ 2-points and $f'$ 3-points). Then $h^1(\mathcal{I}_{Z \cup A}(a, b)) = 0$ and $h^0(\mathcal{I}_{Z' \cup A'}(a, b)) = 0$. 
Theorem 2. Fix non-negative integers $c,d,a,b,e,e'$ such that $a \geq c+2$, $b \geq d+2$, $e \leq \left|\left((a+1)(b+1)-(c+1)(d+1)\right)/3\right|$, and $e' \geq \left|\left((a+1)(b+1)-(c+1)(d+1)\right)/3\right|$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such that $h^1(I_Z(c,d)) = 0$ and $h^0(I_{Z'}(c,d)) = 0$. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of $e$ (resp. $e'$) 2-points. Then $h^1(I_{Z\cup A}(a,b)) = 0$ and $h^0(I_{Z'\cup A'}(a,b)) = 0$.

Theorem 3. Fix integers $c,d,a,b$ such that $a \geq c+2 \geq 0$ and $b \geq d \geq 0$. Let $Z \subset Q$ be a zero-dimensional scheme such that $h^1(I_Z(c,d)) = 0$ and $h^0(I_Z(c,d-1)) = 0$. Set $\gamma := h^0(I_Z(c,d))$. If either $c$ is odd and $\gamma = c$ or $c$ is even and $\gamma \in \{c-1,c\}$, then assume $(a,b) \neq (c,d+2)$. Fix an integer $e$ such that $0 \leq 3e \leq (a+1)(b+1)-(c+1)(d+1)+\gamma$. Let $A \subset Q$ be a general union of $e$ 2-points. Then $h^1(I_{Z\cup A}(a,b)) = 0$.

We work over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$. See Remark 2 for the restrictions on $\text{char}(\mathbb{K})$ used in the proofs of some of the statements.

2. Proofs for 2-Points

Remark 1. It is easy to check that a 3-point of any smooth surface $X$ is a flat limit of a family of disjoint unions of pairs of 2-points of $X$ ([6]).

Lemma 1. Fix integers $c \geq 2$ and $d \geq 0$. Let $Z \subset Q$ be a zero-dimensional scheme. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of $\left\lfloor 2(c+1)/3 \right\rfloor$ (resp. $\left\lfloor 2(c+1)/3 \right\rfloor$) 2-points. Then $h^1(I_{Z\cup A}(c,d+2)) \leq h^1(I_Z(c,d))$ and $h^0(I_{Z'\cup A'}(c,d+2)) \leq h^0(I_{Z'}(c,d))$.

Proof. Set $e := \left\lfloor (c+1)/3 \right\rfloor$. Take a line $L \subset Q$ of type $(0,1)$ such that $Z \cap L = \emptyset$. Fix $S \cup S' \subset L$ such that $S \cap S' = \emptyset$ and $\sharp(S) = \sharp(S') = e$ and general $o,o' \in L \setminus (S \cup S')$. We degenerate $e$ of the 2-points of $A$ and $A'$ to the 2-points of $2S$. We apply $e$ times the Differential Horace Lemma ([1], Lemme 1.3, [3], Lemma 5) with respect to each point of $S'$. If $2e = \left\lfloor 2(c+1)/3 \right\rfloor$, i.e. if $c+1 \equiv 0,1,2 \pmod{3}$ we get $h^1(I_{Z\cup A}(c,d+2)) \leq h^1(I_Z(c,d))$ and $h^0(I_{Z'\cup A'}(c,d+2)) \leq h^0(I_{Z'}(c,d)) + 2$. Now assume $c+1 \equiv 1 \pmod{3}$. Adding also $2o'$ we get $h^0(I_{Z'\cup A'}(c,d)) \leq h^0(I_{Z'}(c,d))$. Now assume $c \equiv 2 \pmod{3}$, i.e. $c = 3e+1$. We first insert $2o$ and get $h^1(I_{Z\cup A}(c,d+2)) \leq h^1(I_Z(c,d))$, because at the first step in $L$ we get a scheme $(2S \cup S' \cup 2O) \cap L$ of degree $3e+2 = c+1$ and during the second step a scheme $S \cup (2S' \cap L) \cup \{o\}$ of degree $3e+1$. To get $h^0(I_{Z'\cup A'}(c,d)) \leq h^0(I_{Z'}(c,d))$ we add $2o'$ at the second step.

Lemma 2. Fix integers $c,d,e,e'$ such that $c \geq 2$, $d \geq 0$, $0 \leq e \leq c+1$ and $e' \geq c+1$. Fix zero-dimensional schemes $Z \subset Q$ and $Z' \subset Q$ such
that $h^1(I_Z(c,d)) = 0$ and $h^0(I_{Z'}(c,d)) = 0$. Let $A \subset Q$ (resp. $A' \subset Q$) be a general union of $e$ (resp. $e'$) 2-points. Then $h^1(I_{Z\cup A}(c,d+3)) = 0$ and $h^0(I_{Z\cup A'}(c,d+3)) = 0$.

**Proof.** Since a 3-point is a flat limit of a family of disjoint unions of two 2-points (Remark 1), this lemma is a particular case of Lemma 8. \qed

**Lemma 3.** Fix integers $e \geq 0$, $d \geq 0$ and $c \geq 2$. Let $Z \subset Q$ be a zero-dimensional scheme such that $h^1(I_Z(c,d)) = 0$ and $h^0(I_Z(c,d-1)) = 0$. Set $\gamma := h^0(I_Z(c,d))$, $u := [(c+1)/2]$, and $\epsilon := c + 1 - 2u$. Let $A \subset Q$ be a general union of $e$ 2-points.

(a) If $e \leq u$, then $h^1(I_{Z\cup A}(c,d+1)) \leq \max\{0, e - \gamma\}$.

(b) If $e$ is even, $e = u + 1$ and either $\operatorname{char}(\mathbb{K}) = 0$ or $\operatorname{char}(\mathbb{K}) > c$, then $h^1(I_{Z\cup A}(c,d+1)) \leq \max\{0, u + 2 - \gamma\}$.

**Proof.** Since $h^1(I_Z(c,d)) = 0$, we have $\deg(Z) = (c+1)(d+1) - \gamma$. Since $h^0(I_Z(c,d-1)) = 0$, we have $0 \leq \gamma \leq c+1$ and $h^1(I_Z(c,d-1)) = c+1 - \gamma$. Fix a line $L \subset Q$ of type $(0,1)$ such that $L \cap Z = \emptyset$. First assume $c$ even and $e = u+1$. Fix $S \subset Q$ such that $\sharp(S) = u$ and $o \in L \setminus S$. Let $E \subset L$ be the 2-point of $L$ with $o$ as its reduction. Since $\deg(L \cap (2S \cup \{o\})) = c+1$, By the Differential Horace Lemma for double points ([1], Lemme 1.3, [3], Lemma 5) to prove the lemma it is sufficient to prove that $h^1(I_{Z\cup S \cup E}(c,d)) = \max\{0, u + 2 - \gamma\}$. Let $W \subset H^0(O_L(c))$ be the image of the restriction map $\rho : H^0(I_Z(c,d)) \to H^0(O_L(c))$. Since $h^0(I_Z(c,d-1)) = 0$, $\rho$ is injective. Hence $\dim(W) = \gamma$. Hence to prove part (b) it is sufficient to prove that $S \cup E$ imposes $\min\{\gamma, \deg(S \cup E)\}$ independent conditions to the linear system $W$. In arbitrary characteristic $S$ imposes independent conditions to any linear system $V$ on $L$ with dimension $\geq \sharp(S)$. This is also true for $S \cup E$ if either $\operatorname{char}(\mathbb{K}) = 0$ or $\operatorname{char}(\mathbb{K}) > c$, because $E$ is a general tangent vector of $L$ and our assumption on $\operatorname{char}(\mathbb{K})$ implies that the rational map induced by $V$ is separable.

The case $e \leq u$ is easier. Indeed, we do not use Differential Horace and in the residual we only have $Z \cup S$ with $\sharp(S) = e$, instead of $Z \cup S \cup E$. \qed

**Lemma 4.** Assume $\operatorname{char}(\mathbb{K}) = 0$. Fix integers $d \geq 0$ and $c \geq 2$. Let $Z \subset Q$ be a zero-dimensional scheme such that $h^1(I_Z(c,d)) = 0$ and $h^0(I_Z(c,d-1)) = 0$. Set $\gamma := h^0(I_Z(c,d))$ and $e' := [(2c + 2 + \gamma)/3]$. If $c$ is odd and $\gamma = c$, then set $e := e' - 1$. If $c$ is even and $\gamma \in \{c-1, c\}$, then set $e := e' - 1$. In all other cases set $e := e'$. Let $U \subset Q$ be a general union of $e - [(c+1)/2]$ 2-points. Then $h^1(I_{Z\cup U}(c,d+2)) = 0$.
Proof. Since $h^0(I_Z(c, d-1)) = 0$, we have $\gamma \leq c + 1$. Moreover, $\gamma = c + 1$ if and only if $h^1(I_Z(c-1, d)) = 0$. Hence the case $\gamma = c+1$ is true by Lemma 2. From now on we assume $\gamma \leq c$. Since each connected component of $U$ contains a general tangent vector of $Q$ at its support and $U_{\text{red}}$ is general, it is sufficient to prove that $f := \sharp(U_{\text{red}}) \geq \gamma/2$. Set $f' := f + e' - c$. Let $g \in \{0, 1, 2\}$ the congruence class of $2c+2+\gamma$ modulo 3. First assume that $c$ is odd. In this case we get $f' = (2c+2+\gamma)/3 - g/3 - (c+1)/2$, i.e. $2f' = c/3 + 2\gamma/3 - 2g/3 + 1/3$. We have $2f' \geq \gamma$ if and only if $c/3 - 2g/3 + 1/3 \geq \gamma/3$. The last inequality is obviously true if $\gamma \leq c - 1$. Now assume that $c$ is even. In this case we have $f' = (2c+2+\gamma)/3 - g/3 - c/2 - 1$, i.e. $2f' = c/3 + 2\gamma/3 - 2g/3 - 1/3$. We conclude if $\gamma \leq c - 2$. \hfill \Box

Lemma 5. Fix integers $d \geq 0$ and $c \geq 2$. Let $Z \subset Q$ be a zero-dimensional scheme such that $h^1(I_Z(c, d)) = 0$ and $h^0(I_Z(c, d-1)) = 0$. Set $\gamma := h^0(I_Z(c, d))$. Take an integer $e$ such that $0 \leq 3e \leq 2c + 2 + \gamma$. Let $A \subset Q$ be a general union of $e$ 2-points. If $c$ is odd assume that either $\text{char}(\mathbb{K}) = 0$ or $\text{char}(\mathbb{K}) > c$. Then $h^1(I_{Z\cup A}(c, d+2)) = 0$.

Proof. Since $h^1(I_Z(c, d)) = 0$, we have $\deg(Z) = (c + 1)(d + 1) - \gamma$. Since $h^0(I_Z(c, d-1)) = 0$, we have $0 \leq \gamma \leq c + 1$ and $h^1(I_Z(c, d-1)) = c + 1 - \gamma$. Since the case $\gamma = c + 1$ is true by Lemma 2 applied to the integers $(c, d-1)$, we may assume $\gamma \leq c$. Hence $e \leq c$. It is sufficient to do the case $e = [(2c+2+\gamma)/3]$. In particular we assume $e \geq [(c+1)(2)]$. Fix a line $L \subset Q$ of type $(0, 1)$ such that $L \cap Z = \emptyset$. First assume that $c$ is odd. Let $A' \subset Q$ be a general union of $e - (c + 1)/2$ 2-points. Take $S \subset L$ such that $\sharp(S) = (c + 1)/2$. It is sufficient to prove that $h^1(I_{Z\cup A'\cup S}(c, d+2)) = 0$. Since $\deg(2S \cap L) = c + 1$, it is sufficient to prove that $h^1(I_{Z\cup A'\cup S}(c, d+1)) = 0$. Since $3e \leq 2c + 2 + \gamma$, we have $\deg(A') \leq c + 1 + \gamma$. Lemma 3 gives $h^1(I_{Z\cup A'}(c, d+1)) = 0$. Hence $h^0(I_{Z\cup A'}(c, d+1)) = (c+1)(d+2) - \deg(Z) - \deg(A') - \gamma + c + 1 - 3e + 3(c+1)/2 \geq (c+1)/2$ and to prove the lemma for $c$ odd it is sufficient to prove that $S$ imposes $\sharp(S)$ independent conditions to $H^0(I_{Z\cup A'}(c, d+1))$. Let $W \subset H^0(\mathcal{O}_L(c))$ be the image of the restriction map $\rho : H^0(I_{Z\cup A'}(c, d+1)) \to H^0(\mathcal{O}_L(c))$. Since $h^0(I_{Z\cup A'}(c, d)) = 0$ (Lemma 4), $\rho$ is injective. Hence $\dim(W) \geq (c+1)/2 - \sharp(S)$. Hence it is sufficient to use that $S$ is general in $L$.

Now assume that $c$ is even. Let $A'' \subset Q$ be a general union of $e - c/2 - 1$ 2-points of $Q$. Take a general $S' \cup \{o\} \subset L$ such that $\sharp(S') = c/2$ and $o \notin S'$. Let $E \subset L$ be the 2-point of $L$ with $o$ as its support. Since $\deg(2S' \cap L) + \deg(\{o\}) = c + 1$, the Differential Horace Lemma for 2-points shows that to prove that a general union $Y$ of $Z \cup 2S \cup A''$ and a 2-point satisfies $h^1(I_Y(c, d+2)) = 0$ (and hence to prove the lemma in the case $c$ even), it is sufficient to prove
that \( h^1(\mathcal{I}_{Z\cup A'}\cup S\cup E(c, d + 1)) = 0\). Lemma 3 gives \( h^1(\mathcal{I}_{Z\cup A'}(c, d + 1)) = 0\). Let \( V \subset H^0(\mathcal{O}_L(c)) \) be the image of the restriction map \( \rho' : H^0(\mathcal{I}_{Z\cup A'}(c, d + 1)) \to H^0(\mathcal{O}_L(c)) \). Since \( h^0(\mathcal{I}_{Z\cup A'}(c, d)) = 0 \) (Lemma 4), \( \rho \) is injective. Hence \( \dim(V) \geq \#(S) + \deg(E) \). Hence it is sufficient to use that \( S \cup \{o\} \) is general in \( L \) and hence that \( E \) is a general tangent vector of \( L \setminus S \).

**Lemma 6.** Fix integers \( c \geq 2 \) and \( d \geq 0 \). Let \( Z \subset Q \) be a zero-dimensional scheme. Let \( A \subset Q \) (resp. \( A' \subset Q \)) be a general union of \( [4(c + 1)/3] \) (resp. \( [4(c + 1)/3] \)) 2-points. Then \( h^1(\mathcal{I}_{Z\cup A}(c, d + 4)) \leq h^1(\mathcal{I}_Z(c, d)) \) and \( h^0(\mathcal{I}_{Z\cup A'}(c, d + 4)) \leq h^0(\mathcal{I}_Z(c, d)) \).

**Proof.** Adapt either the proof of Lemma 1 or the one of Lemma 5 quoting Lemmas 2 and 1 for the residual with respect to \( L \) instead of Lemmas 4 and 3.

**Lemma 7.** Fix integers \( c \geq 2 \) and \( d \geq 2 \). Let \( Z \subset Q \) be a zero-dimensional scheme. Set \( \gamma := h^0(\mathcal{I}_Z(c, d)) \). Assume \( \gamma \leq c \), \( h^0(\mathcal{I}_Z(c - 1, d - 2)) = 0 \) and \( h^1(\mathcal{I}_Z(c, d)) = 0 \). Let \( U \subset Q \) be a general union of \( [(2c + d + 4 + \gamma)/3] \) 2-points of \( Q \). Then \( h^1(\mathcal{I}_{Z\cup U}(c + 1, d + 2)) = 0 \).

**Proof.** Let \( C \subset Q \) be a smooth curve of type \((1, 2)\) such that \( C \cap Z = \emptyset \). We have \( (c+2)(d+3) - (c+1)(d+1) = 2c + d + 4 \). Since \( \gamma \leq c \), \( c \geq 2 \) and \( d \geq 2 \), we have \( [(2c + d + 4 + \gamma)/3] \leq [(2c + d + 1)/2] \). Take a general \( S \subset C \) such that \( \#(S) = [(2c + d + 4 + \gamma)/3] \). It is sufficient to prove that \( h^1(\mathcal{I}_{Z\cup 2S}(c+1, d+2)) = 0 \). Since \( \deg(2S \cap C) \leq 2c + d + 1 \) and \( Z \cap C = \emptyset \), it is sufficient to prove that \( h^1(\mathcal{I}_{Z\cup S}(c, d)) = 0 \). Let \( V \subset H^0(\mathcal{O}_C(c)) \) be the image of the restriction map \( \rho' : H^0(\mathcal{I}_Z(c, d)) \to H^0(\mathcal{O}_C(c, d)) \). Since \( h^0(\mathcal{I}_Z(c - 1, d - 2)) = 0 \) and \( Z \cap C = \emptyset \), \( \rho \) is injective. Since \( \dim(V) = \gamma \) and \( S \) is general in \( C \), \( S \) imposes \( \gamma \) independent conditions to \( V \). Since \( \rho \) is injective, we get \( h^0(\mathcal{I}_{Z\cup S}(c, d)) = 0 \).

**Proof of Theorem 3.** If \( a = c \) and \( b = d + 2 \), then we quote Lemma 5. Now assume \( a = c \) and \( b \geq d + 3 \). In this case we copy the proof of Lemma 5 and induction on \( d \). If \( a = c + 1 \) and \( b = d + 2 \), then we quote Lemma 7. Then for a fixed \( a \) we get all cases with \( b > d + 2 \) using induction on \( b \) and the Horace Differential lemma for double points with respect to a general \( L \in |\mathcal{O}_Q(1, 0)| \).

**Proof of Theorem 2.** Taking a union of \( Z \) and \( \gamma := h^0(\mathcal{I}_Z(a, b)) \) general points of \( Q \) we reduce to the case \( \gamma = 0 \). Then we apply Theorem 3.
Proposition 1. Fix integers \( m \geq 3, e > 0, a > m, b > m \) such that \((a, b) \neq (m + 1, m + 1)\). Let \( A \subset Q \) be a general union of two \( m \)-points and \( e \) 2-points. Then either \( h^1(\mathcal{I}_A(a, b)) = 0 \) or \( h^0(\mathcal{I}_A(a, b)) = 0 \).

Proof. Just using Bezout theorem we get

\[ h^i(\mathcal{O}_{mP \cup mO}(m, m - 1)) = h^i(\mathcal{O}_{mP \cup mO}(m - 1, m)), \]

\( i = 0, 1 \). Apply Theorem 2. \( \square \)

3. The Proofs for 3-Points

Lemma 8. Fix non-negative integers \( c, d, e, f, e', f' \) such that \( c \geq 2, d \geq 0, f \leq 2[(c + 1)/4], f' \leq 2[(c + 1)/4], e \leq c + 1 - 2f \) and \( e' \geq c + 1 - 2f' \). Fix zero-dimensional schemes \( Z \subset Q \) and \( Z' \subset Q \) such that \( h^1(\mathcal{I}_Z(c, d)) = 0 \) and \( h^0(\mathcal{I}_{Z'}(c, d)) = 0 \). Let \( A \subset Q \) (resp. \( A' \subset Q \)) be a general union of \( e \) 2-points and \( f \) 3-points (resp. \( e' \) 2-points and \( f' \) 3-points). Then \( h^1(\mathcal{I}_{Z \cup A}(c, d + 3)) = 0 \) and \( h^0(\mathcal{I}_{Z' \cup A'}(c, d + 3)) = 0 \).

Proof. Set \( x := \lfloor (c + 1)/4 \rfloor \). Since any 3-point of \( Q \) is a flat limit of a family of disjoint unions of pairs of 2-points of \( Q \) (Remark 1), the semicontinuity theorem for cohomology shows that it is sufficient to do the case \( f = f' = 2x \). Fix a line \( L \subset Q \) of type \((0, 1)\) such that \( L \cap Z = L \cap Z' = \emptyset \). Fix a general \( S \cup S' \subset L \) such that \( S \cap S' = \emptyset \) and \( \sharp(S) = \sharp(S') = x \). Fix general \( o, o', o'' \in L \setminus S \cup S' \). Fix \( o'_1 \in S' \) and set \( S'_1 := S' \setminus \{o'_1\} \).

First assume \( c \equiv 3 \pmod{4} \). In this case we apply 3 times the Horace Differential Method for 3-points to each point of \( S' \) with respect to the integers \( 3 > 2 \) so that \((1, 2, 3)\) are the degrees of the intersection with \( L \) of the first, second and third trace ([5], [7], [8], [9]), while we specialize \( x \) of the 3-points of \( A \) and \( A' \) to 3-points with a point of \( S \) as their support. At each step in \( L \) we have a scheme of degree \( c + 1 \).

Now assume \( c \equiv 1 \pmod{4} \). In the first step we also add \( 2o \). In this step we have a scheme whose intersection with \( L \) has degree \( c + 1 \), while in the second step the scheme on \( L \) has only degree \( c \), because \( \text{Res}_L(2o) = \{o\} \). Hence at the second step we also use the Horace Differential lemma for double points with respect to \( o' \), so that in \( L \) we have a degree \( c + 1 \) scheme, while at the third step we have a scheme sitting of degree \( c + 1 \) inside \( L \), with no connected component with \( o \) as its reduction and with a degree 2 component with \( o' \) as its reduction.
Now assume $c \equiv 0 \pmod{4}$. We take $3S$ and apply the Differential Horace Method for $3$-points with respect to the integers $3 > 2$, i.e. with traces of degrees $(1, 2, 3)$, for each point of $S'_1$ and with respect to the sequence $2$ (i.e. with traces of degrees $(2, 1, 3)$) with respect to $o'_1$; hence the intersection of $L$ with this virtual scheme has degree $3x + (x - 1) + 2 = c + 1$. ($x$ components of degree $3$, $x - 1$ components of degree $1$ and one of degree $2$). At the second step we also add the scheme $2o$. The new virtual scheme intersects $L$ in a scheme with degree $c + 1$ ($2x + 1$ of its connected components have degree $2$, the one supported by $o'_1$ has degree $1$). In the last step, after taking the virtual residual scheme, we get a scheme of degree $c + 1$ ($x + 1$ components of degree $1$, $x$ components of degree $3$).

Now assume $c \equiv 2 \pmod{4}$. We first add $3S$, apply the Differential Horace lemma for $3$-points with respect to the integers $3 > 2$, i.e. with traces of degrees $(1, 2, 3)$, at each point of $S'_1$, the Differential Horace points for $3$-points with intersections with $L$ of degree $(3, 1, 2)$ (it is the example done in the introduction of [5]) at $o'_1$ and the differential Horace lemma for double points with respect to $o'$. The intersection of the virtual residual scheme with $L$ has only degree $4x + 1 = c - 1$ ($2x$ connected components of degree $2$ and one of degree $1$). Therefore in the second step we also apply the Differential Horace lemma for $2$-points at $o'$ and $o''$; in this way the restriction of this scheme to $L$ has degree $c + 1$ ($2x$ components of degree $2$ and $3$ components of degree $1$). The virtual residue scheme with respect to $L$ is contained in $L$ and it has degree $c + 1$ (it has $x$ components of degree $1$ with $S$ as the union of their reductions, $x - 1$ components of degree $x - 1$ with $S'_1$ as the union of their reductions and $3$ components of degree $2$ with $o'_1$, $o'$ and $o''$ as their reduction). \qed

Proof of Theorem 1. We have $h^1(I_Z(a, x)) = 0$ for all $x \geq d$. We apply $\lfloor (d - b - 2)/3 \rfloor$ times Lemma 8, first with respect to the pair $(c, d) := (a, b)$, then with respect to the pairs $(c, d) := (a, b - 3)$, and so on. Then we apply the lemmas for $2$-points with respect to the integers $(a, b) := (a, b - 3)(d - b - 2)/3 \rfloor$). \qed

Remark 2. The quoted parts of [8] and [9] and the Differential Horace lemma for double points ([3]) are characteristic free. We only use the condition $\text{char}(K) > \max\{a, b, 2c + d\}$ (the only problem is to check Lemma 4 when $\text{char}(K) > 2c + d$).
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